REAL PLANE ALGEBRAIC CURVES

TRISTRAM DE PIRO

Definition 0.1. *We work in the structure* $(\mathcal{R},+,\cdot,0,1,<)$, *consisting of the reals considered as a real closed ordered field, with the associated language* $\mathcal{L}_{RCF}$. *Then, it is well known that;*

$Th((\mathcal{R},+,\cdot,0,1,<))$

has quantifier elimination and is O-minimal, in the sense that every definable subset U *of* $\mathcal{R}$ *in the language* $\mathcal{L}_{RCF}$ *is a finite union of points and intervals, possibly infinite. We let;*

$(\mathcal{R}^*,+,\cdot,0,1,<)=\prod_D(\mathcal{R},+,\cdot,0,1,<)$

be the real closed field defined as the ultraproduct of $(\mathcal{R},+,\cdot,0,1,<)$, *with respect to a non-principal ultrafilter* D *on* $\mathcal{N}$. *By Los's Theorem, we have that;*

$(\mathcal{R},+,\cdot,0,1,<)\prec(\mathcal{R}^*,+,\cdot,0,1,<)$

via the diagonal embedding;

$r\mapsto(r)_n$

We define an infinitesimal $\epsilon\in\mathcal{R}^*$ *by the requirement that* $0<|\epsilon|<r$, *for all* $r\in\mathcal{R}_{>0}$. *By Los's Theorem again, and considering the sequence* $(\frac{1}{n})_{n\in\mathcal{N}}$

$\mathcal{R}^*$ *contains infinitesimals. We define a finite number* $r'\in\mathcal{R}^*$ *by the requirement that* $-r_1<r'<r_2$, *for some* $\{r_1,r_2\}\subset\mathcal{R}$. *For every finite number* r', *it is easy to show that there exists a unique* $r\in\mathcal{R}$, *with* $r'-r$ *an infinitesimal. We say that* $r'\simeq r$. *We define the standard part mapping* $st:\mathcal{R}^*\cup\{+\infty,-\infty\}$ *by;*

$st(r')=r$, *where* $r'\simeq r$, *if* r' *is finite*

$st(r')=\infty$ if r' is infinite and $r'>0$

$st(r')=-\infty$ if r' is infinite and $r'<0$

By an indefinite polynomial, $f\in\mathcal{R}[x,y]$, we mean a polynomial with the property that there exist $\{\overline{a},\overline{b}\}\subset\mathcal{R}^2$ with $f(\overline{a})<0<f(\overline{b})$. For $f\in\mathcal{R}[x,y]$, by $V(f)$, we mean the solution set in $\mathcal{R}^2$ of f. By an algebraic set in $\mathcal{R}^n$, we mean a set definable by a finite set of polynomials $\{f_1,\ldots,f_s\}\subset\mathcal{R}[x_1,\ldots,x_n]$. By an irreducible algebraic set, we mean an algebraic set V with the property that if $V=V_1\cup V_2$, V_1 and V_2 algebraic, then $V=V_1$ or $V=V_2$. The following facts can be found in [2];

(i). If f is of odd degree, then f is indefinite.

(ii). For f irreducible, f is indefinite iff $V(f)$ is infinite.

(iii). If f is irreducible and indefinite, then $V(f)$ is irreducible.

By a real plane algebraic curve we mean $V(f)$, for some $f\in\mathcal{R}[x,y]$ irreducible and indefinite. By a singular point on $V(f)$ for f indefinite, we mean a point p such that $f(p)=0$, $\frac{\partial f}{\partial x}(p)=0$, $\frac{\partial f}{\partial y}(p)=0$. If this condition is not satisfied with $f(p)=0$, we say that p is a nonsingular point.

We define;

$$P^2(\mathcal{R})=\{[a:b:c]:(a,b,c)\in\mathcal{R}^3\setminus(0,0,0)/\sim\}$$

where $(a,b,c)\sim(d,e,f)$ if there exists $\lambda\in\mathcal{R}_{\neq 0}$ such that $\lambda(a,b,c)=(d,e,f)$. Similarly, we define $P^2(\mathcal{R}^*)$.

We define a simple algebraic set $V\subset P^2(\mathcal{R})$ by the requirement that V is the solution set in $P^2(\mathcal{R})$ of a homogeneous *polynomial $f\in\mathcal{R}[X,Y,Z]$. For $g\in\mathcal{R}[x,y]$, we define the homogenisation of g by $Z^m g(\frac{X}{Z},\frac{Y}{Z})$, where $m=deg(g)$. We can consider $\mathcal{R}^2$ as a subset of $P^2(\mathcal{R})$ via the map $(a,b)\mapsto[a:b:1]$. Similarly, we have maps $(a,b)\mapsto[a:1:b]$ and $(a,b)\mapsto[1:a:b]$, which identifies $\mathcal{R}^2$ with the open sets U_1,U_2,U_3 of $P^2(\mathcal{R})$ defined by $X\neq 0$, $Y\neq 0$, $Z\neq 0$ respectively. By the definition of $P^2(\mathcal{R})$, we have that $U_1\cup U_2\cup U_3=P^2(\mathcal{R})$. If $f\in\mathcal{R}[x,y]$ is irreducible and indefinite, then we define $V(f^h)$, the* *projective closure of the real plane algebraic curve $V(f)$, to be the solution set in $P^2(\mathcal{R})$ of the homogenisation f^h of f. We have that, for $(a,b)\in\mathcal{R}^2$;*

$f(a,b)=0$ *iff* $f^h([a:b:1])$

so that, via the identification, $V(f^h)\cap U_1 = V(f)$. We let $l_\infty = (P^2(\mathcal{R})\setminus\mathcal{R}^2)$ be the line at infinity defined by $Z=0$.

Lemma 0.2. *If $f\in\mathcal{R}[x,y]$ is irreducible and indefinite, then f^h is irreducible and indefinite in the sense of homogeneous polynomials and $V(f^h)$ is irreducible in the sense of algebraic sets defined by homogeneous polynomials. $V(f^h)\cap l_\infty$ is finite.*

Proof. We have that, if f is irreducible and indefinite, then f^h is irreducible and indefinite, in the sense of homogeneous polynomials. The second claim can be found in [2]. For the first claim, if f is irreducible and f^h is reducible, then $f^h = u^h v^h$, where u^h and v^h are homogeneous polynomials not equal to constants. Letting u and v be the dehomogenisations defined by $u(x,y) = u^h([x:y:1])$, $v(x,y) = v^h([x:y:1])$, we then have that $f = uv$, so that u or v are constant. This can only happen if u^h or v^h is a function of Z only, in which case $\overline{V}(f)$ contains the line at infinity l_∞, defined by $Z=0$. Then, as u^h and v^h are not constants $Z^k|f^h(X,Y,Z)$, for some $k\geq 1$. Clearly this cannot happen as by the definition above of the homogenisation, f^h contains summands of the form X^iY^j, where $i+j=d$, $d=deg(f)$. It follows that $V(f^h)$ is irreducible in the sense of algebraic sets defined by homogeneous polynomials. This is a simple adaptation of the corresponding fact above for $f\in\mathcal{R}[x,y]$. In particular, using Bezout's theorem and the fact that l_∞ is irreducible as an algebraic set, $V(f^h)\cap l_\infty$ is finite.

