THE RELATIVITY OF ROTATING FRAMES

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Abstract.

Lemma 0.1. *If ρ is a mass density in $\mathcal{R}^3$, then the force on a test particle of mass m at position $\overline{r}$ is given by;*

$\overline{F}(\overline{r}) = -Gm\int_{\mathcal{R}^3}\frac{\rho(\overline{r}')(\overline{r}-\overline{r}')^{\wedge}}{|\overline{r}-\overline{r}'|^2}d\overline{r}'$

In particular, we have that;

$\bigtriangledown\centerdot\overline{F} = -4\pi Gm\rho\ (*)$

where G is the gravitational constant.

If we rotate $\mathcal{R}^3$ about the axis $x = y = 0$, with an angular velocity of ω, then a particle of mass m with coordinates (x, y, z) experiences a force $\overline{F} = -m\omega^2 r\overline{r}_*^{\wedge}$, where $r = \sqrt{x^2+y^2}$, and $\overline{r}_* = (x, y, 0)$.

We have that, in this case, that;

$\bigtriangledown\centerdot\overline{F} = -2m\omega^2\ (**)$

so that $\rho = \frac{\omega^2}{2\pi G}$

Proof. The first claim is a consequence of Newton's universal law of gravitation, the second claim follows from the corresponding result for the force between charges, see [1]. The third claim is a standard result in the theory of circular orbits, see [**?**], the fourth claim follows from the expression of $\bigtriangledown$ in cylindrical coordinates;

$\bigtriangledown\centerdot\overline{F} = \frac{1}{r}\frac{\partial r\overline{F}_r}{\partial r} + \frac{1}{r}\frac{\partial\overline{F}_\theta}{\partial\theta} + \frac{\partial\overline{F}_z}{\partial z}$

$= \frac{1}{r}\frac{\partial r(-m\omega^2 r)}{\partial r}$

$= \frac{1}{r}(-2m\omega^2 r)$

$= -2m\omega^2$

The final claim is just obtained by rearranging $(*), (**)$.

$\square$

Lemma 0.2. *Let $r_0 \in \mathcal{R}_{>0}$ and let $\rho(x,y,z)$ be the smooth mass density, defined by;*

$\rho(x,y,z) = \frac{\omega^2}{2\pi G}$, *for* $0 \leq (x^2+y^2)^{\frac{1}{2}} \leq r_0$

$\rho(x,y,z) = g(r)$

where $r = (x^2+y^2)^{\frac{1}{2}}$, $r \geq r_0$, $g(r_0) = \frac{\omega^2}{2\pi G}$, $|g(r)| \leq \frac{D}{r^2}$, *for* $D \in \mathcal{R}_{>0}$, $r \geq r_0$.

Let $\overline{F}(\overline{r})$ *be the corresponding force for a mass* m*, then for* $0 \leq (x^2+y^2)^{\frac{1}{2}}$;

$\overline{F}(x,y,z) = -m\omega^2|\overline{r}_*|\overline{r}_*^{\wedge}$

Proof. We first claim that $\overline{F}(\overline{r})$ is well defined. We have that;

$|\overline{F}(\overline{r})| = |-Gm\int_{\mathcal{R}^3} \frac{\rho(\overline{r}')(\overline{r}-\overline{r}')^{\wedge}}{|\overline{r}-\overline{r}'|^2} d\overline{r}'|$

$= |-Gm\int_{\mathcal{R}^3} \frac{\rho(\overline{r}-\overline{r}'')(\overline{r}'')^{\wedge}}{|\overline{r}''|^2} d\overline{r}''|$, $(\overline{r}'' = \overline{r}-\overline{r}')$

$\leq Gm\int_{\mathcal{R}^3} \frac{|\rho(r_1-x,r_2-y,r_3-z)|}{|(x,y,z)|^2} dxdydz$

$= Gm\int_{\mathcal{R}^3} \frac{|\rho(r_1+x,r_2+y,r_3+z)|}{|(x,y,z)|^2} dxdydz$

$= Gm\int_{\mathcal{R}^3} \frac{|\rho(r_1+x,r_2+y,z)|}{|(x,y,z)|^2} dxdydz$

$\leq Gm\int_{D(0,r_0+|\overline{r}|)\times\mathcal{R}} \frac{C}{|(x,y,z)|^2} dxdydz + Gm\int_{(D(0,r_0+|\overline{r}|)\times\mathcal{R})^c} \frac{|g(\overline{r}+(x,y,z))|}{|(x,y,z)|^2} dxdydz$

$= Gm\int_{D(0,r_0+|\overline{r}|)\times\mathcal{R}} \frac{Cr}{(r^2+z^2)} drd\theta dz + Gm\int_{(D(0,r_0+|\overline{r}|)\times\mathcal{R})^c} \frac{|g(\overline{r}+(x,y,z))|}{|(x,y,z)|^2} dxdydz$

