SOME NOTES ON THE MORDELL LANG CONJECTURE AND JOUANOLOU'S THEOREM

TRISTRAM DE PIRO

The following statement can be found in [1];

Theorem 0.1. *Mordell Lang Conjecture*

Let K be an algebraically closed field of characteristic zero, let A be an abelian variety defined over K, X a subvariety of A defined over K and Γ a finitely generated subgroup of $A(K)$, then there exist $\{\gamma_1,\ldots,\gamma_m\}\subset\Gamma$, abelian subvarieties $\{B_1,\ldots,B_m\}$ of A such that $\gamma_i+B_i\subseteq X$, for $1\leq i\leq m$ and;

$X(K)\cap\Gamma=\bigcup_{i=1}^m\gamma_i+(B_i(K)\cap\Gamma)$

This will follow from the relative version;

Definition 0.2. *We define a homomorphism $f:A\rightarrow B$ between abelian varieties to be connected if $Ker(f)$ is an abelian subvariety of A. Let $X\subseteq A$ be an irreducible subvariety, then we define;*

$Stab_X=\{a\in A:a+X=X\}$

Lemma 0.3. *The inverse image $f^{-1}(C)$, for C an abelian subvariety of B, $f:A\rightarrow B$ connected, is an abelian variety.*

Proof. As $Ker(f)\subseteq A$ is connected, the fibres of f are equidimensional and connected. If $f^{-1}(C)$ is not irreducible, we can find components $\{W_1,W_2\}$ such that $C=W_1\cup W_2$, with $pr(W_1)=pr(W_2)$. We must have that for generic $a\in f(A)$, that $W_1(a)=W_2(a)$, as $Ker(f)$ is irreducible. The condition on $f(A)$, that $W_1(a')=W_2(a')$ for $a'inf(A)$ is closed and holds for generic a. Hence, as $f(A)$ is irreducible, it holds everywhere on $f(A)$ and $W_1=W_2$.

□

Theorem 0.4. *Relative Mordell Lang Conjecture*

Let $k \subset K$ be algebraically closed fields of characteristic zero. Let A be an abelian variety defined over K and let X be an irreducible subvariety of A defined over K. Let Γ be a finitely generated subgroup of $A(K)$, and suppose that $X \cap \Gamma$ is Zariski dense in X. Then there exists $\gamma \in \Gamma$, an abelian subvariety B of A containing $\gamma + X$, an abelian variety A' defined over k, a subvariety X' of A' defined over k, and a connected homomorphism f from B to A', such that $\gamma + X = f^{-1}(X')$, $f(B \cap \Gamma) \subset A'(k)$.

Lemma 0.5. *Theorem 0.1 follows from Theorem 0.4 and the result in* [3].

Proof. The proof of Theorem 0.1 follows from the fact that Theorem 0.1 is known when $\{X, A\}$ are defined over K having zero transcendence degree over $\mathcal{Q}$, $(*)$, because K can be replaced by a number field $K_0 \subset K$, which defines $\{A, X\}$ and such that Γ is a finitely generated subgroup of $A(K_0)$; the finitely many generators can be chosen over K_0. This result is due to Faltings, see [3]. Then, if $\{X, A\}$ are defined over K an arbitrary algebraically closed field of characteristic zero, Γ is a finitely generated subgroup of A, let $Z = \overline{X \cap \Gamma}$ and $Z = W_1 \cup \ldots \cup W_s$ be its decomposition into irreducibles. Each W_i is defined over K as K is algebraically closed. We have that;

$$Z = \overline{Z \cap \Gamma}$$

$$= \overline{\bigcup_{i=1}^{s} (W_i \cap \Gamma)}$$

$$= \bigcup_{i=1}^{s} \overline{W_i \cap \Gamma}$$

in particularly, $\overline{W_i \cap \Gamma} = W_i$. We can then apply Theorem 0.4, to each W_i, with k having transcendence degree 0 over $\mathcal{Q}$, to find $\{\gamma_1, \ldots, \gamma_s\} \subset \Gamma$, $\{B_1, \ldots, B_s\}$ abelian subvarieties of A, with $\gamma_i + W_i \subseteq B_i$ and $\{A_1, \ldots, A_s\}$ abelian varieties over k, homomorphisms $\{f_1, \ldots, f_s\}$, $f_i : B_i \to A_i$, and $X_i \subset A_i$ subvarieties over k, such that $\gamma_i + W_i = f_i^{-1}(X_i)$, $1 \leq i \leq s$, $f_i(B_i \cap \Gamma) \subset A_i(k)$. We have that the groups $f_i(B_i \cap \Gamma) \subset A_i$ are finitely generated, so we can apply Falting's result, to obtain that there exist $\{\gamma_{i,1}, \ldots, \gamma_{i,m(i)}\} \subset f_i(B_i \cap \Gamma)$, abelian subvarieties $\{C_{i,1}, \ldots, C_{i,m(i)}\}$ of A_i, $1 \leq i \leq s$, such that

$\gamma_{i,j} + C_{i,j} \subseteq X_i$, for $1 \leq i \leq s$, $1 \leq j \leq m(i)$ and;

$X_i(K) \cap f_i(B_i \cap \Gamma) = \bigcup_{j=1}^{m(i)} \gamma_{i,j} + (C_{i,j}(K) \cap f_i(B_i \cap \Gamma))$

Applying f_i^{-1}, using the fact $D_{i,j} = f_i^{-1}(C_{i,j})$ are abelian subvarieties of B_i, as f_i is connected, we obtain that;

$f_i^{-1}(X_i(K) \cap f_i(B_i \cap \Gamma))$

$= \gamma_i + (W_i(K) \cap \Gamma)$

$= f_i^{-1}(\bigcup_{j=1}^{m(i)} \gamma_{i,j} + (C_{i,j}(K) \cap f_i(B_i \cap \Gamma)))$

$= \bigcup_{j=1}^{m(i)} \delta_{i,j} + (D_{i,j}(K) \cap \Gamma)$

where $f_i(\delta_{i,j}) = \gamma_{i,j}$. Then;

$W_i(K) \cap \Gamma = \bigcup_{j=1}^{m(i)} (\delta_{i,j} - \gamma_i) + (D_{i,j}(K) \cap \Gamma)$

and;

$X(K) \cap \Gamma$

$= Z(K) \cap \Gamma$

$= \bigcup_{i=1}^{s} W_i(K) \cap \Gamma$

$= \bigcup_{i=1}^{s} \bigcup_{j=1}^{m(i)} (\delta_{i,j} - \gamma_i) + (D_{i,j}(K) \cap \Gamma)$

Re indexing, we obtain the result.