□

Lemma 0.3. *There exists a standard part mapping $st' : P^2(\mathcal{R}^*)\rightarrow P^2(\mathcal{R})$ with the property that if V is a simple algebraic set and $V(a')$ for some $a'\in P^2(\mathcal{R}^*)$, then $V(st'(a'))$.*

Proof. For a representative (a',b',c') of $d'\in P^2(\mathcal{R}^*)$, assuming that $0\leq|a'|<|b'|<|c'|$, with $|c'|>0$, we have that $0<|\frac{a'}{c'}|<|\frac{b'}{c'}|<1$, so that $\frac{a'}{c'}$ and $\frac{b'}{c'}$ are finite. We define $st'(a',b',c') = [st(\frac{a'}{c'}) : st(\frac{b'}{c'}) : 1]$, where st is the map defined above. We have to check this is well defined. Let $\lambda \in \mathcal{R}^*_{\neq 0}$ have the property that $a'' = \lambda a'$, $b'' = \lambda b'$, $c'' = \lambda c'$, then still $0 \leq |a''| < |b''| < |c''|$, and;

$$st'(a'', b'', c'') = [st(\frac{a''}{c''}) : st(\frac{b''}{c''}) : 1] = [st(\frac{a'}{c'}) : st(\frac{b'}{c'}) : 1] = st'(a', b', c')$$

We repeat the same construction, for the six cases, when $\{|a'|, |b'|, |c'|\}$ are distinct, ordering the tuples in ascending modulus, and dividing by the highest term. We are left with the remaining cases when either 2 of the terms in $\{|a'|, |b'|, |c'|\}$ coincide or all three coincide and are not equal to 0. In the first of these cases, we divide by *either* of the highest modulus terms, and take the standard part as above. Without loss of generality, assuming that $0 \leq |a'| < |b'| = |c'|$, we need to check that;

$$[st(\frac{a'}{c'}) : st(\frac{b'}{c'}) : 1] = [st(\frac{a'}{b'}) : 1 : st(\frac{c'}{b'})]$$

This is obviously true if $b' = c'$, and if $b' = -c'$, then;

$$[st(\frac{a'}{c'}), st(\frac{b'}{c'}) : 1]$$

$$= [st(\frac{a'}{c'}) : -1 : 1]$$

$$= [-st(\frac{a'}{b'}) : -1 : 1]$$

$$= [st(\frac{a'}{b'}) : 1 : -1]$$

$$= [st(\frac{a'}{b'}) : 1 : st(\frac{c'}{b'})]$$

as required. In the second case, we divide by *any* of the three nonzero terms, and again take the standard part as above. The details of checking this is well defined are left to the reader.

Let $f(X, Y, Z) = \sum_{i+j+k=d} a_{ijk} X^i Y^j Z^k$ be a homogeneous polynomial of degree d in $\mathcal{R}[X, Y, Z]$, and let $[a' : b' : c'] \in \mathcal{R}^{*3}$ such that $f([a' : b' : c'])$. Without loss of generality, we may suppose that $0 \leq |a'| < |b'| < |c'|$, in particularly $c' \neq 0$. WE have that;

$$\sum_{i+j+k=d} a_{ijk} a'^i b'^j c'^k = 0$$

$$\text{iff } \sum_{i+j+k=d} a_{ijk} a'^i b'^j c'^{k-d} = 0$$

so that, as $k-d=-i-j$;

$\sum_{i+j\leq d} a_{ij}(\frac{a'}{c'})^i(\frac{b'}{c'})^j = 0$

As $\frac{a'}{c'}$ and $\frac{b'}{c'}$ are finite, we can take the standard part, and use the fact that $st : \mathcal{R}^*_{fin} \rightarrow \mathcal{R}$ is a homomorphism, to obtain that;

$st(\sum_{i+j\leq d} a_{ij}(\frac{a'}{c'})^i(\frac{b'}{c'})^j)$

$= \sum_{i+j\leq d} a_{ij}(st(\frac{a'}{c'}))^i(st(\frac{b'}{c'}))^j = 0$, $(*)$

By the definition of st' above, we have that $st'([a':b':c']) = [st(\frac{a'}{c'}) : st(\frac{b'}{c'}) : 1]$, and clearly, using $(*)$;

$f([st(\frac{a'}{c'}) : st(\frac{b'}{c'}) : 1]) = \sum_{i+j\leq d} a_{ij}(st(\frac{a'}{c'}))^i(st(\frac{b'}{c'}))^j = 0$

so that $f(st'([a':b':c'])) = 0$ as required.

□

Remarks 0.4. *We call $c' \in U_3 \cap P^2(\mathcal{R}^*$ finite in U_1, if it corresponds to a finite element $(a',b') \in \mathcal{R}^{2*}$ via the map $(a,b) \mapsto [a:b:1]$. Similarly, we call an element finite in U_1 or U_2. We have that $st'|_{U_i,fin} = st$, via this identification, for $1 \leq i \leq 3$, as, taking $i=3$, without loss of generality, $st'([a':b':1]) = [st(a') : st(b') : 1]$, as is easily checked. Note that if $c \in U_i \cap P^2(\mathcal{R})$, then $st'^{-1}(c) \subset U_{i,fin}$. If this were not the case, then either we could find $c' \notin U_i$ with $st'(c') = c$, in which case $c' \in X_i = 0$, so that, by the above, $st'(c') \in X_i = 0$ and $c \in X_i = 0$ which is not the case, or we could find $c' \in U_i$ infinite with $st(c') = c$. Then, taking $i=3$, c' is of the form $[a':b':1]$, with a' or b' infinite in $\mathcal{R}^{*2}$, so that if $|a| < |b|$;*

$st'([a':b':1]) = [st(\frac{a'}{b'}) : 1 : st(\frac{1}{b})]$

$= [st(\frac{a'}{b'}) : 1 : 0]$

as $\frac{1}{b}$ is infinitesimal. So that $c \in Z = 0$, which is again a contradiction. Every $c' \in P^2(\mathcal{R}^*)$ is finite in U_i for some $1 \leq i \leq 3$. This follows from the fact that we can without loss of generality suppose $c' = [d':e':f']$, with $|f'| \geq max(|d'|,|e'|) > 0$, so that $c' = [\frac{d'}{f'}, \frac{e'}{f'} : 1] \in U_3$, corresponding to the finite tuple $(\frac{d'}{f'}, \frac{e'}{f'})$.

It follows that we can switch to the affine open sets U_i, when counting points in an infinitesimal neighborhood of $c \in U_i$.

