# SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 9

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**Lemma 0.1.** *Let potentials* $(V,\overline{A})$ *be given in the base frame* $S$*, satisfying the relations for a fixed charge and current* $(\rho,\overline{J})$*;*

$\bigtriangledown\cdot\overline{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}=0$

$\square^2(V)=-\frac{\rho}{\epsilon_0}$

$\square^2(\overline{A})=-\mu_0\overline{J}$

*then in any transformed frame* $S'$ *with velocity vector* $\overline{v}$ *or rotation* $g\in SO(3)$*, if* $(V',\overline{A}',\rho',\overline{J}')$ *are the transformed quantities, then we have the invariance;*

$\bigtriangledown'\cdot\overline{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t'}=0$

$\square^{2'}(V')=-\frac{\rho'}{\epsilon_0}$

$\square^{2'}(\overline{A}')=-\mu_0\overline{J}'$

*If we let* $(\overline{E},\overline{B})$ *be the electromagnetic fields defined by;*

$\overline{E}=-\bigtriangledown(V)-\frac{\partial\overline{A}}{\partial t}$

$\overline{B}=\bigtriangledown\times\overline{A}$

*then we have the further invariance;*

$\overline{E}'=-\bigtriangledown'(V')-\frac{\partial\overline{A}'}{\partial t'}$

$\overline{B}'=\bigtriangledown'\times\overline{A}'$

*for the transformed quantities* $(V',\overline{A}',\overline{E}',\overline{B}')$*. In particular, it follows, that if* $(\rho,\overline{J},\overline{E},\overline{B})$ *satisfy Maxwell's equations in the base frame*

*$S$, then $(\rho',\overline{J}',\overline{E}',\overline{B}')$ satisfy Maxwell's equations in the frame $S'$.*

*Proof.* For the first claim, we have the following transformation rules to the frame $S'$, where $\overline{v}=v\overline{e}_1$, see [2];

$\bigtriangledown'=(\gamma_v(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}),\frac{\partial}{\partial y},\frac{\partial}{\partial z})$

$\frac{\partial}{\partial t'}=\gamma_v(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x})$

and the transformation rules for $(V,A)$, see [2];

$V'=\gamma_v V-\gamma_v v a_1$

$\overline{A}'=(\gamma_v a_1-\frac{\gamma_v v V}{c^2},a_2,a_3)$

where $\overline{A}=(a_1,a_2,a_3)$. We then compute, using the identity;

$\gamma_v^2(1-\frac{v^2}{c^2})=1$

that;

$\bigtriangledown'\cdot\overline{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t'}=\gamma_v(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t})(\gamma_v a_1-\frac{\gamma_v v V}{c^2})+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$

$+\frac{\gamma_v}{c^2}(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x})(\gamma_v V-\gamma_v v a_1)$

$=\gamma_v^2\frac{\partial a_1}{\partial x}-\frac{\gamma_v^2 v}{c^2}\frac{\partial V}{\partial x}+\frac{\gamma_v^2 v}{c^2}\frac{\partial a_1}{\partial t}-\frac{\gamma_v^2 v^2}{c^4}\frac{\partial V}{\partial t}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$

$+\frac{\gamma_v^2}{c^2}\frac{\partial V}{\partial t}-\frac{\gamma_v^2 v}{c^2}\frac{\partial a_1}{\partial t}+\frac{\gamma_v^2 v}{c^2}\frac{\partial V}{\partial x}-\frac{\gamma_v^2 v^2}{c^2}\frac{\partial a_1}{\partial x}$

$=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}+\frac{1}{c^2}\frac{\partial V}{\partial t}$

$=\bigtriangledown\cdot\overline{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}$

$=0$

For the second and third claims, we have, using the invariance of $\square^2$, the facts that $\square^2(V)=-\frac{\rho}{\epsilon_0}$ and $\square^2(\overline{A})=-\mu_0\overline{J}$ in the base frame $S$, and the transformation rule for $\rho$, that;

$\square^{2'}(V')=\square^{2'}(\gamma_v V-\gamma_v v a_1)$

$= \gamma_v \Box^2(V) - \gamma_v v \Box^2(a_1)$

$= -\frac{\gamma_v \rho}{\epsilon_0} - \gamma_v v(-\mu_0 j_1)$

$= -\frac{1}{\epsilon}(\gamma_v(\rho - \frac{v j_1}{c^2}))$

$= -\frac{\rho'}{\epsilon_0}$

and, using the transformation rules for $\overline{J}$;

$\Box^{2'}(\overline{A}') = \Box^2(\gamma_v a_1 - \frac{\gamma_v v V}{c^2}, a_2, a_3)$

$= (\gamma_v \Box^2(a_1) - \frac{\gamma_v v}{c^2}\Box^2(V), \Box^2(a_2), \Box^2(a_3))$

$= (\gamma_v(-\mu_0 j_1) - \frac{\gamma_v v}{c^2}(\frac{-\rho}{\epsilon_0}), -\mu_0 j_2, -\mu_0 j_3)$

$= -\mu_0(\gamma_v(j_1 - v\rho), j_2, j_3)$

$= -\mu_0(\gamma_v(\overline{J}_{||} - \rho v) + \overline{J}_{\perp})$

$= -\mu_0(\overline{J}')$

The further invariance is checked in [7], while the claim about Maxwell's equations is verified in [6]. For an arbitrary boost $\overline{v}$, choose $g \in SO(3)$, with $g^{-1}(v\overline{e}_1) = \overline{v}$, then by Lemma 1.5 of [7], we have that;

$B_{\overline{v}} = R_g B_{v\overline{e}_1} R_{g^{-1}}$.

It is sufficient to check that the relations;

$\bigtriangledown \cdot \overline{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$\Box^2(V) = -\frac{\rho}{\epsilon_0}$

$\Box^2(\overline{A}) = -\mu_0 \overline{J}$

are preserved by a rotation $g$. This can be achieved using Lemma 1.3 of [7];

$\bigtriangledown' \cdot \overline{A}^g + \frac{1}{c^2}\frac{\partial V^g}{\partial t'}$

$= (\bigtriangledown \cdot \overline{A})^g + \frac{1}{c^2}(\frac{\partial V}{\partial t})^g$

$= 0$

and;

$\square^{2'}(V^g) = \bigtriangledown' \cdot (\bigtriangledown'(V^g)) - \frac{1}{c^2}\frac{\partial^2 V^g}{\partial t'^2}$

$= \bigtriangledown' \cdot ((\bigtriangledown(V))^g) - \frac{1}{c^2}(\frac{\partial^2 V}{\partial t^2})^g$

$= (\bigtriangledown \cdot (\bigtriangledown(V)))^g - \frac{1}{c^2}(\frac{\partial^2 V}{\partial t^2})^g$

$= \square^2(V)^g$

$= -\frac{\rho^g}{\epsilon_0}$

and;

$\square^{2'}(\overline{A}^g) = -\mu_0 \overline{J}^g$

is similar.

The further invariance for rotations is checked in [7] as well, and the claim about Maxwell's equations now follows for an arbitrary boost with velocity vector $\overline{v}$ or a rotation $g \in SO(3)$ by [6] again.

$\square$

**Lemma 0.2.** *We can define a frame $S_{\infty,1}$ by taking the limit as $v \rightarrow \infty$, $\lambda_\infty$, of the standard Lorentz transformation with velocity vector $v\bar{e}_1$, using the choice of square root, $\sqrt{\frac{1}{\frac{-1}{c^2}}} = -ic$;*

$t_\infty = \frac{ix}{c}$, $x_\infty = ict$, $y_\infty = y$, $z_\infty = z$

*With the same convention, we can take the limit of the transformation rules for;*

$(\rho, \overline{J}, \overline{E}, \overline{B}, V, \overline{A}, \bigtriangledown, \frac{\partial}{\partial t})$

*to the frame $S_{v\bar{e}_1}$, as $v \rightarrow \infty$, to obtain;*

$\rho_\infty = \frac{ic\overline{J}_{||}}{c^2} = \frac{ij_1}{c}$

$\overline{J}_\infty = ic\rho\overline{e}_1 + \overline{J}_\perp = (ic\rho, j_2, j_3)$

$\overline{E}_\infty = \overline{E}_{||} - ic(\overline{e}_1 \times \overline{B}) = (e_1, icb_3, -icb_2)$

$\overline{B}_\infty = \overline{B}_{||} + \frac{ic}{c^2}(\overline{e}_1 \times \overline{E}) = (b_1, -\frac{i}{c}e_3, \frac{i}{c}e_2)$

$V_\infty = ic\overline{A}_{||} = ica_1$

$\overline{A}_\infty = \frac{ic}{c^2}V + \overline{A}_\perp = (\frac{iV}{c}, a_2, a_3)$

$\bigtriangledown_\infty = (-\frac{ic}{c^2}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$\frac{\partial}{\partial t}_\infty = -ic\frac{\partial}{\partial x}$

*With the above transformation rules, the invariances in Lemma 0.1 are preserved, in particular Maxwell's equations hold in $S_{\infty,1}$ for the transformed quantities $(\rho_\infty, \overline{J}_\infty, \overline{E}_\infty, \overline{B}_\infty)$.*

*If the quantities and their derivatives in the base frame are real analytic, then the transformed quantities are analytic in an open region $V$ containing $\lambda_\infty(\mathcal{R}^4)$, and we can interpret the transformed derivatives as genuine derivatives on $\mathcal{C}^4$ at the transformed points $\lambda_\infty(\mathcal{R}^4)$. Maxwell's equations then hold on the open region $V$.*

*The relations;*

$\bigtriangledown \cdot (\overline{E} \times \overline{B}) = 0$

*and;*

$(\overline{E}, \overline{J}) = 0$

*transform to;*

$\frac{\partial}{\partial t_\infty}(\overline{E}_\infty \times \overline{B}_\infty)_1 = -\frac{1}{\epsilon_0}(\frac{\partial}{\partial y_\infty}(p_{\infty,12}) + \frac{\partial}{\partial z_\infty}(p_{\infty,13}))$

*and;*

$f_{\infty,1} = 0$

*in* $S_{\infty,1}$*, where* $p_{\infty,12}$ *and* $p_{\infty,13}$ *are components of the stress tensor, and* $\overline{f}_\infty$ *is force density.*

*Similarly, we can define a frame* $S_{\infty,\overline{\theta}}$*, where* $|\overline{\theta}| = 1$*, by taking the limit as* $v \rightarrow \infty$*,* $\lambda_{\infty,\overline{\theta}}$*, of the standard Lorentz transformation with velocity vector* $v\overline{\theta}$*, using the choice of square root,* $\sqrt{\frac{1}{\frac{-1}{c^2}}} = -ic$*.*
*With the same convention, we can take the limit of the transformation rules for;*

$(\rho, \overline{J}, \overline{E}, \overline{B}, V, \overline{A}, \bigtriangledown, \frac{\partial}{\partial t})$

*to the frame* $S_{\infty,\overline{\theta}}$*. With the above transformation rules, the invariances in Lemma 0.1 are again preserved, in particular Maxwell's equations hold in* $S_{\infty,\overline{\theta}}$ *for the transformed quantities* $(\rho_{\infty,\overline{\theta}}, \overline{J}_{\infty,\overline{\theta}}, \overline{E}_{\infty,\overline{\theta}}, \overline{B}_{\infty,\overline{\theta}})$*. The claim on analytic functions holds as before.*

*Proof.* The standard Lorentz transformation to the frame $S_{v\overline{e}_1}$ is given by, see [8];

$t' = \gamma_v(t - \frac{vx}{c^2}),\ x' = \gamma_v(x - vt),\ y' = y,\ z' = z$

The transformation rules to the frame $S_{\overline{v}}$ are given by, see [2];

$\rho' = \gamma_v(\rho - \frac{v\overline{J}_{||}}{c^2})$

$\overline{J}' = \gamma_v(\overline{J}_{||} - \rho\overline{v}) + \overline{J}_\perp$

$\overline{E}' = \overline{E}_{||} + \gamma_v(\overline{E}_\perp + \overline{v} \times \overline{B})$

$\overline{B}' = \overline{B}_{||} + \gamma_v(\overline{B}_\perp - \frac{\overline{v} \times \overline{E}}{c^2})$

$V' = \gamma_v(V - v\overline{A}_{||})$

$\overline{A}' = \gamma_v(\overline{A}_{||} - \frac{v}{c^2}V) + \overline{A}_\perp$

$\bigtriangledown' = \gamma_v(\bigtriangledown_{||} + \frac{v}{c^2}\frac{\partial}{\partial t}) + \bigtriangledown_\perp$

$\frac{\partial}{\partial t}' = \gamma_v(\frac{\partial}{\partial t} + v|\bigtriangledown_{||}|)$

Taking $\overline{v} = v\overline{e}_1$, $v \rightarrow \infty$ and using the fact that, with the convention, $lim_{v\rightarrow\infty}\gamma_v = 0$, $lim_{v\rightarrow\infty}v\gamma_v = -ic$, we obtain the first result.