□

We now claim that we can strengthen the hypotheses and weaken the conclusion of Theorem 0.4 to;

Theorem 0.6. *Let $k \subset K$ be algebraically closed fields of characteristic zero. Let A be an abelian variety defined over K and let X be an irreducible subvariety of A defined over K, such that $Stab_X$ is finite. Let Γ be a finitely generated subgroup of $A(K)$, and suppose that $X \cap \Gamma$ is Zariski dense in X. Then there exists $\gamma \in \Gamma$, an abelian subvariety B of A containing $\gamma + X$, an abelian variety A' defined over k, a subvariety X' of A' defined over k, and a bijective homomorphism f from*

B to A', such that $\gamma+X=f^{-1}(X')$, $f(B\cap\Gamma)\subset A'(k)$.

Lemma 0.7. *If Theorem 0.6 is true, then so is Theorem 0.4.*

Proof. This is essentially proved in [1]. Assume that Theorem 0.6 holds. Let $\{X,A,\Gamma\}$ be given as in the hypotheses of Theorem 0.4. Then $Stab_X$ is an algebraic subgroup of A defined over K. Let S be its connected component, then S is irreducible and so defines an abelian subvariety of A. Let $A_1=\frac{A}{S}$ be the quotient abelian variety, with canonical projection $\pi:A\rightarrow A_1$. Then $\{\pi_1(X),A_1\}$ are defined over K and $Stab_{\pi_1(X)}$ is finite, as S has finite index in $Stab_X$. We have that, by continuity of π_1, that;

$$\pi_1(\overline{X\cap\Gamma})$$

$$=\pi_1(X)$$

$$=\overline{\pi_1(X\cap\Gamma)}$$

$$=\overline{\pi_1(X)\cap\pi(\Gamma)}$$

so that $\pi_1(X)\cap\pi(\Gamma)$ is Zariski dense in $\pi_1(X)$. Clearly, $\pi_1(\Gamma)\subset A_1(K)$ is finitely generated. It follows that we can obtain the hypotheses of Theorem 0.6, so that there exists $\gamma\in\Gamma$, an abelian subvariety B_1 of A_1 containing $\pi_1(\gamma)+\pi_1(X)$, an abelian variety A' defined over k, a subvariety X' of A' defined over k, and a bijective homomorphism f from B_1 to A', such that $\pi_1(\gamma)+\pi_1(X)=f^{-1}(X')$, $f(B_1\cap\pi_1(\Gamma))\subset A'(k)$. Then, letting $B=\pi_1^{-1}(B_1)$, so that B is an abelian subvariety as π_1 is connected, which contains $\gamma+X$, $\pi_1\circ f:B\rightarrow A'$ is connected as π_1 is connected and f is a bijective homomorphism, $\gamma+X=(\pi_1\circ f)^{-1}(X')$. Clearly $\pi_1\circ f(B\cap\Gamma)\subset A'(k)$, so the conclusion of Theorem 0.4 is obtained.

□

In order to prove Theorem 0.6, we follow the exposition in [1]. We add a derivation to K and enlarge K to a bigger algebraically closed field which we also denote by K, which is differentially closed. We replace the group Γ by a definable group H which contains it and has finite Morley rank. We arrange that k is the field of constant of K, see the article by [11]. The definable group H cannot be one-based, as $X\cap H$ is a definable subset of H, and by a result in [8], $X\cap H$ would be a finite union of cosets of subgroups of H, and, therefore, $X = \overline{X \cap H}$ would be a finite union of cosets of abelian subvarieties of A, contradicting the assumption that $Stab_X$ is finite? $Stab_X \supset H_1 \subset H$ and H_1 is infinite?

We can find B a strongly minimal δ-definable set such that $H \subset acl(B)$. If B is locally modular, then by [7] or [12], B is one based. It follows that H is one based, as if $\{S, T\}$ are algebraically closed sets in H^{eq}, then, we can find subsets $\{S_1, T_1\} \subset B$, such that $acl^{eq}(S_1) = S$, $acl^{eq}(T_1) = T$, with

$$S_1 \underset{S \cap T}{\downarrow} T_1$$

as for $\overline{s_1} \subset S_1$ finite, $Cb(\overline{s_1}/T_1) \in acl^{eq}(\overline{s_1}) \cap acl^{eq}(T_1)$, and using the finite character of forking independence. By forking symmetry, and the fact that algebraic types have U-rank 0;

$$S \underset{S \cap T}{\downarrow} T$$

This contradiction implies that B is non locally modular, (E). We claim that B is non-orthogonal to the constants, (D). If not, choose $\bar{c}$ in B generic over the definition $\bar{b}$ of B, and consider $tp(\bar{c}\bar{b}/k_0)$, where $k_0 = acl(\bar{b}) \cap k$.