$\leq GmC\int_0^{r_0+|\overline{r}|}\int_0^{2\pi} tan^{-1}(\frac{z}{r})|_{-\infty}^{\infty} drd\theta + Gm\int_{(D(0,r_0+|\overline{r}|)\times\mathcal{R})^c} \frac{D}{|(x,y,z)|^2|\overline{r}+(x,y,z)|^2} dxdydz$

$\leq 2GmC\pi^2(r_0+|\overline{r}|)+Gm\int_{(D(0,r_0+|\overline{r}|)\times\mathcal{R})^c}\frac{D}{|(x,y,z)|^2|\overline{r}+(x,y,z)|^2}dxdydz$

$\leq 2GmC\pi^2(r_0+|\overline{r}|)+Gm\int_{B(\overline{0},r_1)^c}\frac{2D}{|(x,y,z)|^4}dxdydz+Evol(B(\overline{0},r_1)\setminus(D(0,r_0+|\overline{r}|)\times\mathcal{R}))$

$\leq 2GmC\pi^2(r_0+|\overline{r}|)+4D\pi^2Gm\int_{r_1}^{\infty}\frac{drr^2}{r^4}+EF$

$\leq 2GmC\pi^2(r_0+|\overline{r}|)+4D\pi^2Gm\int_{r_1}^{\infty}\frac{dr}{r^2}+EF$

$\leq 2GmC\pi^2(r_0+|\overline{r}|)+\frac{4D\pi^2Gm}{r_1}+EF<\infty$

where $C=max(\frac{\omega^2}{2\pi G},|g(\overline{r}+(x,y,z))||_{D(0,r_0+|\overline{r})\times\mathcal{R}})$, $r_1\in\mathcal{R}_{>0}$ is sufficiently large, $\{E,F\}\subset\mathcal{R}_{>0}$.

By the symmetry of the distribution ρ, it is clear that $F(\overline{r})=h(\overline{r}_*)\overline{r}_*^{\wedge}$, where h is a smooth function on $\mathcal{R}^2$, and by the result of Lemma 0.1, we have that $\bigtriangledown\cdot\overline{F}=-2m\omega^2$ in the region $D(\overline{0},r_0)\times\mathcal{R}$. Using cylindrical coordinates for $\bigtriangledown$ again, we have that;

$\frac{1}{r}\frac{\partial r\overline{F}_r}{\partial r}=\frac{1}{r}(rh(r))'$

$=\frac{h(r)}{r}+h'(r)=-2m\omega^2$

so that, as h is smooth, $h(r)=-m\omega^2r$, and $F(\overline{r})=-m\omega^2|\overline{r}_*|\overline{r}_*^{\wedge}$.

□

Definition 0.3. *We let g_{ω,ω_0} be the metric on $\mathcal{R}^4$, considered as a Lorentzian manifold, satisfying Einstein's field equations, see* [2]*;*

$R_{ij,\omega,\omega_0}=8\pi G(T_{ij,\omega,\omega_0}-\frac{1}{2}T_{\omega,\omega_0}g_{ij,\omega,\omega_0})$

for $0\leq i,j\leq 3$, where R_{ij,ω,ω_0} are the components of the Ricci curvature, G is the gravitational constant, T_{ij,ω,ω_0} are the components of the Maxwell stress tensor for the mass density distribution ρ, defined in Lemma 0.2, rotating at angular velocity ω_0 and T_{ω,ω_0} is the contraction.

Lemma 0.4.

Proof. In the region $D(\overline{0},r_0)\times\mathcal{R}^2$, we have that;

$T_{00}(t,r,\theta,z)=\rho(t,r,\theta,z)=\frac{\omega^2}{2\pi G}$

$T_{0j}=T_{j0}=(\rho(t,r,\theta,z)\overline{v}(t,r,\theta,z))_j=\frac{\omega^2}{2\pi G}(0,r\omega_0,0)_j$

for $1\leq j\leq 3$.

so that;

$T_{01}=T_{10}=0$

$T_{02}=T_{20}=\frac{r\omega^{2}\omega_{0}}{2\pi G}$

$T_{03}=T_{30}=0$

$T_{ij}=T_{ji}=(T_{0i}(t,r,\theta,z)\overline{v}(t,r,\theta,z))_{j}$

for $1\leq i\leq j\leq 3$

so that;