We call a point $[1 : b : 0] \in l_\infty \cap V(f^h)$ nonsingular if it satisfies the definition of non singularity for $V(f^h(1, y, z)$ on U_1 in coordinates (y, z), and a point $[b : 1 : 0]$ nonsingular if it satisfies the definition of non singularity for $V(f^h(x, 1, z)$ on U_2 in coordinates (x, z'). This is a good definition, for $[a : b : 0]$, with $a \neq 0, b \neq 0$, $f^h(1, \frac{b}{a}, 0) = 0$ iff $f^h(\frac{a}{b}, 1, 0) = 0$ and as $y = \frac{Y}{X}$, $z = \frac{Z}{X}$, $x = \frac{X}{Y}$, $z' = \frac{Z}{Y}$, with the change in coordinates given by $x = \frac{1}{y}$, $z' = \frac{z}{y}$, $y \neq 0$, by the chain rule;

$$\frac{\partial f^{h,U_1}}{\partial y} = -\frac{\partial f^{h,U_2}}{\partial x}\frac{1}{y^2} - \frac{\partial f^{h,U_2}}{\partial z'}\frac{z}{y^2}$$

$$\frac{\partial f^{h,U_1}}{\partial z} = \frac{\partial f^{h,U_2}}{\partial z'}\frac{1}{y}$$

so that;

$$\frac{\partial f^{h,U_1}}{\partial y}|_{(1,\frac{b}{a},0)} = \frac{\partial f^{h,U_1}}{\partial z}_{(1,\frac{b}{a},0)} = 0$$

iff

$$\frac{\partial f^{h,U_2}}{\partial x}|_{(\frac{a}{b},1,0)} = \frac{\partial f^{h,U_1}}{\partial z'}_{(\frac{a}{b},1,0)} = 0$$

It follows from Lemma 0.5, applied to the open sets U_1 and U_2, and the dehomgenisations $f^h(1, y, z)$, $f^h(x, 1, z')$, that there can only be finitely many singular points lying on the line l_∞.

Lemma 0.5. *If f is irreducible and indefinite, there can only be finitely many singular points on $V(f)$.*

Proof. Suppose that $\frac{\partial f}{\partial x}$ vanishes on $V(f)$, then $V(f) \subset V(\frac{\partial f}{\partial x})$, and by the real study's lemma, see [2], $f|\frac{\partial f}{\partial x}$. As $\frac{\partial f}{\partial x}$ has lower degree, this is impossible unless $\frac{\partial f}{\partial x} = 0$, in which case $f = q(y)$ for some $q \in \mathcal{R}[y]$. As f is irreducible and indefinite, q is of the form $y - r = 0$, for some $r \in \mathcal{R}$. Then it is easy to see that $\frac{\partial f}{\partial y} = 1$, on $V(f)$, so that every point is nonsingular. We may therefore assume that $\frac{\partial f}{\partial x}$ does not vanish on $V(f)$ and by the same reasoning that $\frac{\partial f}{\partial y}$ does not vanish on $V(f)$. We can assume that $V(f)$ consists of the real points of a complex irreducible curve C. This follows, as if f is reducible over $\mathcal{C}$, we can consider $V(f) \subset V_{\mathcal{C}}(f) \cap V_{\mathcal{C}}(\overline{f})$, so that $V_{\mathcal{C}}(f)$ and $V_{\mathcal{C}}(\overline{f})$ must share a common component defined over $\mathcal{R}$, as $V(f)$ is infinite. This would contradict the fact that f is irreducible. As we can assume that $\frac{\partial f}{\partial x}$ is not a constant, we have that $V(f)$ and $V(\frac{\partial f}{\partial x})$ are the real points of two 1-dimensional, algebraic curves C and D, without a common component. By Bezout's theorem, the number of complex intersections and therefore real intersections is bounded by $deg(C)deg(D)$. It follows that $V(f)\cap V(\frac{\partial f}{\partial x})$ is finite and therefore $V(f)\cap V(\frac{\partial f}{\partial x})\cap V(\frac{\partial f}{\partial y})$ is finite, as required.

□

Definition 0.6. *For f irreducible and indefinite, with $f(a,b)=0$, for some $(a,b)\in\mathcal{R}^2$ and (a,b) a nonsingular point, we define the tangent line l to be the line defined by the equation;*

$$(x-a)\frac{\partial f}{\partial x}|_{(a,b)}+(y-b)\frac{\partial f}{\partial y}|_{(a,b)}=0\ (*)$$

We identify $P^2(\mathcal{R})$ with the parameter space for lines l of the form $aX+bY+cZ=0$, and assume that f does not define a line. We consider the algebraic set $F\subset P^2(\mathcal{R}^*)\times P^2(\mathcal{R}^*)$ defined by;

$$F(d,e)\ \mathit{iff}\ e\in l_d\cap C$$

where $C=V(f^h)$. By Bezout's theorem, F has the property that for $d\in P^2(\mathcal{R})$, the fibre $F(d)$ is finite, possibly empty, with cardinality bounded by $deg(f)$. This property is inherited by $\mathcal{R}^*$, as $\mathcal{R}\prec\mathcal{R}^*$ and the property can be formulated in the language $\mathcal{L}_{RCF}$;

$$\forall(y_1,y_2,y_3)_{y_1\neq0\vee y_2\neq0}[\exists(x_{11},x_{21},\ldots x_{1n},x_{2n})\bigwedge_{i=1}^{n}(f(x_{1i},x_{2i})=0\wedge y_1x_{1i}+y_2x_{2i}+y_3=0)\vee\exists(x_{11},x_{21},\ldots x_{1n-1},x_{2n-1})\exists a\exists b_{a\neq0\vee b\neq0}\bigwedge_{i=1}^{n-1}(f(x_{1i},x_{2i})=0\wedge f^h(a,b,0)\wedge y_1a+y_2b=0)]\wedge\exists(a_1,b_1,\ldots a_n,b_n)_{a_i\neq0\vee b_i\neq0}\bigwedge_{i=1}^{n}f^h(a_i,b_i,0)\ldots.$$

We define;

$$Mult(F;(d,e))=Max\{CardF(d')\cap\mathcal{V}_e:d'\in\mathcal{V}_d\}$$

where $\mathcal{V}_d=st'^{-1}(d)$, $\mathcal{V}_e=st'^{-1}(e)$.

For a nonsingular point $[a:b:0]$ of $V(f^h)$ on the line l_∞, we say that a line l passing through $[a,b:0]$ is algebraically tangent to $V(f^h)$ at $[a:b:0]$ if it satisfies the above definition in either of the affine open sets U_1 or U_2. Again, similar to the above, this is a good definition. If

l is given in coordinates (x,z'), by $\alpha+\beta x+\gamma z'=0$, then l is given in coordinates (y,z) by $\alpha+\beta(\frac{1}{y})+\gamma(\frac{z}{y})=0$, $y\neq 0$ iff $\beta+\alpha y+\gamma z=0$. The algebraic definition of tangency in (x,z') is given by;

$$\gamma\frac{\partial f^{h,U_2}}{\partial x}+\beta\frac{\partial f^{h,U_2}}{\partial z'}=0\ (*)$$

and then;

$$\gamma\frac{\partial f^{h,U_1}}{\partial y}+\alpha\frac{\partial f^{h,U_1}}{\partial z}$$

$$=\gamma(-\frac{\partial f^{h,U_2}}{\partial x}\frac{1}{y^2}-\frac{\partial f^{h,U_2}}{\partial z'}\frac{z}{y^2})+\alpha(\frac{\partial f^{h,U_2}}{\partial z'}\frac{1}{y})$$