We use the representation given in [9]. Replacing $\bar{c}\bar{b}$ with an interdefinable tuple if necessary, there exists a smooth irreducible variety V and an affine subbundle W of $T(V)$ such that;

$$\{\overline{xy} \in V : (\overline{xy}, \overline{x}'\overline{y}') \in W\}$$

is δ-irreducible, defined by $f(\bar{x}, \bar{y}) = 0$, with generic point $\bar{c}\bar{b}$ over k_0, in the sense that any differential polynomial $g(\bar{x}, \bar{y})$, over k_0, vanishing at $\bar{c}\bar{b}$ vanishes on $f(\bar{x}, \bar{y}) = 0$. We have that $f(\bar{x}, \bar{b}) \cap B$ is cofinite in B, as $\bar{c}$ was generic in the sense of Morley rank. Replacing B by $B' = f(\bar{x}, \bar{b}) \cap B$, then B' is strongly minimal and still $H \subset acl(B')$. We claim that if $\bar{e} \in B'$, then $tp(\bar{e}\bar{b}/k_0)$ is a specialisation of $tp(\bar{c}\bar{b}/k_0)$. For suppose that $g \in k_0\{\bar{x}, \bar{y}\}$ is a differential polynomial which has the property that $g(\bar{c}\bar{b}) = 0$, then g vanishes on $f(\bar{x}, \bar{y} = 0$, and it follows that $g(\bar{e}\bar{b}) = 0$, as $f(\bar{e}\bar{b}) = 0$.

As above, replacing $\overline{cb}$ with an interdefinable tuple of length $(N+1)s$, for some N, $s = length(\overline{cb})$, we assume that $t.deg(\overline{cb}_n/\overline{cb}_{n-1}) = d$, forall $n \geq 1$, where where $\overline{cb}_n = (\overline{cb}, \overline{cb}', \ldots, \overline{cb}^{(n)})$. Then, if $tp(\overline{e}\overline{b}/k_0)$ is a proper specialisation, replacing $\overline{cb}$ with the corresponding interdefinable tuple of the same length, we cannot have that $t.deg(\overline{eb}_n/\overline{eb}_{n-1}) \leq d-1$, for any $n \geq 1$, as this is strictly monotone decreasing, and we would have that $U(\overline{c}\overline{b}/k_0) - U(\overline{e}\overline{b}/k_0) = \omega$, which, by additivity of U-rank, implies that $U(\overline{c}/\overline{b}k_0) = U(\overline{e}/\overline{b}k_0) + \omega$, contradicting strong minimality. It follows that, for a proper specialisation;

$t.deg(k_0(\overline{eb}_n)) = t.deg(k_0(\overline{cb}_n)) - r$

where $1 \leq r \leq (N+1)s$.

Moreover, we claim that $tp(\overline{e}\overline{b}/k_0)$ is a proper specialisation iff $\overline{e} \in acl(k_0(\overline{b}))$, $(*)$. We have that there exists N such that for all $n > N$;

$t.deg(k_0(\overline{cb}_n)/k_0(\overline{cb}_{n-1})) = d$

where $\overline{cb}_n = (\overline{cb}, \overline{cb}', \ldots, \overline{cb}^{(n)})$, and $tp(\overline{e}\overline{b}/k_0)$ is a proper specialisation iff;

$t.deg(k_0(\overline{eb}_n)) = t.deg(k_0(\overline{cb}_n)) - r$

where $1 \leq r \leq (N+1)s$, (B), $s = length(\overline{cb})$. In particular, this occurs iff $tp(\overline{e}\overline{b}/k_0) \neq tp(\overline{c}\overline{b}/k_0)$, in which case by strong minimality, $\overline{e} \in acl(k_0(\overline{b}))$ (C)

We claim that for any proper specialisation $tp(\overline{e}\overline{b}'/k_0)$ of $tp(\overline{c}\overline{b}/k_0)$ co-degree r, with $1 < r \leq (N+1)s$, there exists a specialisation;

$tp(\overline{e}'\overline{b}''/k_0)$ of $tp(\overline{c}\overline{b}/k_0)$

of co-degree $r-1$, such that;

$tp(\overline{e}\overline{b}'/k_0)$ is a specialisation of $tp(\overline{e}'\overline{b}''/k_0)$

of co-degree 1. (A)

This follows, as by the representation above, we can find an irreducible subvariety $V_r^0 \subset V$ of algebraic codimension r in V, such that;

$\{\overline{xy} \in V_r^0 : (\overline{xy}, \overline{x}'\overline{y}') \in W\}$

is δ-irreducible, defined by $f_r^0(\overline{x}, \overline{y}) = 0$, with generic point $\overline{e}\overline{b}$ over k_0. Let V_{r-1}^0 be an irreducible subvariety of codimension $r-1$, satisfying the corresponding tangency condition;
...... Tangency condition for codimension 1 on f, choose $f_i \in k_0(V)$ defined over k_0, such that $\{f^1, \ldots, f^d\}$ forms a basis for W

$\sum_{i=1}^{dim(V)} \frac{\partial f}{\partial x_i} f_i^r = h^r f$, for some $h^r \in k_0(V)$, $1 \leq r \leq d$ (UU)

where $f \in k_0[[x_1, \ldots, x_d]] \cap \mathfrak{m}_{\overline{e}\overline{b}}$ in local coordinates $\{x_1, \ldots x_d\}$.

Solve for an etale cover of V. Use uniform bound in etale cohomology of covers of original V, in terms of degree of the cover, and bound in number of specialisations in terms of dimension of cohomology groups. Extend the number of specialisations by taking the images of f under the Galois action.

Given solution in etale cover can push forward to obtain a solution in V?

Need to code the vector bundle generated by W as the kernel of a collection of forms $\{w_1, \ldots, w_p\}$, $w = w_1 \wedge \ldots \wedge w_p$, whose coordinates are rational over $k_0(\overline{c}$, therefore the corresponding minors can be represented by $\{f_i\} \in k_0(V)$, $1 \leq i \leq C_p^{dim(V)}$, so in the Tate algebra. The condition of dependency df on $\{w_1, \ldots, w_p\}$ is defined by the vanishing of the minors of the matrix $(df, w_1, \ldots, w_p\}$. Each condition defines an operator condition T_i on the Tate algebra, rational over $k_0(V)$.