We have that, either directly by the above transformation rules, or using the fact that the limit is distributive in the proof of Lemma 0.1;

$\bigtriangledown_\infty \cdot \overline{A}_\infty + \frac{1}{c^2}\frac{\partial V_\infty}{\partial t_\infty}$

$= (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{iV}{c}, a_2, a_3) + \frac{1}{c^2}(-ic\frac{\partial}{\partial x})(ica_1)$

$= \frac{1}{c^2}\frac{\partial V}{\partial t} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} + \frac{\partial a_1}{\partial x}$

$= \bigtriangledown \cdot \overline{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$

$= 0$

and;

$\Box^2_\infty(V_\infty) = (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (-\frac{i}{c}\frac{\partial}{\partial t}(ica_1), ic\frac{\partial a_1}{\partial y}, ic\frac{\partial a_1}{\partial z}) - \frac{1}{c^2}(-ic\frac{\partial}{\partial x})^2(ica_1)$

$= ic(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}) - \frac{ic}{c^2}\frac{\partial^2 a_1}{\partial t^2}$

$= ic\Box^2(a_1)$

$= ic(-\mu_0 j_1)$

$= -\frac{ij_1}{c\epsilon_0}$

$= -\frac{\rho_\infty}{\epsilon_0}$

and;

$\Box^2_\infty = (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2_\infty}$

$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}(-ic\frac{\partial}{\partial x})^2$

$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$= \Box^2$

so that;

$\Box^2_\infty(\overline{A}_\infty)$

$= (\Box^2_\infty(\frac{iV}{c}), \Box^2_\infty(a_2), \Box^2_\infty(a_3))$

$= (\Box^2(\frac{iV}{c}), \Box^2(a_2), \Box^2(a_3))$

$= (-\frac{i}{c}\frac{\rho}{\epsilon_0}, -\mu_0 j_2, -\mu_0 j_3)$

$= -\mu_0(\frac{i}{c}c^2\rho, j_2, j_3)$

$= -\mu_0(ic\rho, j_2, j_3)$

$= -\mu_0\overline{J}_\infty$

The further invariance can be proved by a similar method, extending the result of [7], the details are left to the reader. The claim about Maxwell's equations then follows from [6] again. If the quantities in the base frame are analytic, then we have, for example, that in the frame $S_{\infty,1}$;

$\rho_\infty(\overline{x}', t') = \frac{ij_1(\lambda_\infty^{-1}(\overline{x'}, t'))}{c}$

at a transformed point $(\overline{x}', t') \in \lambda_\infty(\mathcal{R}^4)$. As $j_1$ is real analytic, it extends to a complex analytic function in an open region $U$ containing $\lambda_\infty^{-1}(\overline{x'}, t')$, corresponding to an open region $\lambda_\infty(U)$ containing $(\overline{x}', t')$. Then, as $\lambda_\infty^{-1}$ is linear and, therefore, analytic, and the composition of analytic functions is analytic, $\rho_\infty$ is analytic on $\lambda_\infty(U)$. A similar result holds for the other transformed quantities by complex linearity of the transformation rules. The claim about the transformed derivatives being genuine derivatives then follows from the transformation rules, the chain rule and the linearity of the derivatives of complex analytic functions. The fact that Maxwell's equations hold on $V$, then follows from the fact that the transformed quantities and their derivatives are analytic and the identity theorem.

By inverting the transformation rules, we have that;

$\overline{E}=\overline{E}_{\infty,||}+ic(\overline{e}_1\times\overline{B}_\infty)=(e_{\infty,1},-icb_{\infty,3},icb_{\infty,2})$

$\overline{B}=\overline{B}_{\infty,||}-\frac{ic}{c^2}(\overline{e}_1\times\overline{E}_\infty)=(b_{\infty,1},\frac{i}{c}e_{\infty,3},-\frac{i}{c}e_{\infty,2})$

$\bigtriangledown=(\frac{ic}{c^2}\frac{\partial}{\partial t_\infty},\frac{\partial}{\partial y_\infty},\frac{\partial}{\partial z_\infty})=(\frac{i}{c}\frac{\partial}{\partial t_\infty},\frac{\partial}{\partial y_\infty},\frac{\partial}{\partial z_\infty})$

so that;

$\bigtriangledown\centerdot(\overline{E}\times\overline{B})=(\frac{i}{c}\frac{\partial}{\partial t_\infty},\frac{\partial}{\partial y_\infty},\frac{\partial}{\partial z_\infty})\centerdot((e_{\infty,1},-icb_{\infty,3},icb_{\infty,2})\times(b_{\infty,1},\frac{i}{c}e_{\infty,3},-\frac{i}{c}e_{\infty,2}))$

$=(\frac{i}{c}\frac{\partial}{\partial t_\infty},\frac{\partial}{\partial y_\infty},\frac{\partial}{\partial z_\infty})\centerdot(-e_{\infty,2}b_{\infty,3}+e_{\infty,3}b_{\infty,2},icb_{\infty,1}b_{\infty,2}+\frac{i}{c}e_{\infty,1}e_{\infty,2},\frac{i}{c}e_{\infty,1}e_{\infty,3}$

$+icb_{\infty,1}b_{\infty,3})$

$=\frac{i}{c}\frac{\partial}{\partial t_\infty}(-e_{\infty,2}b_{\infty,3}+e_{\infty,3}b_{\infty,2})+\frac{\partial}{\partial y_\infty}(icb_{\infty,1}b_{\infty,2}+\frac{i}{c}e_{\infty,1}e_{\infty,2})+\frac{\partial}{\partial z_\infty}(\frac{i}{c}e_{\infty,1}e_{\infty,3}$

$+icb_{\infty,1}b_{\infty,3})$

$=-\frac{i}{c}\frac{\partial}{\partial t_\infty}(\overline{E}_\infty\times\overline{B}_\infty)_1+\frac{i}{c}\frac{\partial}{\partial y_\infty}(e_{\infty,1}e_{\infty,2}+c^2b_{\infty,1}b_{\infty,2})+\frac{i}{c}\frac{\partial}{\partial z_\infty}(e_{\infty,1}e_{\infty,2}$

$+c^2b_{\infty,1}b_{\infty,2})$

so that $\bigtriangledown\centerdot(\overline{E}\times\overline{B})=0$ iff;

$=-\frac{\partial}{\partial t_\infty}(\overline{E}_\infty\times\overline{B}_\infty)_1+\frac{\partial}{\partial y_\infty}(e_{\infty,1}e_{\infty,2}+c^2b_{\infty,1}b_{\infty,2})+\frac{\partial}{\partial z_\infty}(e_{\infty,1}e_{\infty,2}$

$+c^2b_{\infty,1}b_{\infty,2})=0$

iff

$\frac{\partial}{\partial t_\infty}(\overline{E}_\infty\times\overline{B}_\infty)_1=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial y_\infty}(p_{\infty,12})+\frac{\partial}{\partial z_\infty}(p_{\infty,13}))$

By inverting the transformation rules again, we have that;

$\overline{J}=-ic\rho_\infty\overline{e}_1+\overline{J}_{\infty,\perp}=(-ic\rho_\infty,j_{\infty,2},j_{\infty,3})$

so that;

$\overline{E}\centerdot\overline{J}=(e_{\infty,1},-icb_{\infty,3},icb_{\infty,2})\centerdot(-ic\rho_\infty,j_{\infty,2},j_{\infty,3})$

$= -ic\rho_\infty e_{\infty,1} - icj_{\infty,2}b_{\infty,3} - icj_{\infty,3}b_{\infty,2}$

$= -ic((\rho_\infty \overline{E}_\infty + \overline{J}_\infty \times \overline{B}_\infty)_1)$

so that $\overline{E} \cdot \overline{J} = 0$ iff;

$(\rho_\infty \overline{E}_\infty + \overline{J}_\infty \times \overline{B}_\infty)_1 = 0$

iff

$f_{\infty,1} = 0$

For an arbitrary angle $\overline{\theta}$, $|\theta| = 1$, choose $g \in SO(3)$, with $g^{-1}(v\overline{e}_1) = v\overline{\theta}$, then by Lemma 1.5 of [7], we have that;

$B_{v\overline{\theta}} = R_g B_{v\overline{e}_1} R_{g^{-1}}$.

Taking the limit as $v \rightarrow \infty$, we have that;

$B_{\infty,\overline{\theta}} = R_g \lambda_\infty R_{g^{-1}}$.

We have already checked that the relations;

$\bigtriangledown \cdot \overline{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$\Box^2(V) = -\frac{\rho}{\epsilon_0}$

$\Box^2(\overline{A}) = -\mu_0 \overline{J}$

are preserved by a rotation $g$. So it is sufficient to verify that the relations;

$\bigtriangledown_\infty \cdot \overline{A}_\infty + \frac{1}{c^2}\frac{\partial V_\infty}{\partial t_\infty} = 0$

$\Box^2(V_\infty) = -\frac{\rho_\infty}{\epsilon_0}$

$\Box^2(\overline{A}_\infty) = -\mu_0 \overline{J}_\infty$

are preserved by a rotation $g$ as well. The proof is similar to the above, bearing in mind that we are dealing with complex coordinates.
.......

The further invariance for rotations is checked in [7] as well, and the claim about Maxwell's equations now follows for an arbitrary boost with velocity vector $\overline{v}$ or a rotation $g \in SO(3)$ by [6] again.

.......

□

**Definition 0.3.** *Let $(\rho, \overline{J})$ be given in the base frame $S$. We call $S$ a good frame in the direction $\overline{e}_1$ if for all $v \in (-c, c)$, there exist electric and magnetic fields $\{\overline{E}_{v\overline{e_1}}, \overline{B}_{v\overline{e}_1}\}$ in the frame $S_{v\overline{e_1}}$ such that;*

*(i). $(\rho_{v\overline{e_1}}, \overline{J}_{v\overline{e_1}}, \overline{E}_{v\overline{e_1}}, \overline{B}_{v\overline{e}_1})$ satisfy Maxwell's equations in the frame $S_{v\overline{e_1}}$*

*(ii). $(\overline{E}_{v\overline{e_1}}, \overline{J}_{v\overline{e_1}}) = 0$*

*(iii). The transfers of the fields $\{\overline{E}_{v\overline{e_1}}, \overline{B}_{v\overline{e}_1}\}$ back to the base frame $S$ form an analytic family in $\mathcal{R}^4 \times (-c, c)$, with the property that that for any compact subset $D$ of $\mathcal{R}^4$, with open interior, there exists $\epsilon_D > 0$ such that the transfers are analytic on $D^\circ \times (-c - \epsilon_D, c + \epsilon_D)$.*

*We call $S$ an excellent frame in the direction $\overline{e}_1$, if the same conditions are satisfied, with (ii) modified to;*

*(ii)'. $\bigtriangledown_{v\overline{e}_1}(\overline{E}_{v\overline{e_1}} \times \overline{B}_{v\overline{e_1}}) = 0$*

**Lemma 0.4.** *If $S$ is a good frame, then any frame $S'$ connected to $S$ by a velocity $w\overline{e}_1$ is good and, moreover, if $D$ is compact with open interior in $S$, with corresponding $D_{S'}$ in $S'$, we can choose $\epsilon_{D_{S'}}$ to be uniform in $S'$, with the requirement that $0 < \epsilon_{D_{S'}} < min(-c - \theta(-c - \epsilon_D, w), \theta(c+\epsilon_D, w) - c, \frac{c^2}{|w|} - c)$, where $\theta$ is as in the proof. For a compact subset $D$ with open interior in $S$, There exists $\epsilon'_D > 0$ such that in the frame $S_{(-c+\epsilon'_D)\overline{e}_1}$, there exist $\{\overline{E}_{-\infty}, \overline{B}_{-\infty}\}$ in the frame $S_{-\infty,1}$ relative to $S_{(-c+\epsilon'_D)\overline{e}_1}$ as the base frame, such that;*

$(\overline{E}_{-\infty}, \overline{J}_{-\infty}) = 0$

*on the corresponding set $D_{-\infty}$ in $S_{-\infty,1}$, and, therefore on some open $D_{-\infty} \subset V_{-\infty} \subset \mathcal{C}^4$ in $S_{-\infty,1}$.*

*There exist* $\{\overline{E}_\infty,\overline{B}_\infty\}$ *in the frame* $S_{\infty,1}$ *relative to* $S'''_{(-c+\epsilon'_D)\bar{e}_1}$, *as the base frame, which is* $S_{(-c+\epsilon'_D)\bar{e}_1}$, *rotated by* 180 *degrees, such that;*

$$(\overline{E}_\infty,\overline{J}_\infty)=0$$

*on the corresponding set* $D_\infty$ *in* $S_{\infty,1}$, *and, therefore on some open* $D_\infty\subset V_\infty\subset\mathcal{C}^4$ *in* $S_{\infty,1}$.

