DE BROGLIE'S HYPOTHESIS AND THE WAVE EQUATION

TRISTRAM DE PIRO

Abstract.

Lemma 0.1. *We have that;*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{n!\pi}{2^n[(\frac{n}{2})!]^2}$, *if n is even*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{[\frac{n-1}{2}!]^2 2^n}{n!}$, *if n is odd*

Proof. Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$, then for $n \geq 2$, we have that, using integration by parts;

$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$

$= [cos^{n-1}(y)sin(y)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)sin^2(y)dy$

$= \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)(1 - cos^2(y))dy$

$= (n-1)I_{n-2} - (n-1)I_n$

so that, rearranging;

$I_n = \frac{n-1}{n} I_{n-2}$

and, using the fact $I_0 = \pi$, $I_1 = 2$, we have that, for n even;

$I_n = \frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$

and, for n odd;

$I_n = \frac{[\frac{n-1}{2}!]^2 2^n}{n!}$

□

Lemma 0.2. *Let $n \in \mathcal{N}$, $\epsilon > 0$, and let $\gamma_{n,\epsilon}$ be defined by;*

$\gamma_{n,\epsilon}(x) = \frac{1}{\epsilon}cos^n(\frac{\pi x}{2\epsilon})$, *for* $x \in [-\epsilon, \epsilon]$

$\gamma_{n,\epsilon}(x) = 0$, *otherwise*

Then $\gamma_{n,\epsilon}$ has the following properties;

(i). $\gamma_{n,\epsilon} \in C^{n-1}(\mathcal{R})$.

(ii). $\gamma_{n,\epsilon} \geq 0$.

(iii). $\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}$, *n even*

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{[\frac{n-1}{2}!]^2 2^{n+1}}{\pi n!}$, *n odd*

(iv) $\gamma_{n,\epsilon}$ *is supported on* $[-\epsilon, \epsilon]$.

Proof. (ii) is clear as $cos(y) \geq 0$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, (iv) is clear by the definition of $\gamma_{n,\epsilon}$. To prove (i), it is sufficient to show that;

$$cos^n(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon) = cos^n(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon) = 0$$

for $0 \leq m \leq n-1$. We can prove this by induction on n, as for $n = 1$, we have that;

$$cos(\frac{\pi x}{2\epsilon})(\epsilon) = cos(\frac{\pi}{2}) = cos(\frac{\pi x}{2\epsilon})(-\epsilon) = cos(-\frac{\pi}{2}) = 0$$

and, if the inductive hypothesis holds for $n \in \mathcal{N}$, then, for $1 \leq m \leq n$;

$$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon)$$

$$= -[\frac{\pi(n+1)}{2\epsilon}cos^n(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})]^{(m-1)}(\epsilon)$$

$$= -\frac{\pi(n+1)}{2\epsilon}[\sum_{k=0}^{m-1} C_k^{m-1} cos^n(\frac{\pi x}{2\epsilon})^{(m-1-k)} sin(\frac{\pi x}{2\epsilon})^{(k)}](\epsilon)$$

$$= 0$$

and similarly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon) = 0$

while, clearly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})(\epsilon) = cos^{n+1}(\frac{\pi x}{2\epsilon})(-\epsilon) = 0$

To prove (iii), we have that, for $n \in \mathcal{N}$;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx$

$= \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} cos^n(\frac{\pi x}{2\epsilon})$

$= \frac{1}{\epsilon}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)\frac{2\epsilon}{\pi}dy$, $(y = \frac{\pi x}{2\epsilon})$

$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$

so that, using Lemma 0.1, for n even;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$

$= \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}$

and, for n odd;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{[\frac{n-1}{2}!]^2 2^n}{n!}$

$= \frac{[\frac{n-1}{2}!]^2 2^{n+1}}{\pi n!}$

□

Lemma 0.3. *Let $\delta_{n,\epsilon}(x)$ be defined by;*

$\delta_n(x) = \frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\gamma_{n,\epsilon}$, *for n even*

and by;

$\delta_n(x) = \frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n,\epsilon}$, *for n odd*

Then the properties $(i), (ii), (iv)$ of Lemma 0.2 hold, with (iii) changed to;

(iii)'. $\int_{\mathcal{R}}\gamma_{n,\epsilon}(x)dx=1$, $n\in\mathcal{N}$

and, for $n\in\mathcal{N}$*;*

$lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon}=\delta$

in the sense of distributions, where δ *is the Dirac delta function on* $\mathcal{R}$*.*

Proof. The first claim is clear as we have just normalised $\gamma_{n,\epsilon}$. For the remaining claim, let $f\in C_c^\infty(\mathcal{R})$, and write;

$f=f^{+}+f^{-}$

where;

$f^{+}(x)=f(x)$, if $f(x)\geq 0$

$f^{+}(x)=0$ otherwise

$f^{-}(x)=f(x)$, if $f(x)\leq 0$

$f^{-}(x)=0$ otherwise

Then, using properties $(ii),(iii)',(iv)$ of $\delta_{n,\epsilon}$ and continuity of f;

$min_{[-\epsilon,\epsilon]}f^{+}+min_{[-\epsilon,\epsilon]}f^{-}\leq\delta_{n,\epsilon}(f)=\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^{+}(x)dx+\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^{-}(x)dx$

$\leq max_{[-\epsilon,\epsilon]}f^{+}+max_{[-\epsilon,\epsilon]}f^{-}$

with;

$lim_{\epsilon\rightarrow 0}min_{[-\epsilon,\epsilon]}f^{+}+min_{[-\epsilon,\epsilon]}f^{-}=lim_{\epsilon\rightarrow 0}max_{[-\epsilon,\epsilon]}f^{+}+max_{[-\epsilon,\epsilon]}f^{-}$

$=f(0)$

so that $lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon}(f)=f(0)=\delta(f)$, as required.

□

Lemma 0.4. *We define the time derivative* δ'_t *of the delta function* δ *to be;*

$\frac{d}{dt}\delta(x-vt)$

where v *is the velocity, so that;*

$\delta'_t=-v\delta'$

in the sense of distributions. Similarly, we define the time derivative $\delta'_{n,\epsilon,t}$ *of the approximations by;*

$\frac{d}{dt}\delta_{n,\epsilon}(x-vt)$

so that, by the chain rule;

$\delta'_{n,\epsilon,t}(x)=-v\delta'_{n,\epsilon}(x)$

Then;

$lim_{\epsilon\rightarrow 0}\delta'_{n,\epsilon,t}=\delta'_t$

in the sense of distributions.

Proof. For the claim, let $f\in C_c^\infty(\mathcal{R})$, then, using integration by parts, (iv) of Lemma 0.3;

$\delta'_{n,\epsilon,t}(f)=-v\int_{-\epsilon}^{\epsilon}\delta'_{n,\epsilon}(x)f(x)dx$

$=-v([\delta'_{n,\epsilon}(x)f(x)]_{-\epsilon}^{\epsilon}-\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx)$

$=v\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx$

so that, using the main result of Lemma 0.3;

$lim_{\epsilon\rightarrow 0}\delta'_{n,\epsilon,t}(f)$

$=vlim_{\epsilon\rightarrow 0}\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx$

$=vf'(0)$

$= -v\delta'(f)$

$= \delta'_t(f)$

as required.

□

....

References

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: t.depiro@curvalinea.net