Rewrite the differential equation (UU) as;

$\sum_{i=1}^{dim(V)} \frac{\partial log(f)}{\partial x_i} f_i^r = h^r$, for some $h^r \in k_0(V)$, $1 \leq r \leq d$ (UUU)

and define h^r in terms of the given f^r and $log(f)$. Clear then that we can find local analytic solutions to (UU)?? Rational approximation to the logarithm, solve UU to obtain approximate algebraic solutions?

Generic case, codim 2, can choose differential specialisation $\overline{e}$ to lie on intersection $f \cap g$ in the Tete algebra, such that $Rad(f,g) = I(\overline{e})$, and df, dg are tangent, with nonzero differentials at $\overline{e}$. Suppose that;

$\sum_{i=1}^{dim(V)} \frac{\partial f}{\partial x_i} f_i^r = h^r f + s^r g$, for some $h^r, s^r \in k_0(V)$, $1 \leq r \leq d$ $(UU)'''$

$\frac{f}{g}$ and $\frac{g}{f}$ define a regular functions at $\overline{e}$, using fact that f is tangent to g at $\overline{e}$. Divide by f or$g$;

$\sum_{i=1}^{dim(V)} \frac{\partial log(f)}{\partial x_i} f_i^r = h^r + s^r \frac{g}{f}$

$\sum_{i=1}^{dim(V)} \frac{\partial log(g)}{\partial x_i} f_i^r = h^r \frac{f}{g} + s^r$

In both cases, right hand side is regular at $\overline{e}$, forces $f^r(\overline{e}) = 0$, for some subset of the f^r corresponding to one $\frac{\partial f}{\partial x_i}$, use to construct non-trivial solution in codimension 1, on an etale cover.

Exclude case in codim 2;

$[\sum_{i=1}^{dim(V)} \frac{\partial f}{\partial x_i} f_i^r]^{s_r} = h^r f + s^r g$

$s_r \in \mathcal{Z}_{>1}$.

Generic case, codim 2, can choose differential specialisation $\overline{e}$ to lie on intersection $f \cap g$ in the Tete algebra, such that $(f,g) = I(\overline{e})$, and df, dg are transverse, with nonzero differentials at $\overline{e}$; use fact that $\overline{e}$ has codimension 2 and jet spaces argument, to exclude case where higher differentials are all tangent. As we can complete f, g to a set of uniformisers $\{u_1, \ldots, u_n\}$ at $\overline{e}$ f and g generate a subalgebra $k_0[[f,g]]^{alg}$ of the full algebra $K[[u_1, \ldots, u_n]]^{alg}$. We can construct;

$f_1 = (\alpha f + \beta g) + (\gamma f + \delta g)^2 + (\epsilon f + \xi g)^3$

$f_2 = (\alpha f + \beta g) + (\gamma f + \delta g)^2$

where $\{\alpha, \beta, \gamma, \delta, \epsilon, \xi\} \subset k_0$, so that $f_1(\overline{e}) = f_2(\overline{e} = 0$, and $\frac{f_1}{f_2}$ is regular at $\overline{e}$, have that $Rad(f_1, f_2) = I(\overline{e})$ again.
Then;

$[\sum_{i=1}^{dim(V)} \frac{\partial f_1}{\partial x_i} f_i^r]^{s_r} = h^r f_1 + s^r f_2$

$s_r \in \mathcal{Z}_{>1}$.

again. As we must have that $[\sum_{i=1}^{dim(V)} \frac{\partial f_1}{\partial x_i} f_i^r](\overline{e}) = 0$, (VV) as $\overline{ee}' \in W$ by differential specialisation, $df_1(\overline{e})(\overline{e}') = 0$, as $f_1(\overline{e}) = 0$, so $\overline{ee}' \in Ker(df_1) \cap W$, and if one of the conditions in (VV) fails, we would have that $Ker(df_1) \cap W \subsetneq W$, contradicting the requirement that $(\overline{ee}') \in W_{\overline{e}}$ is generic, otherwise U-rank drops, see above.

Let L_i denote the derivations of $k_0[[f,g]]^{alg}$ defined by;

$\sum_{i=1}^{dim(V)} \frac{\partial g}{\partial x_i} f_i^r$

for $1 \leq r \leq C_{p+1}^{dim(V)}$, the components of $df \wedge \omega$. Let $\mathfrak{g}$ denote the Lie algebra generated by the L_i, $1 \leq i \leq r$. Then, letting $m_{f,g}$ denote the maximal ideal of $k_0[[f,g]]^{alg}$, we have by assumption that each L_i maps $m_{f,g}$ to itself, and as the L_i are derivations, we have that if $h \in m_{f,g}^2$, $h = \sum_j = 1^w s_j t_j$, $\{s_j, t_j\} \subset m_{f,g}$, then;

$L_i(h) = L_i(\sum_j = 1^w s_j t_j)$

$= \sum_{j=1}^w L_i(s_j)t_j + L_i(t_j)s_j \in m_{f,g}^2$

Similarly, each L_i maps $m_{f,g}^r$ to $m_{f,g}^r$, for $r \geq 1$. Consider the representation ϕ_2 of the derived algebra $[\mathfrak{g},\mathfrak{g}]$ on the 2 dimensional vector space $V_2 = \frac{m_{f,g}}{m_{f,g}^2}$ over k_0. Then, we have that $\phi_2(\}) \subset sl_2(k_0)$ and using the fact that $\{f,g\}$ are transverse, so generate V_2 over k_0, we can assume that;