*Proof.* Let $S'$ be connected to $S$ by velocity $w\bar{e}_1$, $w\in(-c,c)$, and let $S''$ be connected to $S'$ by velocity $v\bar{e}_1$, $v\in(-c,c)$. By Lemma 1.9 of [7], we have the boost relation;

$$B_{v\bar{e}_1}B_{w\bar{e}_1}=B_{v\bar{e}_1*w\bar{e}_1}$$

where;

$$v\bar{e}_1*w\bar{e}_1=\frac{v\bar{e}_1+w\bar{e}_1}{1+\frac{v\bar{e}_1\cdot w\bar{e}_1}{c^2}}=\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1$$

Fixing $w$ and letting $v$ vary, we have that $\alpha=\frac{v+w}{1+\frac{vw}{c^2}}$ defines an increasing invertible analytic function from $(-c,c)$ onto $(-c,c)$, with inverse $v=\frac{\alpha-w}{1-\frac{\alpha w}{c^2}}=\theta(\alpha,w)$. By the fact that $S$ is a good frame, we can find $\{\overline{E}_{\alpha\bar{e}_1},\overline{B}_{\alpha\bar{e}_1}\}$, such that $(i),(ii)$ are satisfied and the transfers from $S_{\alpha\bar{e}_1}$ back to $S$ satisfy $(iii)$. We transfer $\overline{E}_{\alpha\bar{e}_1}$ and $\overline{B}_{\alpha\bar{e}_1}$ from $S$ back to $S'$ by the transfer rules;

$$\overline{E}'=\overline{E}_{\alpha\bar{e}_1,||}+\gamma_w(\overline{E}_{\alpha\bar{e}_1,\perp}+\overline{w}\times\overline{B}_{\alpha\bar{e}_1})$$

$$\overline{B}'=\overline{B}_{\alpha\bar{e}_1,||}+\gamma_w(\overline{B}_{\alpha\bar{e}_1,\perp}-\frac{\overline{w}\times\overline{E}_{\alpha\bar{e}_1}}{c^2})$$

Then, by the facts that for fixed $w$, the transformation rules are linear, and composition of analytic functions is analytic;

$$\overline{E}'_{v\bar{e}_1}=\overline{E}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1,||}+\gamma_w(\overline{E}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1,\perp}+\overline{w}\times\overline{B}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1})$$

$$\overline{B}'_{v\bar{e}_1}=\overline{B}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1,||}+\gamma_w(\overline{B}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1,\perp}-\frac{\overline{w}\times\overline{E}_{\frac{v+w}{1+\frac{vw}{c^2}}\bar{e}_1}}{c^2})$$

is a family of fields in $S'$, indexed by $v$, satisfying the first part of $(iii)$, whose corresponding fields in the varying frame $S''$, indexed by $v$, satisfy $(i),(ii)$. It remains to prove the last part of $(iii)$. Let $D'$ be compact in $S'$, with open interior, then as the Lorentz transformation connecting $S'$ to $S$ is linear, therefore continuous, and invertible, the corresponding $D$ in $S$ is compact with open interior. As $S$ is good, we can extend the analytic family $\{\overline{E}_{\alpha\bar{e}_1},\overline{B}_{\alpha\bar{e}_1}\}$ in $S$ to $D^\circ\times(-c-\epsilon_D,c+\epsilon_D)$, so we can extend the analytic family $\{\overline{E}'_{v\bar{e}_1},\overline{B}'_{v\bar{e}_1}\}$ to $D'^\circ\times(-c-\epsilon_{D'},c+\epsilon_{D'})$, provided $0<\epsilon_{D'}<min(-c-\theta(-c-\epsilon_D,w),\theta(c+\epsilon_D,w)-c,\frac{c^2}{|w|}-c)$, so that we avoid the pole of the analytic function $\frac{v+w}{1+\frac{vw}{c^2}}$, and we remain within the bound for the base frame $S$ after interpolation.

Fixing some compact set $D$ with open interior in the base frame $S$, $\epsilon>0$, let $S'$ travel with velocity $(-c+\epsilon)\bar{e}_1$ relative to the base frame $S$, so that $S$ travels with velocity $(c-\epsilon)\bar{e}_1$ relative to $S'$. Let $S''$ travel with velocity $(-\frac{c^2}{(c-\epsilon)}+\delta)\bar{e}_1$, $0<\delta<\frac{c^2}{c-\epsilon}-c$, relative to $S$. As is shown in [7], by the velocity addition formula, $S''$ travels with velocity;

$$\alpha(\delta,\epsilon))=\frac{[(c-\epsilon)+(-\frac{c^2}{(c-\epsilon)}+\delta)]}{1+\frac{((c-\epsilon)(-\frac{c^2}{(c-\epsilon)}+\delta))}{c^2}}\bar{e}_1$$

$$=\frac{c^2[(c-\epsilon)-\frac{c^2}{(c-\epsilon)}-\delta]}{\delta(c-\epsilon)}\bar{e}_1$$

relative to $S'$, and as $\delta\rightarrow 0$, $\alpha(\delta,\epsilon)\rightarrow-\infty$ and $S''$ converges to the limit frame $S_{-\infty,1}$ relative to $S'$, $(*)$.

Choose $\epsilon>0$ so that;

$$-\epsilon_D<-\frac{c^2}{c-\epsilon}+c<0$$

iff

$$-\epsilon_D<-\frac{\epsilon c}{c-\epsilon}<0$$

iff

$$0<\frac{\epsilon c}{c-\epsilon}<\epsilon_D$$

iff

$$0<\epsilon<\frac{c\epsilon_D}{c+\epsilon_D}$$

then the interval $(-\frac{c^2}{c-\epsilon}, -c) \subset (-c-\epsilon_D, -c)$. By the hypotheses on $S$, we can find a family $\{\overline{E}_{v\overline{e_1}}, \overline{B}_{v\overline{e}_1}\}$ indexed by $(-c-\epsilon_D, -c)$, in the frames $S''$, travelling with velocity $v\overline{e}_1$, relative to $S$, $v \in (-c-\epsilon_D, -c)$, such that the transfers, with the extension of the transformation rules, back to the frame $S$ form an analytic family.

We can transfer $\overline{E}_{v\overline{e}_1}$ and $\overline{B}_{v\overline{e}_1}$ from $S$ back to $S'$ by the transfer rules;

$$\overline{E}' = \overline{E}_{v\overline{e}_1,||} + \gamma_{-c+\epsilon}(\overline{E}_{v\overline{e}_1,\perp} + (-c+\epsilon)\overline{e}_1 \times \overline{B}_{v\overline{e}_1})$$

$$\overline{B}' = \overline{B}_{v\overline{e}_1,||} + \gamma_{-c+\epsilon}(\overline{B}_{v\overline{e}_1,\perp} - \tfrac{(-c+\epsilon)\overline{e}_1 \times \overline{E}_{v\overline{e}_1}}{c^2})$$

Then, by the facts that for the fixed $-c+\epsilon$, the transformation rules are linear, and composition of analytic functions is analytic;

$$\overline{E}'_{\alpha\overline{e}_1} = \overline{E}_{\theta(\alpha,c-\epsilon)\overline{e}_1,||} + \gamma_{-c+\epsilon}(\overline{E}_{\theta(\alpha,c-\epsilon)\overline{e}_1,\perp} + (-c+\epsilon)\overline{e}_1 \times \overline{B}_{\theta(\alpha,c-\epsilon)\overline{e}_1})$$

$$\overline{B}'_{\alpha\overline{e}_1} = \overline{B}_{\theta(\alpha,c-\epsilon)\overline{e}_1,||} + \gamma_{-c+\epsilon}(\overline{B}_{\theta(\alpha,c-\epsilon)\overline{e}_1,\perp} - \tfrac{(-c+\epsilon)\overline{e}_1 \times \overline{E}_{\theta(\alpha,c-\epsilon)\overline{e}_1}}{c^2})$$

is a family of fields in $S'$, indexed by $\alpha(\epsilon, \delta)$.

The statement of Maxwell's equations in the frames $S''$, together with the requirement that $(\overline{E}_{v\overline{e_1}}, \overline{J}_{v\overline{e_1}}) = 0$ in the frames $S''$ transfer to analytic statements in the frame $S$, and hold by analytic continuation, noting that the singularity created by $\gamma_v$, when $v = c$, is isolated, and the statements are identically zero for $v \in (-c, c)$. In particular by $(*)$ we obtain $(\overline{E}_{-\infty}, \overline{J}_{-\infty}) = 0$ in $S_{-\infty,1}$, and noting that the property is invariant by a rotation, and using the conjugation formula $B_\infty = R_g B_{-\infty} R_g^{-1}$ for a 180 degree rotation $g$, we obtain the final result. □

**Lemma 0.5.** *Letting $S$ be the base frame, moving with velocity $(-c+\epsilon)\overline{e}_1$, relative to the original base frame, obtained in the Lemma 0.4, fixing a compact set with open interior, we can find $\{\overline{E}, \overline{B}\}$ in $S$ such that, for the associated force density $\overline{f}$;*

$$\overline{f} = \overline{0}$$

*With the assumption that $\{\rho, \overline{J}\}$ are analytic, we can either find $\{\overline{E}', \overline{B}'\}$ such that;*

$(\overline{E}', \overline{B}') = 0$, $(*)$

*on $D$ in the original base frame or $\rho = 0, \overline{J} = \overline{0}$ on $D$, in the original base frame.*

*We have either the trivial case, that $\rho = 0, \overline{J} = \overline{0}$ in the original base frame $S$, or we obtain that $\overline{f} = \overline{0}$ for fields $\{\overline{E}, \overline{B}\}$ with $(\overline{E}, \overline{B}) = 0$, and $(\overline{E}, \overline{J}) = 0$, in $S$.*

*Proof.* Following the method of [7], fixing a compact subset with open interior in the original base frame, we let $S$ be the base frame $S'''_{(-c+\epsilon'_D)\overline{e}_1}$, obtained in the Lemma 0.4, with the relative frame $S_{\infty,1}$ such that $(\overline{E}_\infty, \overline{J}_\infty) = 0$. Fix a velocity $\overline{u}$ relative to $S$, with $|\overline{u}| < c$. By the velocity composition formula, to find $\overline{w}$ with;

$$B_{\overline{w}} B_{\overline{u}} = R_h B_{s\overline{e}_1}$$

where $h \in SO(3)$, we must have that $\overline{w} = s\overline{e}_1 * (-\overline{u})$, where;

$$\overline{a} * \overline{b} = \frac{\overline{a}+\overline{b}}{1+\frac{\overline{a}.\overline{b}}{c^2}} + \frac{\gamma_a(\overline{a}\times(\overline{a}\times\overline{b}))}{c^2(\gamma_a+1)(1+\frac{\overline{a}.\overline{b}}{c^2})}$$

so that;

$$\overline{w} = \frac{s\overline{e}_1-\overline{u}}{1-\frac{s\overline{e}_1.\overline{u}}{c^2}} - \frac{\gamma_s(s\overline{e}_1\times(s\overline{e}_1\times\overline{u}))}{c^2(\gamma_s+1)(1-\frac{s\overline{e}_1.\overline{u}}{c^2})}$$

and if $u_1 \neq 0$;

$$\overline{w}_\infty = lim_{s\to\infty}\overline{w} = lim_{s\to\infty}[\frac{s\overline{e}_1-\overline{u}}{1-\frac{su_1}{c^2}} - \frac{\gamma_s(s\overline{e}_1(su_1)-\overline{u}s^2)}{c^2(\gamma_s+1)(1-\frac{su_1}{c^2})}]$$

$$= \frac{\overline{e}_1}{-\frac{u_1}{c^2}} - lim_{s\to\infty}\frac{\gamma_s s^2(u_1\overline{e}_1-\overline{u})}{c^2(\gamma_s+1)(1-\frac{su_1}{c^2})}$$

$$= lim_{s\to\infty}\frac{-s^2(u_1\overline{e}_1-\overline{u})}{c^2(1-\frac{su_1}{c^2})}$$

$$= lim_{s\to\infty}\frac{s^2(0,u_2,u_3)}{c^2(1-\frac{su_1}{c^2})}$$

$$= lim_{s\to\infty}\frac{s(0,u_2,u_3)}{\frac{c^2}{s}-u_1}$$

$$= lim_{s\to\infty}\frac{s(0,u_2,u_3)}{-u_1}$$

$$= lim_{s\to\infty} - s(0, u_2, u_3)$$

We have ignored the $\frac{\overline{e}_1}{-\frac{u_1}{c^2}}$ term, as by the definition of the boost matrix, see [7], substituting a finite term is equivalent to substituting a zero vector, in the computation as $s \rightarrow \infty$. This is an easy computation left to the reader.

Then, working in a triangle;

$B_{\overline{u}} = B_{-\overline{w}(s)} R_{h(s)} B_{s\overline{e}_1}$

$= lim_{s\rightarrow\infty}(B_{-\overline{w}(s)} R_{h(s)} B_{s\overline{e}_1})$

$= lim_{s\rightarrow\infty}(B_{s(0,u_2,u_3)}) R_{h_\infty} lim_{s\rightarrow\infty}(B_{s\overline{e}_1})$

By the conjugation result;

$B_{s(0,u_2,u_3)} = R_g B_{s(u_2^2+u_3^2)^{\frac{1}{2}}\overline{e}_1} R_g^{-1}$

where $g \in SO(3)$ is independent of $s$, with the property that $g((u_2^2 + u_3^2)^{\frac{1}{2}}\overline{e}_1) = (0, u_2, u_3)$, so that;

$B_{\overline{u}} = R_g lim_{s\rightarrow\infty}(B_{s(u_2^2+u_3^2)^{\frac{1}{2}}\overline{e}_1}) R_g^{-1} R_{h_\infty} lim_{s\rightarrow\infty}(B_{s\overline{e}_1})$

$= R_g \lambda_\infty R_g^{-1} R_{h_\infty} \lambda_\infty$

By Lemma 0.4, we have a field $\overline{E}_\infty$ in $S_{\infty,1}$ such that $(\overline{E}_\infty, \overline{J}_\infty) = 0$. This property is invariant by rotations so it holds in the frame $S'$ connected to $S_{\infty,1}$ by the rotation $R_g^{-1} R_{h_\infty}$. Then by Lemma 0.2, the relation $(\overline{E}'_\infty, \overline{J}'_\infty)$ transforms to the relation $(\overline{f})_{\infty,1} = 0$ in the frame $S''$ connected to $S'$ by a further infinite boost back $\lambda_\infty$. Choosing $\overline{u} = (\delta, \delta, 0)$, with $g(\overline{e}_1) = \overline{e}_2$, $\overline{u} = (\delta, 0, \delta)$, with $g(\overline{e}_1) = \overline{e}_3$, and noting that for a rotation $R_g$;

$(R_g^{-1}(\rho'_\infty \overline{E}'_\infty + \overline{J}'_\infty \times \overline{B}'_\infty, \overline{e}_1)$

$= (\rho'_\infty \overline{E}'_\infty + \overline{J}'_\infty \times \overline{B}'_\infty, R_g(\overline{e}_1))$

$= (\overline{f}'_{\infty,1}, R_g(\overline{e}_1))$

for quantities $(\rho'_\infty, \overline{J}'_\infty, \overline{E}'_\infty, \overline{B}'_\infty, \overline{f}'_{\infty,1})$ in the frame $S'''$ connected to $S''$ by the rotation $R_g$, we obtain;

$(\overline{f}'_{\infty,1})_2 = 0$ in $S_1'''$ connected to $S$ by $B_{(\delta,\delta,0)}$