For a point $[a:b:0]=[1:\frac{b}{a}:0]$, we have that $\frac{1}{y^2}|_{(\frac{b}{a},0)}=\frac{a^2}{b^2}$, $\frac{1}{y}|_{(\frac{b}{a},0)}=\frac{a}{b}$, $\frac{z}{y^2}|_{(\frac{b}{a},0)}=0$, so that;

$$\gamma\frac{\partial f^{h,U_1}}{\partial y}+\alpha\frac{\partial f^{h,U_1}}{\partial z}$$

$$=\gamma(-\frac{\partial f^{h,U_2}}{\partial x}\frac{a^2}{b^2})+\alpha(\frac{\partial f^{h,U_2}}{\partial z'}\frac{a}{b})$$

$$=-\frac{\gamma a^2}{b^2}\frac{\partial f^{h,U_2}}{\partial x}+\frac{\alpha a}{b}\frac{\partial f^{h,U_2}}{\partial z'}$$

We have that in coordinates (y,z), as $[a:b:0]=[1:\frac{b}{a}:0]$ lies on the line defined by $\beta+\alpha y+\gamma z=0$, that $\beta+\alpha(\frac{b}{a})=0$, so that;

$$\frac{b}{a}=-\frac{\beta}{\alpha}$$

$$\frac{a}{b}=-\frac{\alpha}{\beta}$$

$$(-\frac{\gamma a^2}{b^2},\frac{\alpha a}{b})$$

$$\sim(-\frac{\gamma a}{b},\alpha)$$

$$=(\frac{\gamma\alpha}{\beta},\alpha)$$

$$\sim(\gamma,\beta)$$

so that;

$$-\frac{\gamma a^2}{b^2}\frac{\partial f^{h,U_2}}{\partial x}+\frac{\alpha a}{b}\frac{\partial f^{h,U_2}}{\partial z'}=0$$

iff

$\gamma\frac{\partial f^{h,U_2}}{\partial x}+\beta\frac{\partial f^{h,U_2}}{\partial z'}=0$

which is the case, by $(*)$.

If $l\neq l_\infty$, *we call* l *an asymptote.*

Lemma 0.7. *For a nonsingular point* $p=(a:b:c)$ *with* $f^h(a,b,c)=0$, f *irreducible and indefinite, not a line,* l *is algebraically tangent to* $V(f)$ *at* $(a:b:c)$ *iff* $Mult(F;d,[a;b:1])\geq 2$, *where* d *is the projective coordinate defining* l *and* F *is the cover above.*

Proof. By the above remark, we can work in an affine set U_i, with coordinates (x,y), the point p having coordinates $(0,0)$ and $f(x,y)=0$. By the fact that p is nonsingular, we can without loss of generality, and using a linear transformation, assume that $\frac{\partial f}{\partial y}\neq 0$. Applying the implicit function theorem and using continuity, we can find g analytic defined on an open neighbourhood $B(0,r)$ of 0 in $\mathcal{R}$ such that, for $x\in B(0,r)$, $f(x,g(x))=0$ and $V(f)\cap(B(0,r)\times B(0,s))=graph(g|_{B(0,r)})$, for sufficiently small $r\in\mathcal{R}_{>0}$, $r<s$, $s\in\mathcal{R}_{>0}$. If $Mult(F;d,[a;b:1])\geq 2$, we can vary the line l through $(0,0)$ to l_ϵ and find 2 points intersection $\{(x_1',y_1'),(x_2',y_2')\}$ between l_ϵ and C. such that $\{x_1',y_1',x_2',y_2'\}\subset\mathcal{V}_0$. As $\{x_1',y_1',x_2',y_2'\}$ are infinitesimals, (x_1',y_1') and (x_2',y_2') must lie in the open neighborhood $(B(0,r)\times B(0,s))$. It follows, that without loss of generality $x_1'<x_2'$, $y_1'=g(x_1')$, $y_2'=g(x_2')$, and the equation of the line through (x_1',y_1') and (x_2',y_2') is given by;

$\frac{y-y_1'}{x-x_1'}=\frac{y_2'-y_1'}{x_2'-x_1'}$

$=\frac{g(x_2')-g(x_1')}{x_2'-x_1'}$ $(*)$

By the transfer of the mean value theorem to $\mathcal{R}^*$, and working in the model complete theory $T_{RCF,an}$, with restricted analytic functions, we have that;

$\frac{g(x_2')-g(x_1')}{x_2'-x_1'}=g'(x_3')$

where $x_3'\in(x_1',x_2')$, so that $x_3'\in\mathcal{V}_0$ is an infinitesimal. As g' is continuous $st(g'(x_3'))=g'(0)$, so that the equation of l is given by applying

st to $(*)$ and we obtain that l is given by;

$\frac{y}{x}=g'(0)$

As $f(x,g(x))=0$, by the chain rule, we have that $\frac{\partial f}{\partial x}|_{(0,0)}+\frac{\partial f}{\partial y}|_{(0,0)}g'(0)=0$, so that;

$\frac{y}{x}=g'(0)=-\frac{\frac{\partial f}{\partial x}|_{(0,0)}}{\frac{\partial f}{\partial y}|_{(0,0)}}$

and;

$x\frac{\partial f}{\partial x}|_{(0,0)}+y\frac{\partial f}{\partial y}|_{(0,0)}=0$

defines the line l, so that l is algebraically tangent. For the converse claim, using the implicit function g again, we have that $y=g'(0)x$ is algebraically tangent to $V(f)$. Choose $a'\in\mathcal{V}_0$, with $a'\neq 0$, this is possible as $\mathcal{R}^*$ contains infinitesimal elements. By S-continuity of g, we have that $g(a')\in\mathcal{V}_0$. The points $(0,0)$ and $(a',g(a'))$ are distinct and lie on the curve defined by $V(g)$. Let l' be the line defined by $y=\frac{g(a')}{a'}x$. Then $(0,0)$ and $(a',g(a'))$ lie on the intersection $l'\cap V(g)$. We have that l' is an infinitesimal variation of l as $\frac{g(a')}{a'}\in\mathcal{V}_{g'(0)}$ due to the fact that;

$g'(0)=lim_{h\rightarrow 0}\frac{g(h)}{h}$

as required.

□

Definition 0.8. *Given* $(a'b')\in\mathcal{R}^{*2}$, *we define;*

$dim(a'b'/\mathcal{R})=t.deg(\mathcal{R}(a',b')/\mathcal{R})$

and call (a',b') *generic over* $\mathcal{R}$ *if* $dim(a',b'/\mathcal{R})=2$.