$L(f) = \lambda f + \mu g \bmod m_{f,g}^2$

$L(g) = \nu f - \lambda g \bmod m_{f,g}^2$

where $L \in [\mathfrak{g},\mathfrak{g}]$. Then considering the representation of $[\mathfrak{g},\mathfrak{g}]$ on $V_3 = \frac{m_{f,g}^2}{m_{f,g}^3}$, we have by the derivation property that;

$L(f^2) = 2f(\lambda f + \mu g) = 2\lambda f^2 + 2\mu fg \bmod m_{f,g}^3$

$L(g^2) = 2g(\nu f - \lambda g) = -2\lambda g^2 + 2\nu fg \bmod m_{f,g}^3$

$L(fg)=f(\nu f-\lambda g)+g(\lambda f+\mu g)=\nu f^2-\lambda fg+\lambda fg+\mu g^2=\nu f^2+\mu g^2$ mod $m_{f,g}^3$ (RR)

Now consider the representation $\phi_{1,3}$ of $[\mathfrak{g},\mathfrak{g}]$ on $\frac{m_{f,g}}{m_{f,g}^3}$, then the image $\phi_{1,3}([\mathfrak{g},\mathfrak{g}])$ must be a subalgebra of $sl_5(k_0)$.... However, using the fact that $[M_{23},M_{32}]=M_{22}-M_{33}$ and (RR), with $M_{22}=0$, we obtain a contradiction unless $\lambda=0$ or $\mu=0$, so the image $\phi_{1,3}([\mathfrak{g},\mathfrak{g}])$ must be solvable, with an upper triangular representation. Assume that $\nu\neq 0$, then using the fact that $M_{13}=[M_{12},M_{23}]$, and (RR) again, with $M_{13}=0$, we must have that $\mu=0$ as well...(no). It follows that we can assume that;

$L(f)=\lambda f$ mod $m_{f,g}^2$

$L(g)=-\lambda g$ mod $m_{f,g}^2$

Exclude the case of a remainder......(no)
...... Need case of infinite co degree specialisations, generated by pairs (f_i,g_i). Consider the divisor group Σ generated by the $\{f_i\}$, we can define a map for an unobstructed subgroup $\Sigma'\leq\Sigma$;

$\Psi:\Sigma\rightarrow\frac{H^0(U,\Omega_X^1)}{H^0(X,\Omega_X^1)}$

$\Psi((f_\alpha))=\frac{df_\alpha}{f_\alpha}-\nu_{f_\alpha}+H^0(X,\Omega_X^1)$

where ν_{f_α} is a regular one form on U_α. Ψ is injective, see [9].

and;

$\Phi:\Sigma\rightarrow\frac{H^0(X,\Omega_X^{p+1}\otimes L)}{(w\wedge H^0(X,\Omega_X^1))}$

$f_i\mapsto(\frac{df_i\wedge w}{f_i}-\nu_i\wedge w)^{twist}+H^0(X,\Omega_X^1)$

As Σ' has arbitrarily large rank and the rank of $H^0(X,\Omega_X^{p+1}\otimes L)$ is finite, we can find, for $f_i\in Ker(\Phi)$, θ_{f_i}, such that;

$(\frac{df_i\wedge w}{f_i}-\nu_i\wedge w)^{twist}=w\wedge\theta_{f_i}$ $(*)$

$(df_i\wedge w-f_i\nu_i\wedge w)=w\wedge\theta_{f_i}$ $(*)$

$(df_i - f_i\nu_i - \theta_{f_i}) \wedge w = 0$ $(*)$

$(df_i - f_i\nu_i - \theta_{f_i}) \in H^0(X, \Omega_X^{p+1} \otimes L)$

....
.... so that;

$(\frac{df_i \wedge w}{f_i} - \nu_i \wedge w - \theta_{f_i} \wedge w)^{twist} = 0$ $(**)$

$(\frac{df_i \wedge w}{f_i} - \nu_i \wedge w - \theta_{f_i} \wedge w) = 0$

$w \wedge \overline{\Psi}(w) = 0$

where $\overline{\Psi}(f_i) = (\frac{df_i}{f_i} - \nu_i - \theta_{f_i})$ and f_i is a codegree 1 specialisation. Follow argument in [**?**].

....
Need case of infinite co degree specialisations, generated by pairs (f_i, g_i). Consider the divisor group Σ generated by the f_i, we can define a map;

Choose N large with $\{f_1, \ldots f_N\}$ linearly independent in $< f_1, \ldots f_N > \subset Div \otimes k_0$. Let Σ' be the group generated.

We have the unobstructed map $\Psi : \Sigma \rightarrow \frac{H^0(U,\Omega_1)}{H^0(X,\Omega_1)}$ defined by;

$\Psi((f_\alpha)) = \frac{df_\alpha}{f_\alpha} - \nu_{f_\alpha}$

Define a map;

$\Phi : \Sigma' \rightarrow \frac{H^0(X,\Omega_X^{p+1} \otimes L)}{w \wedge H^0(X,\Omega_X^1)}$

$f_{1,\alpha}^{m_1} \ldots f_{N,\alpha}^{m_N} \mapsto f_{1,\alpha}^{\delta(m_1)} \ldots f_{N,\alpha}^{\delta(m_n)} (\frac{df_{1,\alpha}^{m_1} \ldots f_{N,\alpha}^{m_N} \wedge w}{f_{1,\alpha}^{m_1} \ldots f_{N,\alpha}^{m_N}} - \nu_{f_{1,\alpha}^{m_1} \ldots f_{N,\alpha}^{m_N}} \wedge w)$

where $\delta(m) = 0$ if $m = 0$ and $\delta(m) = 1$ if $m \geq 1$. Generically, $(\frac{df_{1,\alpha}^{m_1} \ldots f_{N,\alpha}^{m_N} \wedge w}{f_{i,\alpha}} - \nu_{f_{i,\alpha}})$ will be a rational section of the sheaf $\Omega_X^{p+1} \otimes L$, Let M be the sheaf defined by the transition functions;

$\frac{f_{1,\alpha}^{\delta(m_1)} \ldots f_{N,\alpha}^{\delta(m_n)}}{f_{1,beta}^{\delta(m_1)} \ldots f_{N,\beta}^{\delta(m_n)}}$

and we obtain a global section of the sheaf $\Omega_X^{p+1} \otimes L \otimes M$, (AB).