$T_{11}=0(0,r\omega_{0},0)_{1}$

$=0$

$T_{12}=T_{21}=0(0,r\omega_{0},0)_{2}$

$=0$

$T_{13}=T_{31}=0(0,r\omega_{0},0)_{3}$

$=0$

$T_{22}=\frac{r\omega^{2}\omega_{0}}{2\pi G}(0,r\omega_{0},0)_{2}$

$=\frac{r^{2}\omega^{2}\omega_{0}^{2}}{2\pi G}$

$T_{23}=T_{32}=\frac{r\omega^{2}\omega_{0}}{2\pi G}(0,r\omega_{0},0)_{3}$

$=0$

$T_{33}=0(0,r\omega_{0},0)_{3}$

$=0$

$T=T_{00}+T_{11}+T_{22}+T_{33}=\frac{\omega^{2}}{2\pi G}+\frac{r^{2}\omega^{2}\omega_{0}^{2}}{2\pi G}$

$= \frac{\omega^2(1+r^2\omega_0^2)}{2\pi G}$

□

Lemma 0.5.

Proof. We have that;

$R_{ij} = \sum_{a=0}^{3} \frac{\partial \Gamma^a_{ij}}{\partial x^a} - \sum_{a=0}^{3} \frac{\partial \Gamma^a_{ai}}{\partial x^j} + \sum_{a,b=0}^{3} \Gamma^a_{ab}\Gamma^b_{ij} - \Gamma^a_{ib}\Gamma^b_{aj}$

where;

$\Gamma^c_{ab} = \frac{1}{2}\sum_{d=0}^{3}(\frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d})g^{cd}$

By the cylindrical symmetry, we can assume that the metric g is of the form;

$g = \alpha(r)dt^2 + \beta(r)dr^2 + \gamma(r)d\theta^2 + \delta(r)dz^2$

so that;

$\Gamma^0_{01} = \Gamma^0_{10} = \frac{\alpha'(r)}{2\alpha(r)}$

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\gamma'(r)}{2\gamma(r)}$

$\Gamma^3_{13} = \Gamma^3_{31} = \frac{\delta'(r)}{2\delta(r)}$

$\Gamma^1_{11} = \frac{\beta'(r)}{2\beta(r)}$

$\Gamma^1_{00} = -\frac{\alpha'(r)}{2\beta(r)}$

$\Gamma^1_{22} = -\frac{\gamma'(r)}{2\beta(r)}$

$\Gamma^1_{33} = -\frac{\delta'(r)}{2\beta(r)}$

$\Gamma^c_{ab} = 0$ otherwise

We have that $\frac{\partial \Gamma^a_{ij}}{\partial x^a} = 0$, unless $a = 1$, and if $i \neq j$, $\Gamma^1_{ij} = 0$. Similarly, we have that;

$\sum_{a=0}^{3} \frac{\partial \Gamma^a_{ai}}{\partial x^j} = 0$

unless $j=1$, and then, if $i\neq j$, we have $i\in\{0,2,3\}$. We then have that $\Gamma^a_{ai}=0$, for $0\leq a\leq 3$. We have that, for $0\leq a\leq 4$, $\Gamma^a_{ab}\neq 0$ iff $b=1$, and then $\Gamma^b_{ij}=0$, if $i\neq j$. We have that, by the enumeration above;

$\{\Gamma^0_{01}\Gamma^1_{0j},\Gamma^1_{00}\Gamma^0_{1j}\}$, $j\neq 0$

$\{\Gamma^0_{10}\Gamma^0_{0j},\Gamma^2_{12}\Gamma^2_{2j},\Gamma^3_{13}\Gamma^3_{3j},\Gamma^1_{11}\Gamma^1_{1j}\}$, $j\neq 1$

$\{\Gamma^2_{21}\Gamma^1_{2j},\Gamma^1_{22}\Gamma^2_{1j}\}$, $j\neq 2$

$\{\Gamma^3_{31}\Gamma^1_{3j},\Gamma^1_{33}\Gamma^3_{1j}\}$, $j\neq 3$

are all equal to $\{0\}$, so that, for $0\leq a,\leq 3$, $\Gamma^a_{ib}\Gamma^b_{aj}=0$, if $i\neq j$. It follows that $R_{ij}=0$, if $i\neq j$.