Given $c'\in P^2(\mathcal{R}^*)$, *let* c' *be finite with respect to one of the open sets* U_i, $1\leq i\leq 3$, *so that it corresponds to some* $(a',b')\in(a'b')\in\mathcal{R}^{*2}$. *Then we define* $dim(c'/\mathcal{R})=dim(a',b'/\mathcal{R})$, *and call* c' *generic if* $dim(c'/\mathcal{R})=2$

This is a good definition, as suppose that c' is finite with respect to the open sets U_2 and U_3, then;

$c' = [a' : b' : 1] = [e' : 1 : f']$

with $\{a', b', e', f'\}$ finite. By the definition of $P^2(\mathcal{R}^*)$, we have that $a' = \frac{e'}{f'}$, $b' = \frac{1}{f'}$ and;

$tdeg(a'b'/\mathcal{R}) = tdeg(\frac{e'}{f'}, \frac{1}{f'})$

The tuples (e', f') and $(\frac{e'}{f'}, \frac{1}{f'}$ are interdefinable in the sense that $\mathcal{R}(e', f') = \mathcal{R}(\frac{e'}{f'}, \frac{1}{f'})$, so that;

$tdeg(\mathcal{R}(\frac{e'}{f'}, \frac{1}{f'})/mathcalR) = tdeg(\mathcal{R}(e', f')/\mathcal{R})$

and, therefore;

$dim(a', b'/\mathcal{R}) = dim(e', f'/\mathcal{R})$

Given an irreducible indefinite $f \in \mathcal{R}[x, y]$, we say that $c' \in V(f^h)$ is generic over $\mathcal{R}$ if $dim(c'/\mathcal{R}) = 1$.

Definition 0.9. *Given f irreducible and indefinite, we call a point $(a, b) \in V(f)$ isolated, if there exists an open ball $B(\overline{0}, \epsilon)$, $\epsilon \in \mathcal{R}_{>0}$, such that $V(f) \cap B(\overline{0}, \epsilon) = (a, b)$. Similarly, we say that a point $a \in V(f^h) \cap l_\infty$ is isolated if in some affine open U_i, $1 \leq i \leq 3$, a is isolated with respected to the dehomogenised variety $V(f_i^h)$, $1 \leq i \leq 3$.*

Lemma 0.10. *The above definition is good. There exist finitely many isolated points on $V(f^h)$, and they all belong to $Sing(V(f^h))$.*

Proof. The first claim is easily checked by continuity, and the coordinate change given in Remark 0.4. If a is nonsingular, it cannot be isolated due to the implicit function theorem and the fact the implicit function g is continuous. As we have seen, $Sing(V(f^h))$ is finite, so there are finitely many isolated points.

□

Lemma 0.11. *Given an irreducible indefinite $f \in \mathcal{R}[x, y]$, if $c \in V(f^h) \cap P^2(\mathcal{R})$ is not isolated, then we can find $c' \in V(f^h) \cap \mathcal{V}_c$ with c' generic over $\mathcal{R}$. In particular $c' \notin Sing(V(f^h))$.*

Proof. The final claim follows from the first, we have that $Sing(V(f^h))$ is a finite set algebraic set defined over $\mathcal{R}$ by homogeneous polynomials.. This follows from Lemma 0.5, applied to $V(f^h) \cap U_3$ and Remark 0.4, for $1 \leq i \leq 3$, to cover the case of $V(f^h) \cap l_\infty$. The same considerations apply when we work over $\mathcal{C}$. If $c' \in Sing(V(f^h)$, we can work in affine coordinates, to find $(a'b')$ generic over $\mathcal{R}$, witnessing a finite algebraic in $\mathcal{C}^2$ set defined over $\mathcal{R}$. By classical results, it follows that $\{a, b\} \subset \mathcal{C}$ as $\mathcal{C}$ is algebraically closed. However, without loss of generality, a is transcendental over $\mathcal{R}$, contradicting the fact that $\mathcal{C}$ is algebraic over $\mathcal{R}$. For the first claim, we can work in affine coordinates (x, y), with an irreducible indefinite $g \in \mathcal{R}[x, y]$ and suppose that $c = (0, 0)$. If g defines a line l in $\mathcal{R}^2$, the result is obvious. We can choose $a' \in \mathcal{V}_0$, with a' transcendental. This follows as $\mathcal{R}^* \cap \mathcal{C} = \mathcal{R}$, otherwise, we can find $r_1, r_2 \subset \mathcal{R}$, $r_2 \neq 0$, with $b = r_1 + ir_2 \in \mathcal{R}^*$, so $\frac{b-r_1}{r_2} = i \in \mathcal{R}^*$, contradicting the fact that $\mathcal{R}^* \models \neg\exists z(z^2 + 1 = 0)$, so it is sufficient to choose $a' \in \mathcal{R}^* \setminus \mathcal{R}$. Then if the line is defined by $y = mx$, $m \in \mathcal{R}$, $dim(a', ma'/\mathcal{R}) = 1$ and (a', ma') lies on l. If the line is defined by $x = 0$, then $(0, a')$ with $a' \in \mathcal{V}_0$ generic works. We can therefore suppose that $V(g) \cap (B(0, r) \times B(0, r))$, some $r \in \mathcal{R}_{>0}$ defines a finite cover of $pr(V(g) \cap (B(0, r) \times B(0, r)))$, where $B(0, r)$ defines a closed ball. We can assume that $V(g) \cap (B(0, r) \times B(0, r))$ is infinite, otherwise c would be isolated. Moreover, as the cover is finite, we can assume that $pr^{-1}(0) \cap V(g) \cap (B(0, r) \times B(0, r)) = 0$, $(*)$. By O-minimality, we can suppose that $pr(g \cap (B(0, r) \times B(0, r)))$ consists of a finite union of points and intervals including 0. We can assume the projection is infinite, otherwise we would obtain an infinite fibre. It follows that, without loss of generality, we can suppose that there exists an open interval $I \subset pr(g \cap (B(0, r) \times B(0, r)))$ such that $0 \in cl(B(0, r))$, in the real topology. By the cell decomposition, we can assume that there exists a continuous semialgebraic function h defined on I_1, with $(a, h(a)) \in V(g)$ for $a \in I$. Choose $a' \in \mathcal{V}_0$ generic, with $a' \in I$, then $(a', h(a')) \in V(g)$. We have that $dim(a', h(a')) = 1$. By specialisation, we have that $(st(a'), st(h(a'))) \in V(g) \cap (B(0, r) \times B(0, r))$, as $B(0, r)$ is closed, so that $(0, st(h(a'))) \in V(g) \cap (B(0, r) \times B(0, r))$. By $(*)$, it follows that $st(h(a')) = 0$. Therefore, we have found $(a', b') \in V(g) \cap \mathcal{V}_{(0,0)}$ with (a', b') generic, as required.