To see (AB), consider the simplest case;

$(\frac{w\wedge df_{1,\alpha}}{f_{1,\alpha}} - w\wedge\nu_{1,\alpha} + \frac{w\wedge df_{2,\alpha}}{f_{2,\alpha}} - w\wedge\nu_{2,\alpha})^{twist}$

We claim this belongs to $H^0(X, \Omega_X^{p+1} \otimes L \otimes M)$, for the line bundle M defined by the transition functions $f_{1,\alpha}f_{2,\alpha}$. We have that $(w\wedge\nu_{1,\alpha})$ and $(w\wedge\nu_{2,\alpha})$ are regular on U_α, and;

$\frac{w\wedge df_{1,\alpha}}{f_{1,\alpha}} - w\wedge\nu_{1,\alpha} + \frac{w\wedge df_{2,\alpha}}{f_{2,\alpha}} - w\wedge\nu_{2,\alpha}$

$= \frac{1}{f_{1,\alpha}f_{2,\alpha}}[f_{2,\alpha}(f_{1,\alpha}w_1 + g_{1,\alpha}w_2) - f_{1,\alpha}f_{2,\alpha}(w\wedge\nu_{1,\alpha})$

$+f_{1,\alpha}(f_{2,\alpha}w_3 + g_{2,\alpha}w_4) - f_{1,\alpha}f_{2,\alpha}(w\wedge\nu_{2,\alpha})]$

Assume for contradiction that;

$f_{1,\alpha}f_{2,\alpha}|[f_{2,\alpha}(f_{1,\alpha}w_1+g_{1,\alpha}w_2)-f_{1,\alpha}f_{2,\alpha}(w\wedge\nu_{1,\alpha})+f_{1,\alpha}(f_{2,\alpha}w_3+g_{2,\alpha}w_4)$

$-f_{1,\alpha}f_{2,\alpha}(w\wedge\nu_{2,\alpha})]$

Then;

$f_{1,\alpha}f_{2,\alpha}|[f_{2,\alpha}g_{1,\alpha}w_2 + f_{1,\alpha}g_{2,\alpha}w_4$

$f_{1,\alpha}|f_{2,\alpha}g_{1,\alpha}w_2$

$f_{2,\alpha}|f_{1,\alpha}g_{2,\alpha}w_4$

Assuming that $g_{1,\alpha}$ and $g_{2,\alpha}$ are irreducible, see note below, and working in local coordinates for $\{w_2, w_4\}$, we obtain that $f_{1,\alpha} \sim f_{2,\alpha}$, contradicting the presentation below.

.... Consider the case of a codegree 2 specialisation on X, defined by irreducible divisors $\{D, F\}$, with the divisor D defined in local coordinates by f_α. Consider the short exact sequence;

$0 \to \mathfrak{L}(-D) \to O_X \to O_D \to 0$

We have that $\mathfrak{L}(D)\otimes\Omega_X^{p+1}$ is locally free as $\mathfrak{L}(D)$ is invertible and X is nonsingular. It follows that, tensoring with $\mathfrak{L}(D)\otimes\Omega_X^{p+1}$, we obtain a short exact sequence;

$0\rightarrow\mathfrak{L}(-D)\otimes\mathfrak{L}(D)\otimes\Omega_X^{p+1}\rightarrow O_X\otimes\mathfrak{L}(D)\otimes\Omega_X^{p+1}\rightarrow O_D\otimes\mathfrak{L}(D)\otimes\Omega_X^{p+1}\rightarrow 0$

$0\rightarrow\Omega_X^{p+1}\rightarrow\mathfrak{L}(D)\otimes\Omega_X^{p+1}\rightarrow O_D\otimes_{O_X}\mathfrak{L}(D)\otimes\Omega_X^{p+1}\rightarrow 0\ (BV)$

By the above calculation (AB), we have that in local coordinates;

$f_\alpha(w\wedge\frac{df_\alpha}{f_\alpha}-w\wedge\nu_\alpha)$

is a global section σ of $\mathfrak{L}(D)\otimes\Omega_X^{p+1}$ and we compute its restriction $i^{-1}\sigma$ in $O_D\otimes_{O_X}\mathfrak{L}(D)\otimes\Omega_X^{p+1}$. We have that;

$i^{-1}(f_\alpha(w\wedge\frac{df_\alpha}{f_\alpha}-w\wedge\nu_\alpha))$

$=i^{-1}(w\wedge df_\alpha-f_\alpha(w\wedge\nu_\alpha))$

$=i^{-1}(w\wedge df_\alpha)$, as f_α vanishes on $f_\alpha=0$

$=i^*w$ (as df_α vanishes on the tangent vectors of f_α)

The residue doesn't have to be zero. The global section functor is left exact, so we obtain from (BV);

$0\rightarrow\Gamma(X,\Omega_X^{p+1})\rightarrow\Gamma(X,\mathfrak{L}(D)\otimes\Omega_X^{p+1})\rightarrow\Gamma(X,O_D\otimes_{O_X}\mathfrak{L}(D)\otimes\Omega_X^{p+1})$ No.

....

