SIMPLE PROOFS OF THE RIEMANN-LEBESGUE LEMMAS USING NONSTANDARD ANALYSIS

TRISTRAM DE PIRO

Abstract. We give simple proofs of the Riemann-Lebesgue Lemmas for Fourier series and Fourier transforms using nonstandard analysis.

1. Riemann Lebesgue Lemma

Definition 1.1. *We let $\eta \in {}^*\mathcal{N}$ be infinite, and $\overline{V_\eta} = {}^*[-1, 1)$, with the $*$-algebra $\mathfrak{B}_\eta$ generated by;*

$\{[\frac{i}{\eta}, \frac{i+1}{\eta}) : -\eta \leq i \leq \eta - 1\}$

and measure μ_η defined by $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, for $-\eta \leq i \leq \eta - 1$

We let $\{e_i : -\eta \leq i \leq \eta - 1\}$ be the standard orthonormal basis defined by;

$e_i(\frac{j}{\eta}) = \sqrt{\eta}\delta_{ij}$, *for* $-\eta \leq j \leq \eta - 1$

with respect to the inner product $<,>_\eta$, defined by;

$< f, g >_\eta = \int_{\overline{V}_\eta} f\overline{g} d\mu_\eta$, *for* $\{f, g\} \subset V(\overline{V_\eta})$

We recall the inversion theorem from [2], *that for $f \in V(\overline{V_\eta})$;*

$f(x) = \frac{1}{2}{}^* \sum_{m \in \mathfrak{Z}_\eta} (\mathcal{F}_\eta(f))(m) exp_\eta(\pi i m x)$

where $\mathfrak{Z}_\eta = {}^*\mathcal{Z} \cap [-\eta, \eta)$

$exp_\eta(\pi i m x) = {}^*exp(\pi i m \frac{[x\eta]}{\eta})$, *for* $-\eta \leq m \leq \eta - 1$

and $(\mathcal{F}_\eta(f))(m)$

$= \int_{\overline{V}_\eta} f(x) exp_\eta(-\pi imx) d\mu_\eta(x)$

$=< f, exp_\eta(\pi imx) >_\eta$, *for* $-\eta \leq m \leq \eta - 1$

We define the nonstandard derivative on $V(\overline{V_\eta})$ *by;*

$f'(\frac{j}{\eta}) = \eta(f(\frac{j+1}{\eta}) - f(\frac{j}{\eta}))$, *for* $-\eta \leq j \leq \eta - 2$

$f'(\frac{\eta-1}{\eta}) = \eta(f(-1) - f(\frac{\eta-1}{\eta}))$

Lemma 1.2. *If* $f \in C([-1,1])$, *with corresponding* $f_\eta \in V(\overline{V_\eta})$, *then, for infinite* $m \in \mathfrak{Z}_\eta$, *we have;*

$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

Proof. Let $\{a, b\} \subset \overline{V_\eta}$, and let $\chi_{[a,b],\eta} \in V(\overline{V_\eta})$ be defined by;

$\chi_{[a,b],\eta}(x) = 1$ if $\frac{[a\eta]}{\eta} \leq x < \frac{[b\eta]+1}{\eta}$

$\chi_{[a,b],\eta}(x) = 0$ otherwise

We claim that, for infinite m, $\mathcal{F}_\eta(\chi_{[a,b]})(m) \simeq 0$, $(*)$

We have, as in [2], that for $m \in \mathfrak{Z}_\eta$, $x \in \overline{V_\eta}$, that;

$(exp_\eta(-\pi imx))'$

$= exp_\eta(-\pi imx)\theta_\eta(m)$

where $\theta_\eta(m) = \eta(exp_\eta(-\pi i\frac{m}{\eta}) - 1)$

and;

$exp_\eta(-\pi imx) = \frac{(exp_\eta(-\pi imx))'}{\theta_\eta(m)}$

It follows that;

$(\mathcal{F}_\eta(\chi_{[a,b]}))(m)$

$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) exp_\eta(-\pi imx) d\mu_\eta(x)$

