CIRCUIT ANALYSIS OF A SIMPLE TRANSISTOR RADIO

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Abstract. According to [2], a simple AM transistor radio is made up of 4 components, the tuning circuit, the detector circuit, the filter circuit and the amplifier circuit. We analyse each of these components in turn, calculate their Thenevin equivalents, and use this to justify the values of resistors, capacitors, inductors, transistors and batteries in the radio, to pick up a range of AM frequencies.

Definition 0.1. *A tuning circuit consists of an aerial attached to a series LC circuit, which we assume provides the driving voltage $V_0cos(\omega t)$.*

Lemma 0.2. *For a series LC circuit with a forcing voltage;*

$V(t) = V_0cos(\omega t)$

the general solution to the response current, is given by;

$I(t) = \frac{V_0\omega sin(\omega t)}{L(\omega^2-\frac{1}{LC})} + Acos(\frac{t}{\sqrt{LC}}) + Bsin(\frac{t}{\sqrt{LC}})$

and this agrees with the phaser current solution, calculated using the method of impedance. In particularly, we obtain a resonant current when the frequency;

$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$

Proof. Let V_L and V_C be the voltages across the inductor and capacitor respectively. By Kirchoff's voltage law;

$V_L + V_C - V_0cos(\omega t) = 0\ (A)$

and, by the rules for the inductor and capacitor respectively;

$V_L = L\frac{dI}{dt}\ (B)$

$I = C\frac{dV_C}{dt}$ (C)

From (B, C, A);

$\frac{dV_L}{dt} = L\frac{d^2I}{dt^2}$ (D)

$\frac{dV_C}{dt} = \frac{I}{C}$ (E)

$\frac{dV_L}{dt} + \frac{dV_C}{dt} = -V_0\omega sin(\omega t)$ (F)

Substituting (D, E) into (F);

$L\frac{d^2I}{dt^2} + \frac{I}{C} = -V_0\omega sin(\omega t)$

or equivalently;

$\frac{d^2I}{dt^2} + \frac{I}{LC} = -\frac{V_0\omega}{L}sin(\omega t) = g(t)$ $(*)$

This is a second order ODE, with homogeneous equation given by;

$I'' + \frac{I}{LC} = 0$

the general solution for which is;

$Ay_1(t) + By_2(t)$

where;

$y_1(t) = cos(\frac{t}{\sqrt{LC}})$

$y_2(t) = sin(\frac{t}{\sqrt{LC}})$

The Wronskian $W(y_1, y_2)$ is given by;

$y_1(t)y_2'(t) - y_2(t)y_1'(t)$

$= \frac{1}{\sqrt{LC}}(cos^2(\frac{t}{\sqrt{LC}}) + sin^2(\frac{t}{\sqrt{LC}}))$

$= \frac{1}{\sqrt{LC}}$

By Lagrange's variation of parameters, a particular solution to $(*)$ is given by;

$$I(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt$$

$$= \frac{\sqrt{LC}}{L}[-cos(\frac{t}{\sqrt{LC}})\int -sin(\frac{t}{\sqrt{LC}})V_0\omega sin(\omega t)dt + sin(\frac{t}{\sqrt{LC}})\int -cos(\frac{t}{\sqrt{LC}})V_0\omega sin(\omega t)dt]$$

$$= V_0\omega\sqrt{\frac{C}{L}}[cos(\frac{t}{\sqrt{LC}})\int sin(\frac{t}{\sqrt{LC}})sin(\omega t)dt - sin(\frac{t}{\sqrt{LC}})\int cos(\frac{t}{\sqrt{LC}})sin(\omega t)dt]$$

$$= \frac{V_0\omega\sqrt{\frac{C}{L}}}{2}[cos(\frac{t}{\sqrt{LC}})\int(cos(t(\frac{1}{\sqrt{LC}}-\omega)) - cos(t(\frac{1}{\sqrt{LC}}+\omega)))dt - sin(\frac{t}{\sqrt{LC}})\int(sin(t(\frac{1}{\sqrt{LC}}$$