Take a system of local coordinates $\{x_1,\ldots x_n\}$ on X. As $\Omega^1(X)$ is locally free because X is non-singular, we can find an open set U such that the differentials $\{dx_1,\ldots dx_n\}$ remain independent on U. Without loss of generality, we can assume that infinitely many co-degree 2 specialisations lie on U. Otherwise, we find infinitely many codegree specialisations on $Z=X\setminus U$ and we can use induction.....

Using the fact that the derivation $df\wedge w(\xi_1,\ldots,\xi_{p+1})$ has an eigenvector if the representation is semisimple, we can patch the forms;

$(\frac{df_\alpha}{f_\alpha}-\nu_\alpha)\wedge w\wedge dx_{p+2}\wedge\ldots\wedge dx_n$

to give a global section of the canonical sheaf Ω^n_X, possibly with a twist.

Then use cohomological arguments to show that we can find a meromorphic integral $dg\wedge w\wedge dx_{p+2}\wedge\ldots\wedge dx_n=0$. Repeating for the finitely many permutations $dx_{\sigma(p+2)}\wedge\ldots\wedge dx_{\sigma(n)}$, we can find a meromorphic integral;

$dg\wedge w\wedge dx_{\sigma(1)}\wedge\ldots\wedge dx_{\sigma(p+1)}=0$

for all permutations σ, to give that;

$dg\wedge w=0$

.... Case of the remainder, consider the representation $df\wedge w(\xi_1,\ldots,\xi_{p+1})$ on $k_0[f,g]$ and then $\frac{m_{f,g}}{m^N_{f,g}}$ for large N. Again we can find an eigenvector r, to obtain that;

$dr\wedge w\wedge dx_{p+2}\wedge\ldots\wedge dx_n=(\lambda r+\theta)dx_1\wedge\ldots\wedge dx_n$

where $\theta\in m^{N+1}_{f,g}$. Choose a uniformiser s such that r,s are uniformisers for $m_{f,g}$, so that $m^{N+1}_{f,g}=m^{N+1}_{s,r}$, and the intersection product;

$(s=0,r=0)\geq(K,r=0)$ (HU)

where K is the canonical class.

Then;

$\frac{dr}{r}\wedge w\wedge dx_{p+2}\wedge\ldots\wedge dx_n=\frac{s^{N+1}}{r}w_1+w_2\in h^0(\Omega^n_X\otimes L)$

(L defined by $r=0$)

where w_1 and w_2 are regular local sections of the canonical sheaf.

Restricting to the divisor $r=0$, we obtain that;

$$deg(K|_{r=0})=(K,r=0)=(N+1)(s=0,r=0)$$

which contradicts (HU)...no

....

$$(\frac{df_\alpha}{f_\alpha}-\nu_\alpha)\wedge w\wedge dx_{p+2}\wedge\ldots\wedge dx_n$$

gives a rational section of the canonical sheaf Ω^n_X
By the representation, this equals;

$$\frac{g_\alpha}{f_\alpha}\omega+\omega_1$$

where ω_1 is regular and ω is a rational section of Ω^n_X. Twisting by $f=0$, we obtain a rational section σ of $\Omega^n_X\otimes L$, locally of the form;

$$g_\alpha\omega+f_\alpha\omega_1\ (UO)$$

Restricting to $f=0$, defining the divisor L, and taking the degree, we obtain that;

$$(K+L,L)=(K+(g=0),L)$$

$$(L,L)=(g=0,L)$$

If $(L,L)\leq 0$, we obtain a contradiction, as $g=0$ intersects L properly. If $(L,L)>0$, we can use the adjunction formula;

$$\sigma|_L=\omega_L$$

where ω_L is a section of the canonical sheaf Ω^{n-1}_L on L. By the local representation (UO), we have that;

$$g_\alpha\omega|_L\in h^0(\Omega^{n-1}_L)$$

so that;

$$g_\alpha\in h^0(\Omega^{n-1}_L\otimes(\omega|^n_X)|^{-1}_L)$$

Using the short exact sequence of sheaves on L;

$0 \rightarrow \frac{J}{J^2} \rightarrow \Omega_X \otimes O_L \rightarrow \Omega_L \rightarrow 0$

and taking exterior powers;

$\Omega_X^n|_L \cong \Omega_L^{n-1} \otimes (\frac{J}{J^2})$

see [4], p182

$(\frac{J}{J^2}) \cong \Omega_X^n|_L \otimes (\Omega_L^{n-1})^{-1}$

$Hom(\frac{J}{J^2}, O_L) = (\frac{J}{J^2})^{-1} \cong \Omega_L^{n-1} \otimes (\omega|_X^n)|_L^{-1}$

so that, twisting the sheaf of differentials if necessary, the g_α patch to form a global section of the normal bundle $Hom_{O_L}(\frac{J}{J^2}, O_L)$. By a result in [4], L admits an infinitesimal deformation in X over the ring of dual numbers. As $g = 0$ passes through the generic point $\overline{e}$, we can use Schlessinger's criteria to generate a global deformation over P^1, fixing $\overline{e}$. In this case, we can assume that the differential specialisation $\overline{e}$ is defined as the intersection of two linearly equivalent divisors $\{L, L'\}$. We can then remove the twist in L, as locally, we obtain that;

$(\frac{df_\alpha}{f_\alpha} - \nu_\alpha) \wedge w \wedge dx_{p+2} \wedge \ldots \wedge dx_n$

$= \frac{g_\alpha}{f_\alpha}\omega + \omega_1$

$= h\omega + \omega_1$

is a rational section of Ω_X^n, where h is a rational function with $(h) = (f = 0) - (g = 0)$, and we obtain a global section, independently of the divisors $f = 0$, twisting the sheaf Ω_X^n.