□

Lemma 0.12. *Let f be irreducible and indefinite, $deg(f) = d$, $V(f^h)$ not a line, f^h defined by $\overline{a} \in P^{\frac{d(d+3)}{2}}(\mathcal{R})$. If p is a nonsingular point of $V(f^h)$, we have that a line l is algebraically tangent to $V(f^h)$ at p iff* *there exists a variation* $f^{h,\overline{a}'}$ *of* f^h *in the space of homogeneous polynomials of degree* d*, such that* $Card((V(f^{h,\overline{a}'}))\cap l\cap \mathcal{V}_p)\geq 2$*. If* p *is a singular point of* $V(f^h)$*, we have that for* any *line* l *passing through* p *there exists a variation* $f^{h,\overline{a}'}$ *of* f^h *in the space of homogeneous polynomials of degree* d*, such that* $Card((V(f^{h,\overline{a}'}))\cap l\cap \mathcal{V}_p)\geq 2$*.*

Proof. We can work in affine coordinates with $p=(0,0)$, and a representation g of f^h with $deg(g)=d$. Suppose p is nonsingular and let $y=h'(0)x$ be algebraically tangent to $V(g)$ at $(0,0)$, with $h(x)$ an implicit representation of g. Then, as g is not a line, $p(x)=g(x,h'(0)x)$ is not identically zero, and $deg(p)\leq deg(g)=d$. We have that;

$$p(0)=g(0,h'(0)0)=g(0,0)=0$$

$$p'(0)=\tfrac{\partial g}{\partial x}|_{(0,0)}+\tfrac{\partial g}{\partial y}|_{(0,0)}h'(0)$$

Moreover;

$g(x,h(x))=0$, so that $\tfrac{\partial g}{\partial x}|_{(0,0)}+\tfrac{\partial g}{\partial y}|_{(0,0)}h'(0)=0$

and $x^2|p(x)$. We can therefore write $p(x)$ in the form;

$p(x)=x^n r(x)$, with $n\geq 2$, $r(0)\neq 0$.

Consider the variation of g, given by $g(x,y)+\epsilon x^{n-1}$, then $V(g+\epsilon x^{n-1})$ intersects the line l when;

$$g(x,h'(0)x)+\epsilon x^{n-1}=0$$

iff $x^n r(x)+\epsilon x^{n-1}=0$

iff $x^{n-1}(xr(x)+\epsilon)=0$

iff $x=0$ or $xr(x)=-\epsilon$

We have that;

$$(xr(x))(0)=0.r(0)=0$$

$$(xr(x))'(0)=r(0)+0.r'(0)=r(0)\neq 0.$$

so by the inverse function theorem, we can find $\delta \in \mathcal{V}_0$, $\delta \neq 0$, with $\delta r(\delta) = -\epsilon$. It follows that $V(g + \epsilon x^{n-1})$ intersects the line l in two distinct points $(0,0)$ and $(\delta, g'(0)\delta)$, as required. If p is singular, we can repeat the calculation for any line l, given by $y = mx$, using the fact that for $p(x) = g(x, mx)$;

$p(0) = g(0,0) = 0$

$p'(0) = \frac{\partial g}{\partial x}|_{(0,0)} + \frac{\partial g}{\partial y}|_{(0,0)} m = 0$

If p is nonsingular and l is not algebraically tangent, then if l is defined by $y = mx$, we have that, letting $p(x) = g(x, mx)$;

$p(0) = g(0,0) = 0$

$p'(0) = \frac{\partial g}{\partial x}|_{(0,0)} + \frac{\partial g}{\partial y}|_{(0,0)} m \neq 0$

so we can write $p(x) = xr(x)$, with $r(0) \neq 0$. It follows that for a variation $g_{\overline{\epsilon}}(x,y)$, with $\epsilon \in \mathcal{V}_{\overline{0}}$, the solutions to $V(g_{\overline{\epsilon}}) \cap l$ are given by;

$g_{\overline{\epsilon}}(x, mx) = 0$

$p_{\overline{\delta}}(x) = 0$

$xr(x) = \sum_{i=0}^{d} \delta_j x^i$

where $\delta_i \in \mathcal{V}_0$, for $0 \leq i \leq d$. We can rewrite this equation as;

$x(r(x) - \sum_{i=1}^{d} \delta_i x^{i-1}) = \delta_0 \ (*)$

and observe that $(r(x) - \sum_{i=1}^{d} \delta_i x^{i-1})|_0 = r(0) - \delta_1 \neq 0$, as $r(0) \in \mathcal{R}_{\neq 0}$. We have that;

$[x(r(x) - \sum_{i=1}^{d} \delta_i x^{i-1})]'|_0 = (r(x) - \sum_{i=1}^{d} \delta_i x^{i-1})|_0 + x(r(x) - \sum_{i=1}^{d} \delta_i x^{i-1})'|_0$

$= r(0) - \delta_1 + 0$

$= r(0) - \delta_1 \neq 0$

By the inverse function theorem, $(*)$ has a unique solution $x_0 \in \mathcal{V}_0$. More specifically, we can find $(\epsilon, \delta, \theta) \in \mathcal{R}^3_{>0}$, such that the statement;

$\forall y_1 \ldots \forall y_d (\bigwedge_{i=1}^{d} |y_i| < \epsilon \rightarrow \forall w(|w| < \delta) \exists! z(|z| < \theta)$

$(z(r(z) - \sum_{i=1}^{d} y_i z^{i-1}) = w))$ $(**)$

in $\mathcal{L}_{RCF}$, defined over $\mathcal{R}$, is true in $\mathcal{R}$ by the inverse function theorem. The existence of θ follows from the fact that we can extend the domain of the inverse function to the interval $(-\theta, \theta)$, provided that $(xs(x))'(y) \neq 0$, for $y \in (-\theta, \theta)$. This follows by continuity and the intermediate value theorem;

$|(xs(x))'(y) - (xs(x))'(0)| \leq |y| max_{|v| \leq \theta} |(xs(x))''(v)|$

the fact that $(xs(x))'(0)$ is bounded away from zero, $|(xs(v))''|$ is uniformly bounded on the interval $(-\theta, \theta)$, for the given variation;

$xs(x) = x(r(x) - \sum_{i=1}^{d} y_i x^{i-1})$

For the existence, we can guarantee solutions in the interval $(-\theta, \theta)$, provided we choose $\delta \in \mathcal{R}_{>0}$ last with;

$|\delta| < |\theta| max_{|v| \leq \theta} |(xs(x))'(v))|$

for the variation s.

The statement transfers to $\mathcal{R}^*$, and the hypotheses are automatically satisfied for infinitesimals, to find a solution $|z| < \theta$. This solution must lie in $\mathcal{V}_0$, as applying the standard part mapping we have that $st(z)r(st(z)) = 0$, so that $st(z) = 0$ or $r(st(z)) = 0$. By specialisation again, we have that $|st(z)| \leq \theta$, and we can choose θ small enough so that $r|_{[-\theta,\theta]} \neq 0$, by continuity and the fact that $r(0) \neq 0$. It follows that $st(z) = 0$ as required.

□

Remarks 0.13. *If $s = ord_x g(x, mx)$, for the representation g of f in affine coordinates with $p = (0, 0)$, l defined by $y = mx$, then $s \leq d$, and there exists a variation $f^{h,\overline{a}'}$ of f^h in the space of homogeneous polynomials of degree d, such that $Card((V(f^{h,\overline{a}'})) \cap l \cap \mathcal{V}_p) = s$.*

....