$$+\omega)) + sin(t(\omega - \frac{1}{\sqrt{LC}})))]$$

$$= \frac{V_0\omega\sqrt{\frac{C}{L}}}{2}[\frac{cos(\frac{t}{\sqrt{LC}})sin(t(\frac{1}{\sqrt{LC}}-\omega))}{(\frac{1}{\sqrt{LC}}-\omega)} - \frac{cos(\frac{t}{\sqrt{LC}})sin(t(\frac{1}{\sqrt{LC}}+\omega))}{(\frac{1}{\sqrt{LC}}+\omega)} + \frac{sin(\frac{t}{\sqrt{LC}})cos(t(\frac{1}{\sqrt{LC}}+\omega))}{(\frac{1}{\sqrt{LC}}+\omega)}$$

$$+\frac{sin(\frac{t}{\sqrt{LC}})cos(t(\omega-\frac{1}{\sqrt{LC}}))}{(\omega-\frac{1}{\sqrt{LC}})}]$$

$$= \frac{V_0\omega\sqrt{\frac{C}{L}}}{2}[\frac{sin(t(\frac{1}{\sqrt{LC}}-\omega-\frac{1}{\sqrt{LC}}))}{(\frac{1}{\sqrt{LC}}-\omega)} - \frac{sin(t(\frac{1}{\sqrt{LC}}+\omega-\frac{1}{\sqrt{LC}}))}{(\frac{1}{\sqrt{LC}}+\omega)}]$$

$$= \frac{V_0\omega\sqrt{\frac{C}{L}}}{2}[-\frac{sin(\omega t)}{(\frac{1}{\sqrt{LC}}-\omega)} - \frac{sin(\omega t)}{(\frac{1}{\sqrt{LC}}+\omega)}]$$

$$= -\frac{V_0\omega sin(\omega t)\sqrt{\frac{C}{L}}}{2}[\frac{1}{(\frac{1}{\sqrt{LC}}-\omega)} + \frac{1}{(\frac{1}{\sqrt{LC}}+\omega)}]$$

$$= -\frac{V_0\omega sin(\omega t)\sqrt{\frac{C}{L}}}{2}[\frac{\frac{2}{\sqrt{LC}}}{(\frac{1}{LC}-\omega^2)}]$$

$$= \frac{V_0\omega sin(\omega t)}{L(\omega^2-\frac{1}{LC})}$$

and the general solution to $(*)$ is given by;

$$I(t) = \frac{V_0\omega sin(\omega t)}{L(\omega^2-\frac{1}{LC})} + Acos(\frac{t}{\sqrt{LC}}) + Bsin(\frac{t}{\sqrt{LC}}) \ (**)$$

For the method of impedance, see [1], we have the total impedance Z, in a series LC circuit, is given by;

$$Z = Z_L + Z_C = i\omega L + \frac{1}{i\omega C}$$

so that the phaser current, $I' = \frac{V'}{Z}$, where V' is the phaser voltage $V' = V_0e^{i\omega t}$ and the real forcing voltage is $Re(V')$;

$I' = \frac{V_0 e^{i\omega t}}{i\omega L + \frac{1}{i\omega C}}$

$= \frac{V_0 i e^{i\omega t}}{-\omega L + \frac{1}{\omega C}}$

$= \frac{V_0 \frac{\omega}{L} i e^{i\omega t}}{\frac{1}{LC} - \omega^2}$

and taking real parts $I = Re(I')$;

$I = \frac{V_0 \omega sin(\omega t)}{L(\omega^2 - \frac{1}{LC})}$

which agrees with $(**)$ as a particular solution.