$(\overline{f}'_{\infty,1})_3 = 0$ in $S_2'''$ connected to $S$ by $B_{(\delta,0,\delta)}$

so that, letting $\delta \rightarrow 0$, by continuity, there exist $\{\overline{E}, \overline{B}\}$ in $S$ with;

$(\overline{f})_2 = (\overline{f})_3 = 0$

Remembering, that $S$ is connected to the frame $S_{\infty,1}$, by the infinite boost $\lambda_\infty$, and using the fact that we have $(\overline{E}_\infty, \overline{J}_\infty) = 0$, by Lemma 0.2 again, we obtain the transformed relation in $S$;

$(\overline{f})_1 = 0$

Combining the results, we then have $\{\overline{E}, \overline{B}\}$ in $S$, with $\overline{f} = \overline{0}$, $(AA)$. Observing that the statement $\overline{f} = \overline{0}$ is rotation invariant, we have that $\overline{f} = \overline{0}$ in the frame $S'$, connected to the original base frame $S$ by the velocity vector $(-c + \epsilon_D)\overline{e}_1$. By continuity, we can obtain the same result, that there exist $\{\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}}\}$ in $S_{\overline{v}}$, connected to $S'$ by a velocity $\overline{v}$ with $|\overline{v}| < \delta$, for sufficiently small $\delta$, such that the associated force density $\overline{f}_{\overline{v}} = \overline{0}$. We then have that, by the transformation rules and the definition of force density;

$(\overline{f}_{\overline{v}}, \overline{B}_{\overline{v}}) = \rho_{\overline{v}}(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$

$= \gamma_v(\rho - \frac{(\overline{v}, \overline{J})}{c^2})(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ (in $S'$)

$= 0$ $(**)$

for all $\overline{v} \in B(\overline{0}, \delta)$. If the statement $(*)$ in the lemma fails to hold, then by $(**)$, the invariance of $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$, the assumption of analytic, and the analytic nature of the transformation rules, we immediately obtain that $\rho = 0$ on $D'$ in $S'$ and;

$\gamma_v(\rho - \frac{(\overline{v}, \overline{J})}{c^2}) = -\gamma_v \frac{(\overline{v}, \overline{J})}{c^2} = 0$

for all $\overline{v} \in B(\overline{0}, \delta)$, so that $\overline{J} = 0$ as well, on $D'$ in $S'$. We can then transfer these results back to $D$ in the original base frame.

Assume that, working in the unrotated frame $S_{(-c+\epsilon'_D)\overline{e}_1}$, we have fields $\{\overline{E},\overline{B}\}$ with $(\overline{E},\overline{B})=0$, and for which the associated force density $\overline{f}=\overline{0}$. Letting $S_{\infty,1}$ be connected by an infinite boost in the direction $\overline{e}_1$, then as $(\overline{E},\overline{B})$ is an invariant, we have that $(\overline{E}_\infty,\overline{B}_\infty)=0$, and by Lemma 0.2, $(\overline{E}_\infty,\overline{J}_\infty)=0$. Moreover, it as easy exercise, left to the reader, that $(\overline{f}_\infty)_2=(\overline{f}_\infty)_3=0$. We can apply similar considerations to the frames $\{S_{\infty,2},S_{\infty,3}\}$ connected by infinite boosts in the directions $\overline{e}_2$ and $\overline{e}_3$. As is done in [7], we can obtain a complete characterisation of cases. Using the fact that the data is analytic, we have that;

$$\overline{E}_\infty\times(\overline{J}_\infty\times\overline{B}_\infty)=\overline{J}_\infty(\overline{E}_\infty,\overline{B}_\infty)-\overline{B}_\infty(\overline{E}_\infty,\overline{J}_\infty)$$

$$=0$$

in the three frames $\{S_{\infty,1},S_{\infty,2},S_{\infty,3}\}$.

Case 1. $\overline{E}_\infty=\overline{0}$ in all three frames $\{S_{\infty,1},S_{\infty,2},S_{\infty,3}\}$

Then as Maxwell's equations are satisfied in $S_{\infty,i}$, $1\leq i\leq 3$;

$$\rho_\infty=\epsilon_0(\bigtriangledown\centerdot\overline{E}_\infty)=0$$

so that transforming back to $S_{(-c+\epsilon'_D)\overline{e}_1}$, $\frac{ij_1}{c}=0$, $j_1=0$ and similarly $j_2=j_3=0$, so that $\overline{J}=0$. As $\overline{f}=\overline{0}$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$, we either obtain $\rho=0$ or $\overline{E}=\overline{0}$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$, in which case by Maxwell's equations again $\rho=0$ again and, we can transfer to the original base frame, to obtain the trivial case that $\rho=0$ and $\overline{J}=\overline{0}$.

Case 2. $\overline{J}_\infty\times\overline{B}_\infty=\overline{0}$ in some frame $\{S_{\infty,1},S_{\infty,2},S_{\infty,3}\}$

If this occurs in $S_{\infty,1}$, (the same argument applies with the corresponding symmetry in the other two cases, then as $(\overline{f}_\infty)_2=(\overline{f}_\infty)_3=0$, we must have that $\rho_\infty e_{2,\infty}=\rho_\infty e_{3,\infty}=0$. Then either $\rho_\infty=0$, in which case consider Case 1, or $e_{2,\infty}=e_{3,\infty}=0$, so that as $(\overline{E}_\infty,\overline{B}_\infty)=(\overline{E}_\infty,\overline{J}_\infty)=0$, we must have that $e_{1,\infty}j_{1,\infty}=e_{1,\infty}b_{1,\infty}=0$, so that either $e_{1,\infty}=0$, $\overline{E}_\infty=\overline{0}$, then consider case 1 again, or, $j_{1,\infty}=b_{1,\infty}=0$, so that transforming back to $S_{(-c+\epsilon'_D)\overline{e}_1}$, we have that $\overline{B}=\overline{0}$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$. However, we also have that $\overline{f}=0$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$, so that $\rho\overline{E}=\overline{0}$ and either $\rho=0$ or $\overline{E}=\overline{0}$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$, and then $\rho=0$ in

$S_{(-c+\epsilon'_D)\overline{e}_1}$ by the first of Maxwell's equations.

By considering the first two cases simultaneously, we *either* obtain Case 3 considered below in some frame $\{S_{\infty,1}, S_{\infty,2}, S_{\infty,3}\}$, *or* we obtain the trivial case, *or* we obtain $\overline{B} = 0$ in the frame $S_{(-c+\epsilon'_D)\overline{e}_1}$ together with $\rho = 0$. We can then repeat the argument for frames connected to $S_{(-c+\epsilon'_D)\overline{e}_1}$ by velocity vectors in a neighborhood $B(\overline{0}, \gamma)$ for some $\gamma > 0$, to obtain, using continuty, either the trivial case again, the existence of fields $\overline{B}_{\overline{w}} = 0$, with $\rho_{\overline{w}} = 0$, $\overline{w} \in B(\overline{0}, \gamma)$, in which case, by the transformation rules for $\rho$, in the neighborhood $B(\overline{0}, \gamma)$, we obtain $\overline{J} = \overline{0}$ and we reduce to the trivial case again.

(generic) Case 3. $\overline{E}_\infty = \lambda(\overline{J}_\infty \times \overline{B}_\infty)$ in some frame $\{S_{\infty,1}, S_{\infty,2}, S_{\infty,3}\}$ connected to a frame $S'$, in a neighborhood $B(\overline{0}, \gamma)$ of $S_{(-c+\epsilon'_D)\overline{e}_1}$.

Assume this happens in $S_{\infty,1}$ and $S_{(-c+\epsilon'_D)\overline{e}_1}$, otherwise use the corresponding symmetry. Then, as $(\overline{f}_\infty)_2 = \overline{f}_\infty)_3 = 0$, equating coefficients, we must have that $\lambda = -\frac{1}{\rho_\infty}$ and then $(\overline{f}_\infty)_1 = 0$ and $\overline{f}_\infty = \overline{0}$. Now use the argument above in this Lemma;

$$B_{\overline{u}} = R_g \lambda_\infty R_g^{-1} R_{h_\infty} \lambda_\infty$$

together with the results that $\overline{f}_\infty = \overline{0}$ is rotation invariant, and transforms by $\lambda_\infty$, to a relation $(\overline{E}, \overline{J}) = 0$, which is also rotation invariant, to obtain a field $\overline{E}$ in $S_{(-c+\epsilon'_D)\overline{e}_1}$, with the property that the transfers $\overline{E}_{\overline{u}}$ to the frame $S_{\overline{u}}$ connected to $S_{(-c+\epsilon'_D)\overline{e}_1}$, by the velocity vector $\overline{u}$, $u_1 \neq 0$ satisfy $(\overline{E}_{\overline{u}}, \overline{J}_{\overline{u}}) = 0$. By results of [7] we obtain that the transfers $(\overline{f})_{\overline{u}} = \overline{0}$, so that in the original base frame $S$, we obtain $\overline{f} = \overline{0}$, for $\{\overline{E}, \overline{B}\}$, with $(\overline{E}, \overline{J}) = 0$ and $(\overline{E}, \overline{B}) = 0$.

□

We can repeat the above argument for good frames in the context of excellent frames, and obtain;

**Lemma 0.6.** *Letting $S$ be the base frame, moving with velocity $(-c + \epsilon)\overline{e}_1$, relative to the original base frame, obtained in the Lemma 0.4, fixing a compact set with open interior, we can find $\{\overline{E}, \overline{B}\}$ in $S$ such that, for the associated components of the energy stress tensor $\{p_{12}, p_{13}, p_{23}\}$;*

$$\frac{\partial}{\partial t}(\overline{E} \times \overline{B})_1 = -\frac{1}{\epsilon_0}\left(\frac{\partial}{\partial y}(p_{12}) + \frac{\partial}{\partial z}(p_{13})\right)$$

$\frac{\partial}{\partial t}(\overline{E}\times\overline{B})_2=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial x}(p_{12})+\frac{\partial}{\partial z}(p_{23}))$

$\frac{\partial}{\partial t}(\overline{E}\times\overline{B})_3=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial x}(p_{13})+\frac{\partial}{\partial y}(p_{23}))$

*In particular, we can boost these relations back to any frame at infinity, relative to $S$, such that;*

$\bigtriangledown_\infty(\overline{E}_\infty\times\overline{B}_\infty)=0$

*for the associated fields $\{\overline{E}_\infty,\overline{B}_\infty\}$.*

*We then must obtain that $\overline{E}\times\overline{B}=\overline{0}$ in $S$.*

*For the original base frame, we either obtain $\rho=0$, $\overline{J}=\overline{0}$, or we obtain the relations;*

$\Box^2(\rho)=0$

$\bigtriangledown(\rho)+\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\overline{0}$

$\bigtriangledown\times\overline{J}=\overline{0}$

$\Box^2(\overline{J})=\overline{0}$

*together with $\{\overline{E},\overline{B}\}$ such that $\Box^2(\overline{E})=\overline{0}$, $\Box^2(\overline{B})=\overline{0}$.*

*If we assume that $(\rho,\overline{J})$ have compact supports, then we can find $\{\overline{E},\overline{B}\}$ such that $\bigtriangledown\times\overline{E}=\overline{0}$, $\Box^2(\overline{E})=\overline{0}$ and $\overline{B}=\overline{0}$.*

*Proof.* Follow the steps in the above proof of lemma 0.5, up to $(AA)$ replacing $\overline{f}=\overline{0}$ with the corresponding relations;

$\frac{\partial}{\partial t}(\overline{E}\times\overline{B})_1=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial y}(p_{12})+\frac{\partial}{\partial z}(p_{13}))$

$\frac{\partial}{\partial t}(\overline{E}\times\overline{B})_2=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial x}(p_{12})+\frac{\partial}{\partial z}(p_{23}))$

$\frac{\partial}{\partial t}(\overline{E}\times\overline{B})_3=-\frac{1}{\epsilon_0}(\frac{\partial}{\partial x}(p_{13})+\frac{\partial}{\partial y}(p_{23}))$ $(BB)$

noting that these three relations together become rotation invariant. The following claim follows immediately from Lemma 0.2. We can then use the proof in [7], which contained an error in the missing rotation

$R_f$ when we write;

$$B_{\overline{u}} = R_f B_{u\overline{e}_1} R_f^{-1}$$

where $f \in SO(3)$ has the property that $f(u\overline{e}_1) = \overline{u}$.