Lemma 0.14. *Let f_1 and f_2 be irreducible and indefinite, with $V(f_1^h) \neq V(f_2^h)$, defined by $\{\overline{a}_1, \overline{a}_2\}$, not both lines. Then if $p \in V(f_1^h) \cap V(f_2^h)$*

is nonsingular for both curves, we have that $V(f_1^h)$ *and* $V(f_2^h)$ *are algebraically tangent, in the sense that they share a common algebraically tangent line, iff there exists variations* $f^h_{1,\overline{a}'_1}$ *and* $f^h_{2,\overline{a}'_2}$, *with* $\overline{a}'_1\overline{a}'_2 \in \mathcal{V}_{\overline{a}_1\overline{a}_2}$ *such that* $Card(f^h_{1,\overline{a}'_1} \cap f^h_{2,\overline{a}'_2}) \cap \mathcal{V}_p \geq 2$. *If* p *is singular for either curve, then there exist variations having the same property as above.*

Proof. For the first part, we can assume both varieties are not lines, or reduce to Lemma 0.12. By Lemma 0.12, taking l to be the tangent line to $V(f_1^h)$, there exists a variation $f_1^{h,\overline{a}'_1}$ of f^h in the space of homogeneous polynomials of degree d, such that $Card((V(f^{h,\overline{a}'})) \cap l \cap \mathcal{V}_p) \geq 2$. By the explicit form of the variation, there are two points of intersection $(0,0)$ and (a', ma'), where $a' \in \mathcal{V}_0$ solves $xr_1(x) = -\epsilon$, for $\epsilon \in \mathcal{V}_0$ arbitrary, $r_1(x) = \frac{g_1(x,mx)}{x^{n_1}}$, $n_1 \geq 2$, $r_1(0) \neq 0$, g_1 representing f_1 in affine coordinates around $(0,0)$. Without loss of generality, $|xr_2(x)| < |xr_1(x)|$, and by the same argument, we can solve $xr_2(x) = -\delta$, $|\delta| < |\epsilon|$. Then we construct the variation g_2 representing f_2, of degree e, given by $f_2(x,y) + \delta x^{n_2-1}$, where $x^{n_2}r_2(x) = g_2(x,mx)$, $r_2(0) \neq 0$. The second part is the same, picking l to be the tangent line to the other curve, if p is nonsingular, and any line if p is nonsingular for both curves.

□

Lemma 0.15. *Let* f_1 *and* f_2 *be irreducible and indefinite, with* $V(f_1^h) \neq V(f_2^h)$, *defined by* $\{\overline{a}_1, \overline{a}_2\}$, *not both lines. Then if* $p \in V(f_1^h) \cap V(f_2^h)$ *is nonsingular for both curves, we have that* $V(f_1^h)$ *and* $V(f_2^h)$ *are algebraically tangent, in the sense that they share a common algebraically tangent line, iff there exists a variation* $f^h_{2,\overline{a}'_2}$, *with* $\overline{a}'_2 \in \mathcal{V}_{\overline{a}_2}$ *such that* $Card(f^h_{1,\overline{a}_1} \cap f^h_{2,\overline{a}'_2}) \cap \mathcal{V}_p \geq 2$. *If* p *is singular for* $V(f_1^h)$, *and not isolated, then there exists a variation having the same property as above.*

Proof. If p is nonsingular for f_1, we work in affine coordinates so that $p = (0,0)$ and $\{g_1, g_2\}$ represent $\{f_1^h, f_2^h\}$. Without loss of generality, let f_1 be represented by the implicit function h_1, so that $g_1(x, h_1(x)) = 0$, and let $p_1(x) = g_2(x, h_1(x))$. As before, we have that $p_1 \neq 0$ and $p_1(x) = x^r r_1(x)$, with $r_1(0) \neq 0$, $r \geq 2$. As $deg(f_2) \geq 2$, we have that $g_2(x,y) + \epsilon x$ defined a variation of f_2^h which intersects $V(f_1^h)$ locally when;

$$x^r r_1(x) + \epsilon x = 0$$

$$\text{iff } x(x^{r-1} r_1(x) + \epsilon) = 0$$

iff $x = 0$ or $x^{r-1}r_1(x) = -\epsilon$

Taking $\epsilon < 0$, the second condition holds if $xs_1(x) = \delta$, $(*)$ where $\delta^r = \epsilon$, and $s_1^r = r_1$, s_1 is analytic and $s_1(0) \neq 0$. δ is an infinitesimal as $st(\delta^r) = st(\delta)^r = st(\epsilon) = 0$, so that $st(\delta) = 0$. Applying the inverse funcion theorem, we can find $\delta_1 \neq 0$, solving $(*)$, with $\delta_1 \in \mathcal{V}_0$. Then $(0,0)$ and $\delta_1, h(\delta_1)$ witness $Card(f^h_{1,\overline{a}_1} \cap f^h_{2,\overline{a}'_2}) \cap \mathcal{V}_p \geq 2$.
.... □

Lemma 0.16. *Let f_1 be irreducible and indefinite, f_2 irreducible and indefinite, $f_1 \neq f_2$, such that $deg(f_1) = d_1$, $deg(f_2) = d_2$, $d_1 \geq 2$, $d_2 \geq 2$, with $p \in V(f_1^h) \cap V(f_2^h)$, defined over $\{\overline{a}_1, \overline{a}_2\}$, such that p is nonsingular for $V(f_1^h)$ and $V(f_2^h)$. Then working in affine coordinates such that $p = (0,0)$, with g_1 a representation for f_1 and h_1 an implicit function, g_2 a representation for f_2, we have that $ord_x f_2(x, h_1(x)) = s \leq d_1 d_2$, and for $d_1 d_2 < \frac{d_2(d_2+3)}{2}$, there exists a variation $f^h_{2,\overline{a}'_2}$ of $f^h_{2,\overline{a}_2}$ in the space of homogeneous polynomials of degree d_2, such that $Card(V(f^h_{1,\overline{a}_1}) \cap V(f^h_{2,\overline{a}'_2}) \cap \mathcal{V}_p) = s$. In particular, using the terminology of* [1], *we have that;*

$I_{italian}(V(f_1^h), V(f_2^h), p) = Mult(F, p)$

where $Mult(F,p)$ is the real *multiplicity, for the cover $F \subset V(f_1^h) \cap P^{\frac{d_2(d_2+3)}{2}}$ obtained by intersecting $V(f_1^h$ with a homogeneous polynomial in $\mathcal{R}^*[X,Y,Z]$ of degree d_2, the intersection taken in $P^2(\mathcal{R}^*) \times P^{\frac{d_2(d_2+3)}{2}}(\mathcal{R}^*)$.*

Proof. By the theory developed in [1], we have that $I_{italian}(V(f_1^h), V(f_2^h), p) = ord_x f_2(x, h_1(x)) = s$. We can intersect the curve defined by f_1, with the variation $f_2(x,y)+\epsilon$, where $\epsilon \in \mathcal{V}_0$ is generic, which has solutions when;

$f_2(x, h_1(x)) + \epsilon = 0$

iff

$x^s u(x) + \epsilon = 0$

$u(0) \neq 0$

Taking the s distinct roots $\{\epsilon_1, \ldots \epsilon_s\} \subset \mathcal{V}_0$ of $-\epsilon$, we can solve $u(x)v(x) = \epsilon_i$, for $1 \leq i \leq s$, where $v(x)^s = u(x)$, $v(x)$ $v(0) \leq 0$, v

analytic, using either the inverse function theorem or the method of etale extensions. This gives s distinct $\{(x_1, g_1(x_1)) \ldots, (x_s, g_1(x_s))\} \subset (f_1(x, y) = 0) \cap (f_2(x, y) + \epsilon = 0) \cap (\mathcal{V}_{(0,0)}$. These intersections have multiplicity 1, which gives the result.