□

Lemma 0.3. *Let hypotheses be as in lemma 0.2 and let the driven LC tuning circuit be tapped across the inductor, with inductance L_2 across the tap and L_1 remaining in the original LC circuit, so that $L = L_1 + L_2$. Then, if we attach a new network A across the tap, an equivalent network is given by the driving voltage;*

$V' = \frac{V_0 \omega^2 cos(\omega t)}{L(\omega^2 - \frac{1}{LC})}$

in series with the network A and an impedance;

$Z' = i(\frac{\omega L_2(\omega^2 C L_1 - 1)}{\omega^2 LC - 1})$

Proof. Using Thenevin's theorem, the driving voltage V' is given by the open circuit voltage V_{oc}, between the terminals where A is attached. We let I_1 be the current in the original LC loop and I_2 be the current in the tapping loop. Then;

$V_{oc} = L_2 \frac{d}{dt}(I_1 - I_2)$

$= L_2 \frac{dI_1}{dt}$

as no current is drawn in the open circuit. We clearly have that $I = I_1$, where I was found in Lemma 0.2, so that;

$V_{oc} = \frac{V_0 \omega^2 cos(\omega t)}{L(\omega^2 - \frac{1}{LC})} + A' cos(\frac{t}{\sqrt{LC}}) + B' sin(\frac{t}{\sqrt{LC}})$

where $\{A', B'\}$ are arbitrary constants. We have that $Z' = Z_{th}$, where Z_{th} is the impedance looking into the network from the terminals of A and replacing the original driving voltage V with a short circuit. Z_{th} consists of C and L_1 in parallel with L_2, so that;

$$\frac{1}{Z_{th}} = \frac{1}{\frac{1}{i\omega C}+i\omega L_1} + \frac{1}{i\omega L_2}$$

$$= i\left(\frac{1}{\frac{1}{\omega C}-\omega L_1} - \frac{1}{\omega L_2}\right)$$

$$= i\left(\frac{\omega^2 CL_2-(1-\omega^2 CL_1)}{\omega L_2(1-\omega^2 CL_1)}\right)$$

$$= i\left(\frac{\omega^2 LC-1}{(1-\omega^2 CL_1)\omega L_2}\right)$$

so that;

$$Z_{th} = -i\left(\frac{\omega L_2(1-\omega^2 CL_1)}{\omega^2 LC-1}\right)$$

$$= i\left(\frac{\omega L_2(\omega^2 CL_1-1)}{\omega^2 LC-1}\right)\ (*)$$

Noting that Z_{th} is infinite, when $\omega = \frac{1}{\sqrt{LC}}$, the associated current to a forcing voltage of;

$$A'cos\left(\frac{t}{\sqrt{LC}}\right) + B'sin\left(\frac{t}{\sqrt{LC}}\right)$$

is zero, even including the in series network A, and, by linearity of ODE's, for the impedance connected in series to the network A, the current is determined with $A' = B' = 0$ and a forcing voltage of;

$$V' = \frac{V_0\omega^2 cos(\omega t)}{L(\omega^2-\frac{1}{LC})}$$

through Z', in series with A, given by $(*)$.

□

Definition 0.4. *A detector circuit consists of a diode in series with a resistor and capacitor in parallel. We assume the diode is ideal.*

Lemma 0.5. *Let hypotheses be as in the previous lemma, and let a detector circuit be attached to the tuning circuit as network A. Let the forward resistance of the diode be R_1, the remaining resistance be R_2 and the capacitance be C_1. Let I_1 denote the current flowing through the impedance Z', let I_2 be the current flowing through C_1 and let the* *driving voltage be V'. Then, during the half cycle when the diode is on;*

$$I_1 = Re[\frac{V_0'(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}]$$

$$I_2 = Re[V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}} - \frac{1}{R_2}]e^{i\omega t}]$$

and during the half cycle when the diode is off, letting $t_m = \frac{(m-\frac{1}{2})\pi-\phi}{\omega}$, where;

$$\phi = arg(V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}} - \frac{1}{R_2}])$$

is the phase angle, we have that;

$$I_1 = 0$$

$$I_2 = \frac{-2|V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}-\frac{1}{R_2}]|}{C_1R_2(1-e^{\frac{-\pi}{\omega R_2C_1}})}e^{\frac{-(t-t_{2n+1})}{R_2C_1}}$$

There is no additional resonant current in the circuit.