As above, we have that;

$$B_{\overline{u}} = R_g \lambda_\infty R_g^{-1} R_{h_\infty} \lambda_\infty$$

which becomes;

$$R_f B_{u\overline{e}_1} R_f^{-1} = R_g \lambda_\infty R_g^{-1} R_{h_\infty} \lambda_\infty$$

so that;

$$Id = B_{-u\overline{e}_1} R_f^{-1} R_g \lambda_\infty R_g^{-1} R_{h_\infty} \lambda_\infty R_f$$

We then obtain the relation $\bigtriangledown \cdot (\overline{E}_\infty \times \overline{B}_\infty) = 0$ uniformly in the frames connected to $S$ by $\lambda_\infty R_f$ from the relations $(BB)$. This implies the equations in $S$ which we considered in [7]. We claim there in Lemma 1.23 that they imply $\overline{E} \times \overline{B} = \overline{0}$ in $S$. By the proof in [7], if $\overline{B}$ is not identically zero on the corresponding compact set $D$, then $\overline{E} = \lambda \overline{B}$ and $\rho = 0$. By continuity, we can obtain the same result in frames $S'$ connected to $S$ by a velocity vector $\overline{w}$, with $|\overline{w}| < \delta$. Then we can easily adapt the proof of Lemma 1.33 in [7] to obtain that either $\rho = 0$, $\overline{J} = 0$ in $S$, which we can then transfer to the original base frame, it is not necessary for $\delta = c$, or there exist $\{\overline{E}_{\overline{w}}, \overline{B}_{\overline{w}}\}$, with $\overline{B}_{\overline{w}} = \overline{0}$. By a result of [6], Lemma 2.7, we obtain the relations

$$\square^2(\rho) = 0$$

$$\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$$

$$\bigtriangledown \times \overline{J} = \overline{0}$$

$$\square^2(\overline{J}) = \overline{0} \ (CC)$$

in $S$. As is shown in Lemma 2.5 of [6], we obtain the above relations in the original base frame, together with $\{\overline{E}, \overline{B}\}$ such that $\square^2(\overline{E}) = \overline{0}$, $\square^2(\overline{B}) = \overline{0}$. Patching the data over compact sets $D$ with open interior in the original base frame, and noting that the relations $(CC)$ are trivially satisfied by $\rho = 0$, $\overline{J} = \overline{0}$, we obtain a global solution $(\rho, \overline{J})$ satisfying $(CC)$. If we assume that $(\rho, \overline{J})$ have compact supports and are non-trivial, then by the methods of [6], we can find $\{\overline{E}, \overline{B}\}$ such that $\bigtriangledown \times \overline{E} = \overline{0}$, $\square^2(\overline{E}) = \overline{0}$ and $\overline{B} = \overline{0}$.

$\square$

**Lemma 0.7.** *We can approximate smooth functions locally satisfying Maxwell's equations and relevant derivatives by analytic functions.*

*Proof.* On compact sets $D$ with open interior, we can use the Stone-Weierstrass approximation theorem, to approximate the data and relevant derivatives on $D \times (-c, c)$ by polynomials. We can then use continuity to obtain an approximation $D \times (-c - \epsilon, c + \epsilon)$, choosing $\epsilon$ sufficiently small. The final results with the assumption of analytic then hold if the data is assumed to be smooth, by making the approximations arbitrarily close. More details can be found in [7]. $\square$

**Definition 0.8.** *We call an electromagnetic systems $(\rho, \overline{J})$ with compact supports satisfying the continuity equation viable if for every inertial frame $S'$, and for all $\{\overline{E}', \overline{B}'\}$ such that $(\rho', \overline{J}', \overline{E}', \overline{B}')$, satisfies Maxwell's equations in $S'$;*

$$lim_{r \to \infty} \int_{B(\overline{0}, r)} \bigtriangledown \cdot (\overline{E}' \times \overline{B}') dV$$

*exists uniformly and is independent of time $t$.*

*We call a viable electromagnetic system $(\rho, \overline{J})$ satisfying the continuity equation strongly classically non radiating if for every inertial frame $S'$, and for all $\{\overline{E}', \overline{B}'\}$ such that $(\rho', \overline{J}', \overline{E}', \overline{B}')$, satisfies Maxwell's equations in $S'$;*

$$lim_{r \to \infty} \int_{B(\overline{0}, r)} \bigtriangledown \cdot (\overline{E}' \times \overline{B}') dV = 0$$

*We call a viable electromagnetic system $(\rho, \overline{J})$ satisfying the continuity equation stable if the support of $(\rho, \overline{J})$ is stationary, that is there exists $B(\overline{0}, R)$ for which $Supp_t(\rho, \overline{J}) \subset B(\overline{0}, R)$, for all times $t$.*

**Lemma 0.9.** *A stable viable electromagnetic system, satisfying the Lorentz force law, is strongly classically non-radiating.*

*Proof.* Suppose for contradiction that;

$lim_{r\rightarrow\infty}\int_{B(\bar{0},r)}\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')dV=p$

in some inertial frame $S'$, with $p>0$, and $(\rho',\overline{J}')$ is supported on $B(\bar{0},R')$ at time $t'$. Then we can choose $R''$ sufficiently large, so that;

$(i)$. $B(\bar{0},R')\subset B(\bar{0},R'')$

$(ii)$. $\int_{B(\bar{0},R'')}\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')dV=p-\epsilon>0$, uniformly in $t$.

By the Lorentz force law, see Lemma 0.15 below, and Newton's second law of motion, we have that, as in [1];

$du_{mech}=\overline{f}\centerdot d\bar{l}$

$=\overline{f}\centerdot\bar{v}dt$

$=m\bar{a}\centerdot\bar{v}dt$

$=m\frac{d\bar{v}}{dt}\centerdot\bar{v}dt$

$=m\frac{d}{dt}\frac{\bar{v}\centerdot\bar{v}}{2}dt$

$=d(\frac{mv^2}{2})$

so that $u_{mech}$ is defined as local kinetic energy and $u_{mech}\geq 0$. This idea can be made more precise by considering trajectories, see [5].

By Poynting's theorem, we have that;

$\frac{d}{dt}\int_{B(\bar{0},R'')}(u_{mech}+u_{em})dV$

$=-\int_{B(\bar{0},R'')}\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')dV$

$=-(p-\epsilon)$

$<0$

so that $\int_{B(\bar{0},R'')}(u_{mech}+u_{em})dV$ is strictly decreasing, and as $\int_{B(\bar{0},R'')}(u_{mech}+u_{em})dV$ is finite at time $t'$, for some $t''>t'$, we have that;

$\int_{B(\overline{0},R'')}(u_{mech}+u_{em})dV=0$

at time $t>t''$ so that, as $u_{mech}$ and $u_{em}$ are positive, $u_{mech}|_{B(\overline{0},R'')}=u_{em}|_{B(\overline{0},R'')}=0$ at time $t>t''$. This implies that $\overline{E'}|_{B(\overline{0},R'')}=\overline{B'}|_{B(\overline{0},R'')}=\overline{0}$, at time $t>t''$, so that by Maxwell's equations, $\rho'|_{B(\overline{0},R'')}=0$ and $\overline{J'}|_{B(\overline{0},R'')}=\overline{0}$ at time $t>t''$ as well. It follows that the support of $\{\rho',\overline{J}'\}$ must be disjoint from $B(\overline{0},R'')$ at time $t>t''$, contradicting the fact that the system is stable.

Case $p<0$, time reversal argument.

...............

□

**Definition 0.10.** *We call an electromagnetic systems $(\rho,\overline{J})$ with compact supports satisfying the continuity equation classically non-radiating if for every inertial frame $S'$, there exist $\{\overline{E}',\overline{B}'\}$ such that $(\rho',\overline{J}',\overline{E}',\overline{B}')$, satisfies Maxwell's equations in $S'$ and;*

$lim_{r\rightarrow\infty}\int_{B(\overline{0},r)}\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')dV=0$

*locally uniformly in $t$.*

*We call an electromagnetic systems $(\rho,\overline{J})$ with compact supports satisfying the continuity equation strongly non-radiating if for every inertial frame $S'$, there exist $\{\overline{E}',\overline{B}'\}$ such that $(\rho',\overline{J}',\overline{E}',\overline{B}')$, satisfies Maxwell's equations in $S'$ and;*

*$(i)$. $(\overline{E}',\overline{J}')=0$*

*or*

*$(ii)$. $\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')=0$*

**Lemma 0.11.** *Every electromagnetic systems $(\rho,\overline{J})$ with compact supports satisfying the continuity equation, is classically non-radiating.*

*Proof.* Let $S'$ be an inertial frame, then by the transformation rules, $(\rho'.\overline{J}')$ has compact supports. By the argument in [6], we can construct $\{\overline{E}',\overline{B}'\}$ such that $(\rho'.\overline{J}',\overline{E}',\overline{B}')$ satisfies Maxwell's equations in $S'$ and $\overline{B}'$ has compact support. If the support of $\overline{J}'$ varies continuously with $t$ then the support of $\overline{B}'$ varies continuously with $t$, and then it is clear, by the divergence theorem, that;

$lim_{r\rightarrow\infty}\int_{B(\overline{0},r)}\bigtriangledown\centerdot(\overline{E}'\times\overline{B}')dV$

$=lim_{r\rightarrow\infty}\int_{B(\overline{0},r)}(\overline{E}'\times\overline{B}')\centerdot d\overline{S}$

$=0$

locally uniformly in $t$.

□

**Lemma 0.12.** *The strongly non-radiating electromagnetic systems are either of the form $\overline{f}=\overline{0}$ for some $\{\overline{E},\overline{B}\}$ with $(\overline{E},\overline{J})=0$ and $(\overline{E},\overline{B})=0$ in the base frame $S$ or we obtain the relations;*

$\square^2(\rho)=0$

$\bigtriangledown(\rho)+\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\overline{0}$

$\bigtriangledown\times\overline{J}=\overline{0}$

$\square^2(\overline{J})=\overline{0}$

*and we can find $\{\overline{E},\overline{B}\}$ such that $\bigtriangledown\times\overline{E}=\overline{0}$, $\square^2(\overline{E})=\overline{0}$ and $\overline{B}=\overline{0}$, in the base frame $S$.*

*Proof.* Using Lemmas 0.5 and 0.6.

□

**Definition 0.13.** *We define the electromotive force around a moving closed loop $C$ to be;*

$\epsilon=\int_C\overline{g}\centerdot d\overline{l}$

*where $\overline{g}$ is the force per unit charge and $\rho\overline{g}=\overline{f}$, for the charge density and force density respectively.*

*Faraday's strong law of induction says that for any* moving *loop $C(t)$;*

$\epsilon=-\frac{d}{dt}\int_{S(t)}\overline{B}(\overline{x},t)\centerdot d\overline{S}$

*where $S(t)$ is any set of closed surfaces bounded by the loops $C(t)$.*

*Note that this is well defined as for any two closed surfaces* $\{S_1(t), S_2(t)\}$, *bounded by a loop* $C(t)$ *in the family;*

$\int_{S_1(t)} \overline{B}(\overline{x}, t) \centerdot d\overline{S} - \int_{S_2(t)} \overline{B}(\overline{x}, t) \centerdot d\overline{S}$

$= \int_{S_1(t) \cup S_2(t)} \overline{B}(\overline{x}, t) \centerdot \overline{n} dS$

$= \int_{V(t)} (\bigtriangledown \centerdot \overline{B})(\overline{x}, t) dV$

$= 0$

*by the divergence theorem, Maxwell's equations, for the volume* $V(t)$ *enclosed by the two surfaces* $\{S_1(t), S_2(t)\}$.

*We define the flux* $\Phi(t)$ *through a loop* $C(t)$ *as;*

$\Phi(t) = \int_{S(t)} \overline{B}(\overline{x}, t) \centerdot d\overline{S}$

*so Faraday's strong law takes the form;*

$\epsilon = -\frac{d\Phi}{dt}$

**Definition 0.14.** *We call an electromagnetic system normal if the velocity field* $\overline{u}$ *defined by* $\overline{J} = \rho \overline{u}$, *where* $\{\rho, \overline{J}\}$ *are the charge and current, fails to satisfy the symmetry;*

$u_{3x} u_{1y} u_{2z} = u_{2x} u_{3y} u_{1z}$

*where* $\overline{u} = (u_1, u_2, u_3)$.

**Lemma 0.15.** *For a normal electromagnetic system satisfying Maxwell's equations, the Lorentz force law is a consequence of Faraday's strong law of induction and the transformation of fields and forces according to special relativity.*

*Proof.* Let a loop $C(t)$ move along with the velocity field $\overline{u}(\overline{x}, t)$, defined by the current $\overline{J}(\overline{x}, t)$, we have that the change of flux $d\Phi$ is given by;

$d\Phi = \int_{S(t+dt)} \overline{B}(t + dt) \centerdot d\overline{S} - \int_{S(t)} \overline{B}(t) \centerdot d\overline{S}$

where;

$\int_{S(t+dt)} \overline{B}(t+dt) \cdot d\overline{S} + \int_R \overline{B}(t+dt) \cdot d\overline{S} - \int_{S(t)} \overline{B}(t+dt) \cdot d\overline{S} = 0 \ (*)$

by the divergence theorem, and $R$ is the ribbon attaching the surfaces $S(t)$ and $S(t+dt)$.

It follows that, from $(*)$;

$d\Phi = [\int_{S(t+dt)} \overline{B}(t+dt) \cdot d\overline{S} + \int_R \overline{B}(t+dt) \cdot d\overline{S} - \int_{S(t)} \overline{B}(t+dt) \cdot d\overline{S}]$

$- \int_{S(t)} \overline{B}(t) \cdot d\overline{S} - \int_R \overline{B}(t+dt) \cdot d\overline{S} + \int_{S(t)} \overline{B}(t+dt) \cdot d\overline{S}$

$= \int_{S(t)} \overline{B}(t+dt) \cdot d\overline{S} - \int_{S(t)} \overline{B}(t) \cdot d\overline{S} - \int_R \overline{B}(t+dt) \cdot d\overline{S}$

$= dt \int_{S(t)} \frac{\partial \overline{B}}{\partial t} \cdot d\overline{S} - \int_R \overline{B}(t+dt) \cdot d\overline{S}$

We have that, for the ribbon;

$d\overline{S} = (d\bar{l} \times \overline{u} dt) = (d\bar{l} \times \overline{u}) dt$

so that, using the rule for scalar triple products;

$\int_R \overline{B}(t+dt) \cdot d\overline{S} = \int_R \overline{B}(t+dt) \cdot (d\bar{l} \times \overline{u}) dt$

$= \int_R (\overline{u} \times \overline{B}(t+dt)) \cdot d\bar{l} dt$

and then, using Stoke's theorem and Maxwell's equations;

$\frac{d\Phi}{dt} = \int_{S(t)} \frac{\partial \overline{B}}{\partial t} \cdot d\overline{S} - \int_{S(t)} (\overline{u} \times \overline{B}(t)) \cdot d\bar{l}$

$= - \int_{S(t)} (\bigtriangledown \times \overline{E}) \cdot d\overline{S} - \int_{S(t)} (\overline{u} \times \overline{B}(t)) \cdot d\bar{l}$

$= -(\int_{C(t)} [\overline{E} + \overline{u} \times \overline{B}] \cdot d\bar{l})$

so that by Faraday's strong law and the definition of electromotive force, using the fact that the loop's initial position is arbitrary;

$\overline{g} = \overline{E} + \overline{u} \times \overline{B} + \bigtriangledown(h)$

where $h$ is an undetermined scalar.