$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) \frac{(exp_\eta(-\pi imx))'}{\theta_\eta(m)} d\mu_\eta(x)$

$= \frac{1}{\theta_\eta(m)} \int_{\overline{V}_\eta} \chi_{[a,b]}(x)(exp_\eta(-\pi imx))' d\mu_\eta(x)$

$= \frac{1}{\eta\theta_\eta(m)} * \sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \chi_{[a,b]}(\frac{i}{\eta})(exp_\eta(-\pi im\frac{i}{\eta}))'$

$= \frac{1}{\eta\theta_\eta(m)} * \sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \eta(exp_\eta(-\pi im\frac{i+1}{\eta}) - (exp_\eta(-\pi im\frac{i}{\eta})))$

$= \frac{1}{\theta_\eta(m)}(exp_\eta(-\pi im(\frac{[b\eta]+2}{\eta})) - (exp_\eta(-\pi im(\frac{[a\eta]}{\eta}))))$

As in [2], Lemma 0.12, for $m \in \mathfrak{Z}_\eta$, that;

$2|m| \leq |\theta_\eta(m)|$

It follows that, for m infinite;

$|(\mathcal{F}_\eta(\chi_{[a,b]}))(m)| \leq \frac{1}{2|m|}|(exp_\eta(-\pi im(\frac{[b\eta]+2}{\eta})) - (exp_\eta(-\pi im(\frac{[a\eta]}{\eta}))))|$

$\leq \frac{2}{2|m|} = \frac{1}{|m|} \simeq 0$

Hence, $(*)$ is proved. As $f \in C([-1,1])$, by Darboux's Theorem, there exists a sequence of step functions $\{g_n : n \in \mathcal{N}\}$, such that;

$\int_{-1}^{1} |f - g_n| d\mu < \frac{1}{n}$

where μ denotes Lebesgue measure. We have that;

$g_{n,\eta} = (\sum_{k=1}^{m(n)-1} c_k \chi_{[b_{kn}, b_{(k+1)n}]})_\eta$

$= (\sum_{k=1}^{m(n)-1} c_k (\chi_{[b_{kn}, b_{(k+1)n}]})_\eta$

$= (\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}$, $(**)$

for a partition $-1 \leq b_{1n} \leq \ldots \leq b_{m(n)n} \leq 1$ and $c_k \in \mathcal{R}$, for $1 \leq k \leq m(n) - 1$. We have that, for infinite $m \in \mathfrak{Z}_\eta$, using $(*), (**)$, that;

$\mathcal{F}_\eta(g_{n,\eta})(m)$

$= \mathcal{F}_\eta((\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}))(m)$

$= \sum_{k=1}^{m(n)-1} c_k \mathcal{F}_\eta(\chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]})(m)$

$\simeq 0$

Then, for infinite m, and $n \in \mathcal{N}$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)|$

$\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)|$

$\leq \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta$, $(***)$

As $f \in C[-1,1]$, we have that, f_η is S-continuous, bounded and S-integrable, and ${}^\circ f_\eta = st^* f$, where $st : \overline{V}_\eta \to [-1,1]$ is the standard part mapping. $g_{n,\eta}$ is bounded and S-integrable, $|(f_\eta - g_{n,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

${}^\circ(\int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta)$

$= \int_{\overline{V}_\eta} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta)$

$= \int_{\overline{V}_\eta} |(st^*(f) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$\leq \int_{\overline{V}_\eta} |(st^*(f) - st^* g_n)| dL(\mu_\eta)$

$+ \int_{\overline{V}_\eta} |(st^*(g_n) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$\simeq \int_{-1}^{1} |f - g_n| d\mu < \frac{1}{n}$

Therefore, using $(***)$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

As this holds for all $n \in \mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

□

Remarks 1.3. *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f \in C([-1,1])$, *then;*