Proof. During the half cycle, when the diode is on, let $V' = V_0'e^{i\omega t}$ and let the phaser currents be $\{I_1, I_2\}$, then by Kirchoff's voltage law around the two loops;

$$V' = I_1Z' + I_1R_1 + (I_1 - I_2)R_2$$

$$= I_1(Z' + R_1 + R_2) - I_2R_2 \ (*)$$

$$V_{C_1} = (I_1 - I_2)R_2$$

so that, by the rule for a capacitor;

$$\frac{dV_{C_1}}{dt} = (\frac{dI_1}{dt} - \frac{dI_2}{dt})R_2 = \frac{I_2}{C_1}, \ (**)$$

and rearranging $(*)$;

$$I_2 = I_1(\frac{Z'+R_1+R_2}{R_2}) - \frac{V'}{R_2} \ (A)$$

$\frac{dI_2}{dt}=\frac{dI_1}{dt}(\frac{Z'+R_1+R_2}{R_2})-\frac{i\omega V_0'}{R_2}e^{i\omega t}$

and substituting into $(**)$;

$R_2\frac{dI_1}{dt}-R_2(\frac{dI_1}{dt}(\frac{Z'+R_1+R_2}{R_2})-\frac{i\omega V_0'}{R_2}e^{i\omega t})=\frac{1}{C_1}(I_1(\frac{Z'+R_1+R_2}{R_2})-\frac{V'}{R_2})$

so that;

$\frac{dI_1}{dt}(R_2-(Z'+R_1+R_2))-i\omega V_0'e^{i\omega t}-I_1(\frac{Z'+R_1+R_2}{R_2C_1})=-\frac{V'}{R_2C_1}$

or equivalently;

$\frac{dI_1}{dt}(Z'+R_1)+I_1(\frac{Z'+R_1+R_2}{R_2C_1})=V_0'(\frac{1}{R_2C_1}-i\omega)e^{i\omega t}$

so that;

$\frac{dI_1}{dt}+I_1(\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1})=V_0'(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}$

and multiplying by the integrating factor $e^{\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}$, we obtain;

$\frac{d}{dt}(e^{\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}I_1)=e^{\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}V_0'(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}$

so that;

$e^{\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}I_1=\frac{e^{\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}V_0'(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}+D$

$I_1=\frac{V_0'(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}+De^{-\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}$

where $D\in\mathcal{C}$ is arbitrary, and, using (A);

$I_2=\frac{V_0'(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})e^{i\omega t}}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}-\frac{V_0'e^{i\omega t}}{R_2}+D(\frac{Z'+R_1+R_2}{R_2})e^{-\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}$

$=V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}-\frac{1}{R_2}]e^{i\omega t}+D(\frac{Z'+R_1+R_2}{R_2})e^{-\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}t}$

During the cycle $[t_{2n+1},t_{2n+2}]$, $n\geq 0$ when the diode is off, clearly $I_1=0$, and then;

$V_{C_1}=-R_2I_2$

so that by the rule for a capacitor;

$\frac{dV_{C_1}}{dt}=-R_2\frac{dI_2}{dt}=\frac{I_2}{C_1}$

It follows that, on the cycle when the diode is off;

$I_2=Fe^{\frac{-(t-t_{2n+1})}{R_2C_1}}$ (B)

where $F=-\frac{V_{C_1}}{R_2}|_{t=t_{2n+1}}$

and;

$V_{C_1}|_{t=t_{2n+1}}-V_{C_1}|_{t=t_{2n}}=\int_{t_{2n}}^{t_{2n+1}}\frac{I_2(t)}{C_1}dt$

where $[t_{2n},t_{2n+1}]$, denotes the previous cycle when the diode is on. To obtain equilibrium in the circuit, we require that;