Let $S'$ be an inertial frame moving relative to the base frame $S$ at a velocity of $v\overline{e}_1$. Then, in $S'$, the Lorentz force is given by;

$\overline{g}' = \overline{E}' + \overline{u}' \times \overline{B}' + \bigtriangledown'(h')$

By the transformation rules for $\{\overline{E}', \overline{B}', \overline{u}'\}$;

$\overline{g}' = \overline{E}_{||} + \gamma_v(\overline{E}_\perp + v\overline{e}_1 \times \overline{B})$

$+(\frac{u_1-v}{1-\frac{u_1v}{c^2}}, \frac{u_2}{\gamma_v(1-\frac{u_1v}{c^2})}, \frac{u_3}{\gamma_v(1-\frac{u_1v}{c^2})}) \times (\overline{B}_{||} + \gamma_v(\overline{B}_\perp - \frac{v\overline{e}_1}{c^2} \times \overline{E}))$

$+ \bigtriangledown' (h')$

$= (e_1, \gamma_v e_2 - \gamma_v v b_3, \gamma_v e_3 + \gamma_v v b_2)$

$+(\frac{u_1-v}{1-\frac{u_1v}{c^2}}, \frac{u_2}{\gamma_v(1-\frac{u_1v}{c^2})}, \frac{u_3}{\gamma_v(1-\frac{u_1v}{c^2})}) \times (b_1, \gamma_v b_2 + \frac{\gamma_v v e_3}{c^2}, \gamma_v b_3 - \frac{\gamma_v v e_2}{c^2})$

$+ \bigtriangledown' (h')$

$= (e_1 + \frac{u_2}{1-\frac{u_1v}{c^2}}(b_3 - \frac{ve_2}{c^2}) - \frac{u_3}{1-\frac{u_1v}{c^2}}(b_2 + \frac{ve_3}{c^2}),$

$\gamma_v e_2 - \gamma_v v b_3 + \frac{1}{\gamma_v(1-\frac{u_1v}{c^2})}(u_3b_1) - \frac{(u_1-v)}{1-\frac{u_1v}{c^2}}(\gamma_v b_3 - \frac{\gamma_v v e_2}{c^2}),$

$\gamma_v e_3 + \gamma_v v b_2 - \frac{1}{\gamma_v(1-\frac{u_1v}{c^2})}(u_2b_1) + \frac{(u_1-v)}{1-\frac{u_1v}{c^2}}(\gamma_v b_2 + \frac{\gamma_v v e_3}{c^2}))$

$+ \bigtriangledown' (h')$. $(A)$

By the transformation rules for forces, see [2], we have that;

$\overline{g}' = \overline{g}_{||} + \frac{-\frac{\gamma_v v}{c^2}(\overline{g}_\perp \cdot \overline{u}_\perp)\overline{e}_1 + \overline{g}_\perp}{\gamma_v(1-\frac{u_1v}{c^2})}$

We have that;

$\overline{g}_{||} = \overline{E}_{||} + (\overline{u} \times \overline{B})_{||} + (\bigtriangledown(h))_{||}$

$= (e_1 + u_2b_3 - u_3b_2, 0, 0) + (\bigtriangledown(h))_{||}$

$\overline{g}_\perp = \overline{E}_\perp + (\overline{u} \times \overline{B})_\perp + (\bigtriangledown(h))_\perp$

$= (0, e_2 + u_3b_1 - u_1b_3, e_3 + u_1b_2 - u_2b_1) + (\bigtriangledown(h))_\perp$

$\overline{g}_{\perp} \cdot \overline{u}_{\perp} = u_2(e_2 + u_3b_1 - u_1b_3) + u_3(e_3 + u_1b_2 - u_2b_1) + (\bigtriangledown(h))_{\perp} \cdot \overline{u}_{\perp}$

so that;

$\overline{g}' = (e_1 + u_2b_3 - u_3b_2 - \frac{v}{c^2(1-\frac{u_1v}{c^2})}[u_2(e_2 + u_3b_1 - u_1b_3) + u_3(e_3 + u_1b_2$

$-u_2b_1)], \frac{1}{\gamma_v(1-\frac{u_1v}{c^2})}(e_2 + u_3b_1 - u_3b_3), \frac{1}{\gamma_v(1-\frac{u_1v}{c^2})}(e_2 + u_3b_1 - u_3b_3))$

$+ \bigtriangledown (h)_{||} + \frac{-\frac{\gamma_v v}{c^2}(\bigtriangledown(h)_{\perp} \cdot \overline{u}_{\perp})\overline{e}_1 + \bigtriangledown(h)_{\perp}}{\gamma_v(1-\frac{u_1v}{c^2})}$ $(B)$

Equating $(A)$ and $(B)$, by a straightforward calculation, left to the reader, we obtain that;

$\bigtriangledown'(h') = \bigtriangledown(h)_{||} + \frac{-\frac{\gamma_v v}{c^2}(\bigtriangledown(h)_{\perp} \cdot \overline{u}_{\perp})\overline{e}_1 + \bigtriangledown(h)_{\perp}}{\gamma_v(1-\frac{u_1v}{c^2})}$

Taking the limit as $v \rightarrow \infty$, letting $\overline{s} = \bigtriangledown(h)$, we obtain that;

$\bigtriangledown_{\infty}(h_{\infty}) = \overline{s}_{||} + \frac{\frac{i}{c}(\overline{s}_{\perp} \cdot \overline{u}_{\perp})\overline{e}_1 + \overline{s}_{\perp}}{\frac{iu_1}{c}}$

$= (s_1 + \frac{s_2u_2+s_3u_3}{u_1}, -\frac{s_2ic}{u_1}, -\frac{s_3ic}{u_1})$ $(D)$

We have the transformation rule;

$\bigtriangledown' = \gamma_v(\bigtriangledown_{||} + \frac{\overline{v}}{c^2}\frac{\partial}{\partial t}) + \bigtriangledown_{\perp}$

so that, the operator $\bigtriangledown_{\infty}$ transforms in the limit for $v \rightarrow \infty$ as;

$\bigtriangledown_{\infty} = -\frac{i}{c}\frac{\partial}{\partial t}\overline{e}_1 + \bigtriangledown_{\perp}$

$= (-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

Taking the curl of $(D)$, we obtain;

$(-\frac{i}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (s_1 + \frac{s_2u_2+s_3u_3}{u_1}, -\frac{s_2ic}{u_1}, -\frac{s_3ic}{u_1}) = \overline{0}$

so that;

$\frac{\partial}{\partial y}(\frac{s_3}{u_1}) = \frac{\partial}{\partial z}(\frac{s_2}{u_1})$

$\frac{\partial}{\partial t}(\frac{s_3}{u_1}) = -\frac{\partial}{\partial z}(\overline{s} \cdot \frac{\overline{u}}{u_1})$

$\frac{\partial}{\partial t}(\frac{s_2}{u_1}) = -\frac{\partial}{\partial y}(\overline{s} \centerdot \frac{\overline{u}}{u_1})$

Boosting in the directions $\overline{e}_2$ and $\overline{e}_3$, we obtain, by symmetry the equations;

$\frac{\partial}{\partial y}(\frac{s_3}{u_1}) = \frac{\partial}{\partial z}(\frac{s_2}{u_1}),\ \frac{\partial}{\partial x}(\frac{s_3}{u_2}) = \frac{\partial}{\partial z}(\frac{s_1}{u_2}),\ \frac{\partial}{\partial y}(\frac{s_1}{u_3}) = \frac{\partial}{\partial x}(\frac{s_2}{u_3})$

$\frac{\partial}{\partial t}(\frac{s_3}{u_1}) = -\frac{\partial}{\partial z}(\overline{s} \centerdot \frac{\overline{u}}{u_1}),\ \frac{\partial}{\partial t}(\frac{s_2}{u_1}) = -\frac{\partial}{\partial y}(\overline{s} \centerdot \frac{\overline{u}}{u_1})$

$\frac{\partial}{\partial t}(\frac{s_3}{u_2}) = -\frac{\partial}{\partial z}(\overline{s} \centerdot \frac{\overline{u}}{u_2}),\ \frac{\partial}{\partial t}(\frac{s_1}{u_2}) = -\frac{\partial}{\partial x}(\overline{s} \centerdot \frac{\overline{u}}{u_2})$

$\frac{\partial}{\partial t}(\frac{s_1}{u_3}) = -\frac{\partial}{\partial x}(\overline{s} \centerdot \frac{\overline{u}}{u_3}),\ \frac{\partial}{\partial t}(\frac{s_2}{u_3}) = -\frac{\partial}{\partial y}(\overline{s} \centerdot \frac{\overline{u}}{u_3})$

and, by the fact that $\overline{s} = \bigtriangledown(h)$, we have that the following three equations;

$\frac{\partial}{\partial y}(s_3) = \frac{\partial}{\partial z}(s_2),\ \frac{\partial}{\partial x}(s_3) = \frac{\partial}{\partial z}(s_1),\ \frac{\partial}{\partial x}(s_2) = \frac{\partial}{\partial y}(s_1)$

Combining the first and last three sets of equations, we obtain;

$\frac{s_1}{s_2} = \frac{u_{3x}}{u_{3y}},\ \frac{s_1}{s_3} = \frac{u_{2x}}{u_{2z}},\ \frac{s_2}{s_3} = \frac{u_{1y}}{u_{1z}}$

which implies the symmetry;

$\frac{u_{3x}}{u_{3y}}\frac{u_{1y}}{u_{1z}} = \frac{u_{2x}}{u_{2z}}$

or;

$u_{3x}u_{1y}u_{2z} = u_{2x}u_{3y}u_{1z}$

$\square$

**Remarks 0.16.** *It is probably possible to prove this result without the assumption of normality. Either one can manipulate the additional equations in the proof, or one can use a direct transformation of the* $\bigtriangledown'$ *operator, to the base frame* $S$*, from the frame* $S_{v\overline{e}_1}$*.*

**Lemma 0.17.** *We cannot have in a classically non radiating system, satisfying the Lorentz force law, that the condition of strongly non radiating fails, unless it contradicts the laws of thermodynamics. Hence classically non-radiating systems which conform to the laws of thermodynamics and the Lorentz force law are strongly non radiating and* *classifiable. In particular, any normal electromagnetic system obeying Maxwell's equation, the strong form of Faraday's law of induction and the laws of thermodynamics are strongly non-radiating and classifiable.*

*Proof.* For the first claim, let a classically non radiating system be given. We consider the strong no radiation condition, that in all inertial frames $S$, there exist $\{\overline{E},\overline{B}\}$ such that either $(\overline{E},\overline{J})=0$ or $\bigtriangledown\centerdot(\overline{E}\times\overline{B})=0$.

If this condition fails, we can find an inertial frame $S'$, such that for the fields $\{\overline{E},\overline{B}\}$ in the definition of classically non radiating, there exist $\{(\overline{x}_1,t_1),(\overline{x},t_2)\}$ such that;

$(\overline{E},\overline{J})(\overline{x}_1,t_1)\neq 0$

$\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x}_2,t_2)\neq 0$

Case 1. There exists $(\overline{x},t)$ such that $(\overline{E},\overline{J})(\overline{x},t)\neq 0$ and $\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t)\neq 0$

By a time reversal argument, we can assume that $\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x}_2,t)>0$.

Case 1a. $(\overline{E},\overline{J})(\overline{x}_1,t)<0$

By continuity, we can choose disjoint volumes $V_1$ and $V_2$, containing $\overline{x}_1$ and $\overline{x}_2$, such that;

$\int_{V_1}(\overline{E},\overline{J})(\overline{x},t)d\overline{x}<0$

$\int_{V_2}\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t)>0$

Connecting the....

Case 2. There does not exist $(\overline{x},t)$ such that $(\overline{E},\overline{J})(\overline{x},t)\neq 0$ and $\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t)\neq 0$ $(AA)$

By a time reversal argument, we can assume that $\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x}_2,t_2)>0$.

Case 2a. $(\overline{E},\overline{J})(\overline{x}_1,t)<0$

By continuity, we can choose disjoint volumes $V_1$ and $V_2$, containing $\overline{x}_1$ and $\overline{x}_2$, such that;

$\int_{V_1}(\overline{E},\overline{J})(\overline{x},t_1)d\overline{x}<0$

$\int_{V_2}\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t_2)>0$

By the condition $(AA)$, we can assume that;

$\int_{V_2}(\overline{E},\overline{J})(\overline{x},t_1)d\overline{x}=0$

$\int_{V_1}\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t_2)=0$

so that, connecting the 2 volumes by a thin strip, we can assume that, we have a volume $V_3$, containing $\{\overline{x}_1,\overline{x}_2\}$, for which;

$\int_{V_3}(\overline{E},\overline{J})(\overline{x},t_1)d\overline{x}<0$

$\int_{V_3}\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t_2)>0$

It follows that, in a small time interval $T_1$ about $t_1$, the total kinetic energy of the charge in $V_3$ decreases. Choose a large ball $D_3$ containing $V_3$, so that, using the definition of classically non radiating, there exists a time interval $T_2$ about $t_2$, with;

$\int_{D_3}\bigtriangledown\centerdot(\overline{E}\times\overline{B})(\overline{x},t)=0$

for $t\in T_2$, so no energy leaks from the boundary. It follows that electromagnetic energy is transferred from $V_3$ to the complement $D_3\setminus V_3$, and, by the movement of electrons, transferred to kinetic energy of electrons in $D_3\setminus V_3$, by collisions. If the average velocity of the target is greater than the source, this contradicts a principle of thermodynamics, that heat cannot be transferred from the cooler to the hotter body. If the average velocity of the target is less than the source, we can use a time reversal argument.