When $\omega_0=0$, we are then left with the 4 equations;

$R_{00}=\frac{\partial\Gamma^1_{00}}{\partial x_1}+(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31})\Gamma^1_{00}-2\Gamma^0_{01}\Gamma^1_{00}$

$=-(\frac{\alpha'(r)}{2\beta(r)})'+(\frac{\alpha'(r)}{2\alpha(r)}+\frac{\beta'(r)}{2\beta(r)}+\frac{\gamma'(r)}{2\gamma(r)}+\frac{\delta'(r)}{2\delta(r)})(-\frac{\alpha'(r)}{2\beta(r)})-2\frac{\alpha'(r)}{2\alpha(r)}(-\frac{\alpha'(r)}{2\beta(r)})$

$=-(\frac{\alpha'(r)}{2\beta(r)})'+(-\frac{\alpha'(r)}{2\alpha(r)}+\frac{\beta'(r)}{2\beta(r)}+\frac{\gamma'(r)}{2\gamma(r)}+\frac{\delta'(r)}{2\delta(r)})(-\frac{\alpha'(r)}{2\beta(r)})$

$=4\omega^2-2\omega^2g_{00}$

$=4\omega^2-2\omega^2\alpha(r)$

$R_{11}=\frac{\partial\Gamma^1_{11}}{\partial x_1}-\frac{\partial(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31})}{\partial x_1}+(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31})\Gamma^1_{11}$

$-(\Gamma^0_{10}\Gamma^0_{01}+\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{12}\Gamma^2_{21}+\Gamma^3_{13}\Gamma^3_{31})$

$=-2\omega^2g_{11}$

$R_{22}=\frac{\partial\Gamma^1_{22}}{\partial x_1}+(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31})\Gamma^1_{22}-2\Gamma^2_{21}\Gamma^1_{22}$

$=-(\frac{\gamma'(r)}{2\beta(r)})'+(\frac{\alpha'(r)}{2\alpha(r)}+\frac{\beta'(r)}{2\beta(r)}+\frac{\gamma'(r)}{2\gamma(r)}+\frac{\delta'(r)}{2\delta(r)})(-\frac{\gamma'(r)}{2\beta(r)})-2\frac{\gamma'(r)}{2\gamma(r)}(-\frac{\gamma'(r)}{2\beta(r)})$

$=-(\frac{\gamma'(r)}{2\beta(r)})'+(\frac{\alpha'(r)}{2\alpha(r)}+\frac{\beta'(r)}{2\beta(r)}-\frac{\gamma'(r)}{2\gamma(r)}+\frac{\delta'(r)}{2\delta(r)})(-\frac{\gamma'(r)}{2\beta(r)})$

$$= -2\omega^2 g_{22}$$

$$= -2\omega^2\gamma(r)$$

$$R_{33} = \frac{\partial\Gamma^1_{33}}{\partial x_1} + (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{33} - 2\Gamma^3_{31}\Gamma^1_{33}$$

$$= -(\tfrac{\delta'(r)}{2\beta(r)})' + (\tfrac{\alpha'(r)}{2\alpha(r)} + \tfrac{\beta'(r)}{2\beta(r)} + \tfrac{\gamma'(r)}{2\gamma(r)} + \tfrac{\delta'(r)}{2\delta(r)})(-\tfrac{\delta'(r)}{2\beta(r)}) - 2\tfrac{\delta'(r)}{2\delta(r)}(-\tfrac{\delta'(r)}{2\beta(r)})$$

$$= -(\tfrac{\delta'(r)}{2\beta(r)})' + (\tfrac{\alpha'(r)}{2\alpha(r)} + \tfrac{\beta'(r)}{2\beta(r)} + \tfrac{\gamma'(r)}{2\gamma(r)} - \tfrac{\delta'(r)}{2\delta(r)})(-\tfrac{\delta'(r)}{2\beta(r)})$$

$$= -2\omega^2 g_{33}$$

$$= -2\omega^2\delta(r)$$

By the symmetry of the equations, we can assume that there exists a solution for $\gamma(r) = \delta(r)$, in which case we can reduce the last two equations to;

$$-(\tfrac{\gamma'(r)}{2\beta(r)})' + (\tfrac{\alpha'(r)}{2\alpha(r)} + \tfrac{\beta'(r)}{2\beta(r)})(-\tfrac{\gamma'(r)}{2\beta(r)})$$

$$= -\tfrac{\gamma''(r)}{2\beta(r)} + \tfrac{\gamma'(r)\beta'(r)}{2\beta^2(r)} + (\tfrac{\alpha'(r)}{2\alpha(r)} + \tfrac{\beta'(r)}{2\beta(r)})(-\tfrac{\gamma'(r)}{2\beta(r)})$$

$$= -\tfrac{\gamma''(r)}{2\beta(r)} + (\tfrac{\alpha'(r)}{2\alpha(r)} - \tfrac{\beta'(r)}{2\beta(r)})(-\tfrac{\gamma'(r)}{2\beta(r)})$$

$$= -2\omega^2\gamma(r)$$

$$-\tfrac{\delta''(r)}{2\beta(r)} + (\tfrac{\alpha'(r)}{2\alpha(r)} - \tfrac{\beta'(r)}{2\beta(r)})(-\tfrac{\delta'(r)}{2\beta(r)})$$

$$= -2\omega^2\delta(r)$$

□

References

[1]
[2]

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