.... For infinite co-degree 2 specialisations, have to remove singular points p_i for the defining divisors $\{f_i, g_i\}$ and the cases when $\{f_i, g_i\}$ are tangent. If p_i is singular for one divisor g_i, this can be achieved by a simple change of variables, replacing f_i with $\lambda f_i + \mu g_i$ and noting that $d(\lambda f_i + \mu g_i) = \mu d(g_i) \neq 0$. If p_i is singular for both f_i and g_i, we can blow up the variety X along the subvariety V_i for which p_i is the generic point, to obtain an interdefinable specialisation p_i' and a smooth variety X', and note that by birationality;

$$dim(H^0(X', \Omega^{p+1} \times L)) = dim(H^0(X, \Omega^{p+1} \otimes L))$$

We only need to do this a finite number of times by the effective version of Jouanalou's theorem. If p_i is nonsingular for both f_i and g_i but f_i and g_i are tangent, we can reduce to the case where f_i and g_i are transverse, by considering higher tangent spaces $T^i(X)$ and checking that;

$$dim(H^0(T^i(X), \Omega^{p+1} \otimes L)) = dim(H^0(X, \Omega^{p+1} \otimes L))$$

We can assume that the co-degree 2 specialisations p_i are defined by (f_i, g_i) so that the f_i are distinct. Otherwise we obtain an infinite number of co-degree 2 specialisations on a single divisor f_i. Then we can reverse the roles of g_i and f_i, and if the g_i are not distinct, obtain an infinite number of codegree 2 specialisations on the intersection $f_i \cap g_i$, contradicting algebraicity. To remove the obstruction in the divisor group Σ, we can assume that the generators $\{f_i, g_i\}$ are irreducible and intersect transversely at p_i, so that $I(p_i) =< f_i, g_i >$. Then, by an effective calculation on the obstruction, we can find a subgroup $\Sigma_1 \leq \Sigma$ of arbitrary rank N, which is unobstructed for the map Ψ. Let $\{b_1, \ldots, b_N$ be a basis, and assume that $\{f_i : 1 \leq i \leq r\}$ appear as components of the basis. Then $N \leq r$, as $< b_1, \ldots, b_N > \subset < f_1, \ldots, f_r >$. We can then define the twist using the product of these irreducible components.

In order to prove that that the map Φ above is well defined, we still need to show that that the representation of the derived algebra above $[\mathfrak{g}, \mathfrak{g}]$ is reasonably well behaved, as for example we could have that $f_\alpha | g_\alpha + g_\alpha^2$, even though the reduced intersection f_α and g_α is transverse at p, as g_α may contain a component such that $1 + g_\alpha$ vanishes on $f_\alpha = 0$. (Can assume that g_α is irreducible?)

.... (need the fact that V_r^0 does).... We claim that;

$$f(\overline{x}, \overline{y}) = 0 \cup \{\neg V_r(\overline{x}, \overline{y})\} \cup V_{r-1}^0(\overline{x}, \overline{y}) \ (***)$$

is consistent. If not, then by compactness, we can find a variety W_r of codimension r, such that;

$$f(\overline{x}, \overline{y}) = 0 \cup \neg W_r(\overline{x}, \overline{y}) \rightarrow \neg V_{r-1}^0$$

But W_r defines a Zariski closed subset of V^0_{r-1}, so by the geometric axioms for DCF, we can find a tuple $\overline{e}'\overline{b}'$ satisfying $f(\overline{x},\overline{y})=0$, with $V^0_{r-1}(\overline{e}'\overline{b}')$ and $\neg W_r(\overline{e}',\overline{b}')$. Let $\overline{e}''\overline{b}''$ realise the type $(***)$. Then by construction, we have that $tp(\overline{c}\overline{b})$ specialises to $tp(\overline{e}''\overline{b}'')$ specialises to $tp(\overline{e}\overline{b})$. In particularly, considering differential polynomials $r(\overline{y})$ over k_0, we have that $r(\overline{y})$ vanishes on $\overline{b}$ iff $r(\overline{y})$ vanishes on $\overline{b}''$, so that $tp(\overline{b}/k_0) = tp(\overline{b}''/k_0)$. Using $\aleph_0$-homogeneity of K, we can replace $(\overline{e}''\overline{b}'')$ with a tuple $(\overline{e}'\overline{b})$ in the realisation of $(***)$. It follows that (A) holds.

The following result is due to Hrushovski, [5], which relies on Jouanalou's Theorem, see [6];

If k_0 is an algebraically closed field of constants, and $p(\overline{x}) = tp(\overline{d}/k_0)$, then either;

(a). There is $c \in C \setminus k_0$ with $c \in acl(k_0(\overline{d}))$ (non-orthogonality) or

(b). $p(\overline{x})$ has only finitely many co-degree 1 specialisations. (G)

Assume that (a) does not hold for $tp(\overline{c}\overline{b}/k_0)$. We have (a) does not hold for the proper specialisations, $tp(\overline{e}\overline{b}/k_0)$, as, otherwise, we would have that;

$$c \underset{k_0}{\not\perp} \overline{e}\overline{b}$$

but by the definition of k_0;

$$c \underset{k_0}{\perp} \overline{b}$$

so that, by transitivity;

$$c \underset{k_0(\overline{b})}{\not\perp} \overline{e}$$

which is a contradiction, as by (C), $\overline{e} \in acl(k_0(\overline{b}))$.

As, by (A), we can factor any proper specialisation into a chain of codegree 1 specialisations, and use the bound in (B), it follows, by (G)

applied repeatedly, that $tp(\bar{c}\bar{b}/k_0)$ has only finitely many proper specialisations of the form $tp(\bar{e}\bar{b}/k_0)$, where $\bar{e} \in B'$. It follows by (C), that $acl(k_0(\bar{b})) \cap B'$ is finite and B' is ω-categorical. By a result due to Zilber, see [10], this implies that B' is locally modular, which contradicts (E). Hence, the claim (D) is shown.

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Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: t.depiro@curvalinea.net