.......

The second claim follows from Lemmas 0.15 and 0.11.

$\square$

We change the topic slightly, by giving a quantised example of a series for an electromagnetic system confined to a sphere.

**Lemma 0.18.** *Let the charge and current $(\rho, \overline{J})$ be given, in polar coordinates $(r, \theta, \phi)$ by;*

$\rho(r,\theta,\phi,t) = \frac{sin(kr)}{r}e^{ikct}$

$\overline{J}(r,\theta,\phi) = \frac{ic}{k}(\frac{krcos(kr)-sin(kr)}{r^2})e^{ikct}\hat{\overline{r}}$

*Then $\{\rho, \overline{J}\}$ satisfy the relations;*

$\square^2(\rho) = 0$

$\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

$\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot \overline{J} = 0$

$\square^2(\overline{J}) = \overline{0}$

*and with the choice of $k_0$ such that $tan(k_0 r_0) = kr_0$*

$\overline{J}|_{S(r_0)} = \overline{0}$

*Proof.* For the first claim, using the Laplacian in spherical coordinates;

$\bigtriangledown^2(\rho) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \rho}{\partial r})$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(\frac{kcos(kr)}{r} - \frac{sin(kr)}{r^2})e^{ikct})$

$= \frac{1}{r^2}\frac{\partial}{\partial r}((krcos(kr) - sin(kr))e^{ikct})$

$= \frac{1}{r^2}(kcos(kr) - k^2 rsin(kr) - kcos(kr))e^{ikct}$

$= -\frac{k^2 sin(kr)}{r}e^{ikct}$

$= -k^2\rho$

$= \frac{1}{c^2}\frac{\partial^2 \rho}{\partial^2 t}$

so that $\square^2(\rho) = 0$. For the second claim, we have that;

$\bigtriangledown(\rho) = \frac{\partial \rho}{\partial r}\hat{\overline{r}}$

$= (\frac{kcos(kr)}{r} - \frac{sin(kr)}{r^2})e^{ikct}\hat{\overline{r}}$

$= \frac{krcos(kr)-sin(kr)}{r^2}e^{ikct}\hat{\overline{r}}$

$= -\frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}$

so that $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

For the third claim, we have that;

$\bigtriangledown \cdot \overline{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(\frac{ic}{k}(krcos(kr) - sin(kr))e^{ikct})$

$= \frac{ic}{kr^2}(kcos(kr) - k^2rsin(kr) - kcos(kr))e^{ikct}$

$= \frac{ic}{kr^2}(-k^2rsin(kr))e^{ikct}$

$= -ick\frac{sin(kr)}{r}e^{ikct}$

$= -\frac{\partial \rho}{\partial t}$

where $\overline{J} = A_r\hat{\overline{r}}$, so that $\bigtriangledown \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$

For the fourth claim, we have that;

$\bigtriangledown \times \overline{J} = \overline{0}$

as $\overline{J} = A_r\hat{\overline{r}}$ and, $\frac{\partial A_r}{\partial \theta} = \frac{\partial A_r}{\partial \phi} = 0$.

It follows that, using the second and third claims;

$\bigtriangledown^2(\overline{J}) = \bigtriangledown(\bigtriangledown \cdot \overline{J}) - \bigtriangledown \times (\bigtriangledown \times \overline{J})$

$= \bigtriangledown(-\frac{\partial \rho}{\partial t})$

$= -\frac{\partial}{\partial t}(\bigtriangledown(\rho))$

$= -\frac{\partial}{\partial t}(-\frac{1}{c^2}\frac{\partial \overline{J}}{\partial t})$

$= \frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial^2 t}$

so that $\Box^2(\overline{J}) = \overline{0}$.

The final claim is clear.

$\Box$

**Lemma 0.19.** *Let* $\overline{E} = \frac{i}{\epsilon_0 kc}\overline{J}$ *and* $\overline{B} = \overline{0}$*, then* $(\rho, \overline{J}, \overline{E}, \overline{B})$ *satisfy Maxwell's equations, with* $\Box^2(\overline{E}) = \overline{0}$ *and* $\bigtriangledown \times \overline{E} = \overline{0}$*.*

*Proof.* For the first claim, we have that;

$\bigtriangledown \cdot \overline{E} = \frac{i}{\epsilon_0 kc} \bigtriangledown \cdot \overline{J}$

$= -\frac{i}{\epsilon_0 kc}\frac{\partial\rho}{\partial t}$

$= -\frac{i}{\epsilon_0 kc} ikc\rho$

$= \frac{\rho}{\epsilon_0}$

and, as in the previous lemma;

$\bigtriangledown \times \overline{E} = \frac{i}{\epsilon_0 kc} \bigtriangledown \times \overline{J}$

$= \overline{0}$

$= -\frac{\partial\overline{B}}{\partial t}$

and;

$\bigtriangledown \cdot \overline{B} = 0$

and;

$\bigtriangledown \times \overline{B}$

$= \overline{0}$

$= \mu_0\overline{J} + \frac{1}{c^2}\frac{\partial\overline{E}}{\partial t}$

as $\epsilon_0\mu_0 = \frac{1}{c^2}$, where $\Box^2(\overline{E}) = \overline{0}$ follows as above.

□

**Lemma 0.20.** *Taking the real parts* $\{Re(\rho), Re(\overline{J}), Re(\overline{E}), Re(\overline{B})\}$ *of the quantities found above, we have that all of the above relations are satisfied. The property of the Balmer series is satisfied.*

*Proof.* The first claim is clear, as the relations have real coefficients. We have that, with $k_0$ chosen so that $Re(\overline{J})|_{S(r_0)} = \overline{0}$, in particular $j_1(k_0r_0) = 0$;

$$Re(\overline{E}) = Re(\frac{i}{\epsilon_0 k_0 c}\overline{J})$$

$$= Re(\frac{i}{\epsilon_0 k_0 c}\frac{ic}{k_0}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})e^{ik_0 ct}\hat{\overline{r}})$$

$$= Re(-\frac{1}{\epsilon_0 k_0^2}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})e^{ik_0 ct}\hat{\overline{r}})$$

$$= -\frac{1}{\epsilon_0 k_0^2}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})cos(k_0 ct)\hat{\overline{r}}$$

so that;

$$U_{k_0}(t) = \frac{1}{2}\int_{B(r_0)} \epsilon_0 |Re(\overline{E})|^2 + \frac{1}{\mu_0}|Re(\overline{B})|^2 dV$$

$$= \frac{\epsilon_0}{2}\int_{B(r_0)} |Re(\overline{E})|^2 dV$$

$$= \frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{1}{\epsilon_0^2 k_0^4}\int_{B(r_0)}(\frac{(k_0 r cos(k_0 r) - sin(k_0 r))^2}{r^4})dV$$

$$= 4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{1}{\epsilon_0^2 k_0^4}\int_0^{r_0}(\frac{(k_0 r cos(k_0 r) - sin(k_0 r))^2}{r^2})dr$$

$$= 4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{1}{\epsilon_0^2 k_0^4}\int_0^{r_0}[\frac{d}{dr}(\frac{sin(k_0 r)}{r})]^2 r^2 dr$$

$$= 4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{k_0^2}{\epsilon_0^2 k_0^4}\int_0^{r_0}[\frac{d}{dr}(j_0(k_0 r))]^2 r^2 dr$$

$$= 4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{k_0^2}{\epsilon_0^2 k_0^4}\int_0^{r_0}[\frac{d}{dr}((\frac{\pi}{2k_0 r})^{\frac{1}{2}}J_{\frac{1}{2}}(k_0 r))]^2 r^2 dr$$

$$= \frac{\pi}{2}4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{k_0^2}{\epsilon_0^2 k_0^5}\int_0^{r_0}[\frac{d}{dr}(\frac{J_{\frac{1}{2}}(k_0 r)}{r^{\frac{1}{2}}})]^2 r^2 dr$$

$$= \frac{\pi}{2}4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{k_0^2}{\epsilon_0^2 k_0^5}\int_0^{r_0}[-k_0^{\frac{3}{2}}\frac{J_{\frac{3}{2}}(k_0 r)}{k_0^{\frac{1}{2}} r^{\frac{1}{2}}}]^2 r^2 dr$$

$$= \frac{\pi}{2}4\pi\frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{k_0^4}{\epsilon_0^2 k_0^5}\int_0^{r_0}[J_{\frac{3}{2}}(k_0 r)]^2 r dr$$

$= \frac{\pi^2}{\epsilon_0 k_0} cos^2(k_0 ct) \int_0^{r_0} [(\frac{2k_0 r}{\pi})^{\frac{1}{2}} j_1(k_0 r)]^2 r dr$

$= \frac{2k_0}{\pi} \frac{\pi^2}{\epsilon_0 k_0} cos^2(k_0 ct) \int_0^{r_0} [j_1(k_0 r)]^2 r^2 dr$

$= \frac{\pi r_0^2}{4k_0} J_{\frac{5}{2}}^2(k_0 r_0) \frac{2k_0}{\pi} \frac{\pi^2}{\epsilon_0 k_0} cos^2(k_0 ct)$

$= \frac{\pi^2 r_0^2}{2\epsilon_0 k_0} J_{\frac{5}{2}}^2(k_0 r_0) cos^2(k_0 ct)$

where we have used the definition of the Bessel functions $\{j_0, j_1\}$, Lemma 3.4 from [6] to do the integration and the fact from Wikipedia (Bessel Functions), that;

$(\frac{1}{r}\frac{d}{dr})(\frac{J_{\frac{1}{2}}(r)}{r^{\frac{1}{2}}}) = -\frac{J_{\frac{3}{2}}(r)}{r^{\frac{3}{2}}}$

to do the differentiation in the above calculation.

Using the fact, see [6] on the Balmer series, that for large values of $k_0$;

$J_{\frac{5}{2}}(k_0 r_0) = (P_2(\frac{1}{k_0 r_0}) sin(k_0 r_0) - Q_1(\frac{1}{k_0 r_0}) cos(k_0 r_0))(\frac{2}{\pi k_0 r_0})^{\frac{1}{2}}$

where $P_2$ is a polynomial of degree 2, with even degree terms, $Q_1$ is a polynomial of degree 1, with odd degree terms, $cos(k_0 r_0) \simeq 0$, $sin(k_0 r_0) \simeq (-1)^{n_0}$, $k_0 \simeq \frac{\pi}{r_0}(n_0 + \frac{1}{2})$, we obtain that;

$J_{\frac{5}{2}}(k_0 r_0) \simeq (P_{2,0} + O(\frac{1}{k_0^2 r_0^2}))(\frac{2}{\pi k_0 r_0})^{\frac{1}{2}}$

so that;

$U_{k_0}(t) = \frac{\pi^2 r_0^2}{2\epsilon_0 k_0} \frac{2}{\pi k_0 r_0} cos^2(k_0 ct)(P_{2,0}^2 + O(\frac{1}{k_0^2 r_0^2}))$

$= \frac{\pi r_0}{\epsilon_0 k_0^2} cos^2(k_0 ct)(P_{2,0}^2 + O(\frac{1}{k_0^2 r_0^2}))$

and integrating over a cycle;

$< U_{k_0} >= \frac{1}{2} \frac{\pi r_0 P_{2,0}^2}{\epsilon_0 k_0^2} + O(\frac{1}{k_0^4})$

so that we obtain the property of the Balmer series for large $\{k_0, k_1\}$

$< U_{k_0} > - < U_{k_1} > \simeq \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0}(\frac{1}{k_0^2} - \frac{1}{k_1^2})$

□

**Lemma 0.21.** *With the above electromagnetic configuration, and $k_0$ chosen so that $\overline{J}|_{S(r_0)} = \overline{0}$, we have that;*

$Q = \int_{B(r_0)} \rho d\overline{x} = 0$

*Proof.* We calculate, using integration by parts;

$\int_{B(r_0)} \rho d\overline{x}$

$= \int_{B(r_0)} \frac{sin(k_0 r)}{r} e^{ik_0 ct} dr$

$= 4\pi e^{ik_0 ct} \int_0^{r_0} r sin(k_0 r) dr$

$= 4\pi e^{ik_0 ct} ([-\frac{r cos(k_0 r)}{k_0}]_0^{r_0} + \int_0^{r_0} \frac{cos(k_0 r)}{k_0} dr$

$= 4\pi e^{ik_0 ct} (-\frac{r_0 cos(k_0 r_0)}{k_0} + \frac{sin(k_0 r_0)}{k_0^2})$

$= 4\pi e^{ik_0 ct} (-\frac{k_0 r_0 cos(k_0 r_0) + sin(k_0 r_0)}{k_0^2})$

$= 0$

as $k_0$ is chosen so that $\overline{J}|_{S(r_0)} = \overline{0}$, and $j_1(k_0 r_0) = 0$.

□

The calculation shows that the configuration, confined to the sphere $S(r_0)$, is not ionised. In order to model charge accumulating inside a hollow shell using an electron beam, we need the following lemma.

**Lemma 0.22.** *We can obtain the conclusions of Lemma 0.18, by adding a constant $C$ to $\rho$ and keeping $\overline{J}$ the same. We can obtain the conclusions of Lemma 0.19 by setting;*

$\overline{E} = \frac{i}{\epsilon_0 kc} \overline{J} + \frac{C}{3\epsilon_0} \overline{x} = \frac{i}{\epsilon_0 kc} \overline{J} + \frac{Cr}{3\epsilon_0} \hat{\overline{r}}$

*and* $\overline{B} = \overline{0}$.