$-R_2F_{eq}e^{\frac{-(t_{2n+2}-t_{2n+1})}{R_2C_1}}+\int_{t_{2n+2}}^{t_{2n+3}}\frac{I_2(t)}{C_1}dt=-R_2F_{eq}$

so that;

$F_{eq}=-\frac{\int_{t_{2n+2}}^{t_{2n+3}}\frac{I_2(t)}{C_1}dt}{R_2(1-e^{\frac{-(t_{2n+2}-t_{2n+1})}{R_2C_1}})}$ $V_{C_1,eq}=\frac{\int_{t_{2n+2}}^{t_{2n+3}}\frac{I_2(t)}{C_1}dt}{1-e^{\frac{-(t_{2n+2}-t_{2n+1})}{R_2C_1}}}$

should be independent of n, which we can achieve with $D=0$ and, letting ϕ be the phase angle of I_2, with $D=0$ and the diode on, so that;

$I_2=Ecos(\omega t+\phi)$

$E=|V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}-\frac{1}{R_2}]|$

$\phi=arg(V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}}-\frac{1}{R_2}])$

for sufficiently large $m\geq 0$, making $t_m=\frac{(m-\frac{1}{2})\pi-\phi}{\omega}$ be the times. We then have that;

$F_{eq}=-\frac{\int_{t_{2n+2}}^{t_{2n+3}}\frac{I_2(t)}{C_1}dt}{R_2(1-e^{\frac{-(t_{2n+2}-t_{2n+1})}{R_2C_1}})}$

$$= -\frac{\int_{\frac{2n+\frac{3}{2}-\phi}{\omega}}^{\frac{2n+\frac{5}{2}-\phi}{\omega}} \frac{Ecos(\omega t+\phi)}{C_1}dt}{R_2(1-e^{\frac{-\pi}{\omega R_2 C_1}})}$$

$$= \frac{-2E}{C_1 R_2(1-e^{\frac{-\pi}{\omega R_2 C_1}})}$$

For the final claim, we would obtain additional resonant currents if;

$$i\omega + \frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1} = 0$$

but Z' is purely imaginary and ω is real.

□

Lemma 0.6. *Let hypotheses be as in lemma 0.4, then, if we attach a new network B across the last capacitor, an equivalent network is given by the driving voltage;*

$$V'' = ?$$

in series with the network A and an impedance;

$$Z'' = ?$$

where the network A is now the combination of the tuning and detector circuits.

Proof. Using Thenevin's theorem, the driving voltage V'' is given by the open circuit voltage V_{oc}, between the terminals where B is attached. Clearly, we have that;

$$V_{oc} = V_{C_1}$$

where;

$$V_{C_1} = \frac{2|V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}} - \frac{1}{R_2}]|}{C_1(1-e^{\frac{-\pi}{\omega R_2 C_1}})} e^{\frac{-(t-t_{2n+1})}{R_2C_1}}, \text{ for } t \in [t_{2n+1}, t_{2n+2}]$$

$$V_{C_1} = \frac{2|V_0'[\frac{(\frac{Z'+R_1+R_2}{R_2})(\frac{\frac{1}{R_2C_1}-i\omega}{Z'+R_1})}{i\omega+\frac{Z'+R_1+R_2}{(Z'+R_1)R_2C_1}} - \frac{1}{R_2}]|}{C_1(1-e^{\frac{-\pi}{\omega R_2 C_1}})} e^{\frac{-\pi}{\omega R_2 C_1}} + \int_{t_{2n+2}}^{t} \frac{I_2(t)}{C_1}dt, \text{ for } t \in [t_{2n+2}, t_{2n+3}]$$

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The Thenevin resistance Z'' ??, need a Fourier series to decompose the periodic driving voltage, same phasor analysis for the filter circuit.

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References

[1] Engineering Circuit Analysis Third Edition, W.H.Hayt and J.E.Kemmerly Jr, McGraw Hill, (1978).

[2] Building and Designing Transistor Radios, A Beginner's Guide, R.H.Warring, The Lutterworth Press, (1977).

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