By Bezout's theorem $s \leq d_1 d_2$. As p is non singular for both curves, we can find implicit functions $\{g_1, g_2\}$ which are analytic and can be expanded locally as;

$g_1(x) = \sum_{i=1}^{\infty} b_{i1} x^i$

$g_2(x) = \sum_{i=1}^{\infty} b_{i2} x^i$

By uniqueness of these representations, we have that $g_1 \neq g_2$. Let $s_1 = \mu_i(b_{i1} \neq b_{i2})$. Similarly, we claim that $s_1 = s$. In order to see this, first note that $g_1(x) - g_2(x) = x^{s_1} w(x)$, with $w(0) \neq 0$. Then we have that that $f_2(x, y - \epsilon)$, $\epsilon \in \mathcal{V}_0$ defines a variation of f_2, with implicit function defined by $g_2 + \epsilon$. We have that the solutions to $f_1(x, y) \cap f_2(x, y - \epsilon) \cap \mathcal{V}_{(0,0)}$ are given locally by solutions;

$g_1 - (g_2 + \epsilon) = 0$

iff

$x^{s_1} w(x) = \epsilon$

Using the same trick as above, we can find s_1 distinct intersections in $f_1(x, y) \cap f_2(x, y - \epsilon) \cap \mathcal{V}_{(0,0)}$ and these intersections are transverse, proving that $s_1 = s$.

By repeatedly differentiating the implicit function relation $f_2(x, g_2(x)) = 0$, and evaluating at $(0, 0)$, we obtain a system of algebraic equations $\{H_1, \ldots, H_r\}$ relating the coefficients $(a_{ij}) \subset A^{\frac{(d+1)(d+2)}{2}}(\mathcal{R})$ for polynomials of degree d, with the derivatives $\{g_2^{(1)}(0), \ldots g_2^{(r)}(0)\}$, defining a real algebraic variety $V_r \subset A^r \times A^{\frac{(d+1)(d+2)}{2}}(\mathcal{R})$. The fibres of $pr_1 : V_r \rightarrow A^r$ are hyperplanes defined by r linear equations over $\mathcal{R}$. Using dimension considerations and projectivising the equations, taking $r = s - 1$, $s - 1 \leq de - 1 < de < \frac{d(d+3)}{2}$, pr_1 is onto. The image of $pr_2 : V_r \rightarrow A^{\frac{(d+1)(d+2)}{2}}$ is an open set $U \subset A^{\frac{(d+1)(d+2)}{2}}$ and, by the inverse function theorem, $pr_2|_{pr_2^{-1}(U) \cap V_r}$ is bijective, hence V_r

is irreducible. Let q be the dimension of a generic fibre of pr_2 so that $q = \frac{d(d+3)}{2} - (s-1) > 0$. By linear algebra, the condition that $dim(pr_2(\overline{g})) \geq q+1$ is definable by a set of $C_{s-1}^{\frac{(d+1)(d+2)}{2}}$ equations in $A^{s-1}(\mathcal{R})$, so for sufficiently large d, we expect this to define the empty set by dimension considerations. It follows that the cover pr_2 is equidimensional, and using the cell decomposition, we can vary the $\{g_2^{(1)}(0), \ldots g_2^{(s)}(0)\}$ to $\{g_2^{(1)}(0), \ldots g_2^{(s)}(0)+\epsilon\}$, with $\epsilon \in \mathcal{V}_0$, and obtain a variation $f_{2,\epsilon}$, such that the common solution to $f_1 \cap f_{2,\epsilon} \cap \mathcal{V}_0$ is given by;

$x^s w_1(x) + \epsilon x^{s-1} = 0$

where w_1 is an infinitesimal variation of w, in particular $w_1(0) \neq 0$.

and;

$x^s w_1(x) + \epsilon x^{s-1} = 0$

iff $x^{s-1}(xw_1(x) + \epsilon_1) = 0$

iff $x = 0$ or $xw_1(x) = -\epsilon_1$

and, following the proof above, choosing $\epsilon_1 \in \mathcal{V}_0$ generic, we find $(x_1, h_1(x)) \in V(f^h_{1,\overline{a}_1}) \cap V(f^h_{2,\overline{a}'_2}) \cap \mathcal{V}_p$, with $(x_1, h_1(x))$ distinct from $(0,0)$.

We now consider $V(f^h_{1,\overline{a}'_1})$ as defining a curve over $\mathcal{R}^*$ and work in the ultraproduct;

$(\mathcal{R}^{*,2}, +, \cdot, 0, 1, <) = \prod_D (\mathcal{R}^*, +, \cdot, 0, 1, <)$

so that $\mathcal{R}^* \prec \mathcal{R}^{*,2}$.

We now have $ord_x g_{2,\epsilon_1}(x, h_1(x)) = s-1$, and we consider the variation;

$g_{2,\epsilon_1\epsilon_2},(x,y) = g_2(x,y) + \epsilon_1 x^{s-1} + \epsilon_2 x^{s-2} = 0$

with $\epsilon_2 \in \mathcal{V}_0$ generic, and ϵ_2 infinitesimal with respect to $\mathcal{R}^*$, in particular $0 < |\epsilon_2| < |\epsilon_1|$. The point $(x_1, h_1(x_1))$ moves to a new point

$(x_{12}, h_1(x_{12}))$ under the variation defined by (ϵ_1, ϵ_2), as the intersection was transverse, by the non-existence of coincident mobile points, and clearly $x_{12} \in \mathcal{V}_{x_1}$ is distinct from the new point of intersection, obtained via the method above, $(x_3, g_1(x_3))$, $x_1 \in \mathcal{V}_0$, obtaining at least 3 intersections.

Continuing this process, we can find a tuple $0 < |\epsilon_s| < \ldots |\epsilon_2| < |\epsilon_1|$, such that for this variation, $Card(V(f_1^h) \cap V(f_{2,\bar{\epsilon}}^h \cap \mathcal{V}_p) \geq s$, computed in $\mathcal{R}^{*,s}$. We can use saturation of $\mathcal{R}^*$ to find the variation in $\mathcal{R}^*$. We then, have as;

$$I_{italian}(V(f_1^h), V(f_{2,\bar{\epsilon}}^h, p) = s$$

that, there are exactly s intersections and;

$$Mult(F, p) = s$$

as required.

$\square$

References

[1] A Theory of Branches for Algebraic Curves, Tristram de Piro, available at http://www.curvaline.net, (2009)

[2] Real Plane Algebraic Curves, Master's Thesis, Pedro Gonzales Garcia, The Arctic University of Norway,(2021)

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: t.depiro@curvalinea.net