*Proof.* The first claim is a simple calculation, the relations all involve a derivative of $\rho$.

For the second claim, we have that;

$$\bigtriangledown \cdot (\frac{C}{3\epsilon_0}\overline{x}) = 3\frac{C}{3\epsilon_0} = \frac{C}{\epsilon_0}$$

and;

$$\bigtriangledown \times (\frac{C}{3\epsilon_0}\overline{x}) = \overline{0}$$

and;

$$\frac{\partial(\frac{C}{3\epsilon_0}\overline{x})}{\partial t} = \overline{0}$$

so the result follows by linearity.

□

**Lemma 0.23.** *Taking the real parts* $\{Re(\rho), Re(\overline{J}), Re(\overline{E}), Re(\overline{B})\}$ *of the quantities found in Lemma 0.22, we have that all of the above relations are satisfied. The property of the Balmer series is satisfied again.*

*Proof.* The first claim is again clear, as the relations have real coefficients. We have that, with $k_0$ chosen so that $Re(\overline{J})|_{S(r_0)} = \overline{0}$, in particular $j_1(k_0r_0) = 0$;

$$Re(\overline{E}) = [-\frac{1}{\epsilon_0 k_0^2}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})cos(k_0 ct) + \frac{C}{3\epsilon_0}r]\hat{\overline{r}}$$

so that;

$$U_{k_0}(t) = \frac{\epsilon_0}{2}\int_{B(r_0)} |Re(\overline{E})|^2 dV$$

$$= \frac{\epsilon_0}{2}\int_{B(r_0)} |[-\frac{1}{\epsilon_0 k_0^2}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})cos(k_0 ct) + \frac{Cr}{3\epsilon_0}]\hat{\overline{r}}|^2 dV$$

$$= \frac{\epsilon_0 cos^2(k_0 ct)}{2}\frac{1}{\epsilon_0^2 k_0^4}\int_{B(r_0)}(\frac{(k_0 r cos(k_0 r) - sin(k_0 r))^2}{r^4})dV$$

$$-\epsilon_0 \int_{B(r_0)}(\frac{Cr}{3\epsilon_0^2 k_0^2}(\frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r^2})cos(k_0 ct))dV$$

$$+\frac{\epsilon_0}{2}\int_{B(r_0)}\frac{C^2 r^2}{9\epsilon_0^2}dV$$

and integrating over a cycle, using the previous result;

$$< U_{k_0} >= \frac{1}{2}\frac{\pi r_0 P_{2,0}^2}{\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + \frac{\epsilon_0}{2}\int_{B(r_0)}\frac{C^2 r^2}{9\epsilon_0^2}dV$$

$= \frac{1}{2}\frac{\pi r_0 P_{2,0}^2}{\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + \frac{C^2\epsilon_0}{18\epsilon_0^2}\int_{B(r_0)} r^2 dV$

$= \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + \frac{2C^2\pi\epsilon_0}{9\epsilon_0^2}\int_0^{r_0} r^4 dr$

$= \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + \frac{2C^2 r_0^5\pi\epsilon_0}{45\epsilon_0^2}$

We have that;

$Q = \int_{B(r_0)}(\rho + C)dV$

$= \int_{B(r_0)} CdV$

$= \frac{4\pi C r_0^3}{3}$

so that;

$C = \frac{3Q}{4\pi r_0^3}$

$< U_{k_0} >= \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + 2(\frac{3Q}{4\pi r_0^3})^2 \frac{r_0^5\pi\epsilon_0}{45\epsilon_0^2}$

$= \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} + O(\frac{1}{k_0^4}) + \frac{18Q^2}{16\pi^2 r_0^6}\frac{r_0^5\pi\epsilon_0}{45\epsilon_0^2}$

$= \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} + \frac{Q^2}{40\pi r_0\epsilon_0} + O(\frac{1}{k_0^4})$

and we obtain the property of the Balmer series again, for large $\{k_0, k_1\}$;

$< U_{k_0} > - < U_{k_1} >\simeq \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0}(\frac{1}{k_0^2} - \frac{1}{k_1^2})$

□

We can use this calculation to find the energy required in an electron beam to achieve a standing wave defined by the first fundamental $k_0$.

**Lemma 0.24.** *With a pulse duration of $T = 0.016s$, a steady beam current of $6.25 \times 10^{-4}A$, a varying voltage of $0$ to $10^4V$, a spherical radius of $0.01m$, we can achieve the first fundamental $k_0 = \frac{4.49}{r_0}$. Assuming the power supply is $60Hz$, $120\,V$, we can achieve $T$ and $V$ by attaching the anode and cathode of the gun to the mains with a rectifier and a transformer with a turns ration of $83$ in series.*

*Proof.* Assuming the current in the electron gun is constant, we have that;

$\int_0^t V(t)Idt = \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2} cos^2(k_0 ct) + \frac{Q^2}{40\pi r_0 \epsilon_0} + \epsilon cos(k_0 ct)$ $(*)$

where $\epsilon$ is a constant we haven't determined. For the first fundamental $k_0 = \frac{4.49}{r_0}$, see [4], we choose $r_0 = 0.01$, $\epsilon_0 \simeq 9 \times 10^{-12}$. To compute $P_{2,0}$, use the fact that;

$j_2(x) = (\frac{3}{x^2} - 1)\frac{sin(x)}{x} - \frac{3}{x^2} cos(x)$

$J_{\frac{5}{2}}(x) = (\frac{2x}{\pi})^{\frac{1}{2}} j_2(x)$

$= (\frac{2}{\pi x})^{\frac{1}{2}}((\frac{3}{x^2} - 1)\frac{sin(x)}{x} - \frac{3}{x^2} cos(x))$

$= (\frac{2}{\pi x})^{\frac{1}{2}}[(\frac{3}{x^2} - 1)sin(x) - \frac{3}{x} cos(x)]$

so that $P_2(x) = 3x^2 - 1$, $P_{2,0} = -1$. Then, from $(*)$, differentiating with respect to $t$ and using the FTC;

$V(t)I \simeq -\frac{\pi r_0 k_0 c}{2\epsilon_0 k_0^2} sin(2k_0 ct) + \frac{2QI}{40\pi r_0 \epsilon_0} - \epsilon k_0 c sin(k_0 ct)$ $(**)$

Integrating over a cycle of time $\frac{2\pi}{2k_0 c}$, and assuming that $V(t)$ is approximately constant during this period, we obtain;

$V(t)I = \frac{QI}{20\pi r_0 \epsilon_0}$

so that $V(t) = \frac{Q(t)}{20\pi r_0 \epsilon_0}$

$\simeq Q(t)10^{11}$

For $V_{max} = 10^4 V$ and pulse duration $T = 0.016$s, if $I$ is the steady beam current, then $Q(t) = It$, $V_{max} = 10^{11} I \times 0.016$, so that;

$I = \frac{V_{max}}{10^{11} \times 0.016}$

$= \frac{10^4}{10^{11} \times 0.016}$

$\simeq 6.25 \times 10^{-4}$A

For the last claim, note that the frequency $f = \frac{1}{T} = \frac{1}{0.016} \simeq 60$, which we can achieve with a rectifier attached to the mains power supply, while for a voltage with $V_{max} = 10^4$, we require a transformer with turns ratio $\frac{10^4}{120} \simeq 83.3$, attached to the mains in series with a rectifier.

□

**Remarks 0.25.** *It is not practical to vary the accelerating voltage in an electron gun, this motivates the following calculation.*

**Lemma 0.26.** *Determination of $\epsilon$ and Power Input*

$\epsilon = \frac{-4\pi C r_0^2 sin(k_0 r_0)}{3\epsilon_0 k_0^2}$

*and, using Matlab;*

*At a radius of $0.01m$, the corresponding frequency is $21.4$ GHz, with a beam current pulse width of $0.05s$ and the current varying from $0$ to $1mA$.*

*Proof.* We have that;

$\epsilon = -\frac{C\epsilon_0}{3\epsilon_0^2 k_0^2} \int_{B(r_0)} \frac{k_0 r cos(k_0 r) - sin(k_0 r)}{r} dV$

$= -\frac{4\pi C}{3\epsilon_0 k_0^2} \int_0^{r_0} (k_0 r cos(k_0 r) - sin(k_0 r)) r dr$

$= -\frac{4\pi C}{3\epsilon_0 k_0^2} [[\frac{k_0 r^2 sin(k_0 r)}{k_0}]_0^{r_0} - \int_0^{r_0} \frac{2k_0 r sin(k_0 r)}{k_0} dr + [\frac{r cos(k_0 r)}{k_0}]_0^{r_0} - \int_0^{r_0} \frac{cos(k_0 r)}{k_0}]$

$= -\frac{4\pi C}{3\epsilon_0 k_0^2} [r_0^2 sin(k_0 r_0) - 2([\frac{-r cos(k_0 r)}{k_0}]_0^{r_0} + \int_0^{r_0} \frac{cos(k_0 r)}{k_0} dr) + \frac{r_0 cos(k_0 r_0)}{k_0}$

$-\frac{sin(k_0 r_0)}{k_0^2}]$

$= \frac{-4\pi C}{3\epsilon_0 k_0^2} (r_0^2 sin(k_0 r_0) + \frac{2r_0 cos(k_0 r_0)}{k_0} - \frac{2 sin(k_0 r_0)}{k_0^2} + \frac{r_0 cos(k_0 r_0)}{k_0} - \frac{sin(k_0 r_0)}{k_0^2})$

$= \frac{-4\pi C}{3\epsilon_0 k_0^2} (r_0^2 sin(k_0 r_0) + 3(\frac{k_0 r_0 cos(k_0 r_0) - sin(k_0 r_0)}{k_0^2}))$

$= \frac{-4\pi C r_0^2 sin(k_0 r_0)}{3\epsilon_0 k_0^2}$

as $j_1(k_0 r_0) = 0$

It follows that;

$$\int_0^t V(t)I(t)dt = \frac{\pi r_0 P_{2,0}^2}{2\epsilon_0 k_0^2}cos^2(k_0ct)+\frac{Q^2}{40\pi r_0\epsilon_0}-\frac{4\pi C r_0^2 sin(k_0r_0)}{3\epsilon_0 k_0^2}cos(k_0ct)\ (CC)$$

so that, differentiating;

$$V(t)I(t) = -\frac{\pi r_0 k_0 c}{2\epsilon_0 k_0^2}sin(2k_0ct) + \frac{2QI}{40\pi r_0\epsilon_0} - \frac{3I}{4\pi r_0^3}\frac{4\pi r_0^2 sin(k_0r_0)}{3\epsilon_0 k_0^2}cos(k_0ct)$$

$$+\frac{3Qk_0c}{4\pi r_0^3}\frac{4\pi r_0^2 sin(k_0r_0)}{3\epsilon_0 k_0^2}sin(k_0ct)$$

$$= -\frac{\pi r_0 c}{2\epsilon_0 k_0}sin(2k_0ct) + \frac{QI}{20\pi r_0\epsilon_0} - \frac{I sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2} + \frac{Qc}{k_0r_0\epsilon_0}sin(k_0ct)$$

and keeping $I$ fixed;

$$V(t) = -\frac{\pi r_0 c}{2\epsilon_0 k_0 I}sin(2k_0ct) + \frac{Q}{20\pi r_0\epsilon_0} - \frac{sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2} + \frac{Qc}{k_0r_0\epsilon_0 I}sin(k_0ct)$$

$$= -\frac{\pi r_0 c}{2\epsilon_0 k_0 I}sin(2k_0ct) + \frac{It}{20\pi r_0\epsilon_0} - \frac{sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2} + \frac{tc}{k_0r_0\epsilon_0}sin(k_0ct)$$

$$= -\frac{\pi r_0 c}{2\epsilon_0 k_0 I}sin(2k_0ct) - \frac{sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2} + (\frac{I}{20\pi r_0\epsilon_0} + \frac{c}{k_0r_0\epsilon_0}sin(k_0ct))t$$

while keeping $V$ fixed;

$$I(t)[V - \frac{Q}{20\pi r_0\epsilon_0} + \frac{sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2}] = -\frac{\pi r_0 c}{2\epsilon_0 k_0}sin(2k_0ct) + \frac{Qc}{k_0r_0\epsilon_0}sin(k_0ct)$$

so that;

$$\frac{dQ}{dt} = \frac{\lambda(t)+Q(t)\mu(t)}{V+hQ(t)+\nu(t)}\ (BB)$$

where;

$$\lambda(t) = -\frac{\pi r_0 c}{2\epsilon_0 k_0}sin(2k_0ct)$$

$$\mu(t) = \frac{c}{k_0r_0\epsilon_0}sin(k_0ct)$$

$$h = -\frac{1}{20\pi r_0\epsilon_0}$$

$$\nu(t) = \frac{sin(k_0r_0)cos(k_0ct)}{r_0\epsilon_0 k_0^2}$$

A simple Matlab program can solve $(BB)$, the code for which is provided at [3]. At a radius of 0.01m, the corresponding frequency is 21.4 Ghz, with a beam current pulse width of 0.05s and the current varying from 0 to 1mA, see [3]. It is important to pass the pulse signal through a half wave rectifier (diode); with no current and charge supplied, the charge should drain naturally from the sphere in order to physically solve $(BB)$. □

**Remarks 0.27.** *A frequency of* 21.43 *is just outside the range of commercially available radio transverters and receivers, which maximise at* $20GHz$*, although the signal might still be detectable with a lower gain. Alternatively one can attempt to build an RF circuit, see* [4]*. The pulse width of the corresponding beam current in the transmitter, consisting of an electron beam fired into a hollow sphere of radius* $0.01m$*, is achievable with an electron gun attached to a signal generator.*

## References

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