$lim_{|m|\to\infty}\mathcal{F}(f)(m) = 0$

It is sufficient to show that, given $\epsilon > 0$, *there exists* $M(\epsilon)$, *such that;*

$|\mathcal{F}(f)(m)| < \epsilon$, *for all* $m \in \mathcal{Z}$, $m \geq M(\epsilon)$. *As, for all infinite* $m \in \mathfrak{Z}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

it follows by underflow, that, for all $|m| \geq M(\epsilon)$, $m \in \mathcal{Z}$;

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

The result then follows from the fact, that, for finite $m \in \mathcal{Z}$;

$^\circ(\mathcal{F}_{f_\eta}(m)) = \mathcal{F}(f)(m)$

as $f_\eta(x)$ *and* $exp_\eta(-\pi imx)$ *are* S*-continuous and* S*-integrable on* $\overline{V_\eta}$. *The proof above follows the structure of the standard result.*

Definition 1.4. *We recall the definitions from* [2]. *We let* $\eta \in {}^*\mathcal{N}$ *be infinite and odd, we let;*

$\overline{\mathcal{R}_\eta} = {}^*\bigcup_{-\frac{(\eta-1)}{2}\leq i\leq\frac{(\eta-1)}{2}}[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$

so that $\overline{\mathcal{R}_\eta} = {}^*[-\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}})$. *We let* $\mathcal{D}_\eta$ *denote the associated* $*$*-finite algebra, generated by the intervals* $[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$, *for* $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$, *and* μ_η *the associated counting measure defined by* $\mu_\eta([\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})) = \frac{1}{\sqrt{\eta}}$. *We let* $(\overline{\mathcal{R}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ *denote the associated Loeb space.*

We let $(\mathcal{R} \cup \{+\infty, -\infty\}, \mathfrak{D}, \mu)$ *denote the extended real line, with the completion* $\mathfrak{D}$ *of the extension of the Borel field, and* μ *the extension of Lebesgue measure, with* $\mu(+\infty) = \mu(-\infty) = \infty$. *We let* $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$ *denote the extended real half line, with the completion* $\mathfrak{C}$ *of the extended Borel field, and* λ *the extension of Lebesgue measure, with* $\lambda(+\infty) = \infty$, *see* [**?**], *Chapter 6.*

Given a measurable $f_\eta : \overline{\mathcal{R}_\eta} \to {}^*\mathcal{C}$, *we define the nonstandard Fourier transform* $\mathcal{F}(f_\eta) : \overline{\mathcal{R}_\eta} \to {}^*\mathcal{C}$ *by;*

$\mathcal{F}(f_\eta)(y) = \int_{\overline{\mathcal{R}_\eta}} f_\eta(x) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

With this definition, we have, see [5] *that;*

$f_\eta(x) = \int_{\overline{\mathcal{R}}_\eta} \hat{f}_\eta(y) exp_\eta(2\pi i x y) d\mu_\eta(y)$ $(*)$

We define the nonstandard derivative on $V(\overline{\mathcal{R}}_\eta)$ *by;*

$f'(\frac{j}{\sqrt{\eta}}) = \sqrt{\eta}(f(\frac{j+1}{\sqrt{\eta}}) - f(\frac{j}{\sqrt{\eta}}))$, *for* $-\frac{(\eta-1)}{2} \leq j \leq \frac{\eta-3}{2}$

$f'(\frac{\eta-1}{2\sqrt{\eta}}) = \sqrt{\eta}(f(-\frac{(\eta-1)}{2\sqrt{\eta}}) - f(\frac{\eta-1}{\sqrt{\eta}}))$

Lemma 1.5. *If* $f \in S(\mathcal{R})$, *with corresponding* $f_\eta \in V(\overline{V_\eta})$, *then, for infinite* $y \in \overline{\mathcal{R}_\eta}$, *we have;*

$(\mathcal{F}_\eta(f_\eta))(y) \simeq 0$

Proof. Let $n \in \mathcal{N}$, we first prove that;

$|\int_{(|x| > \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi i x y) d\mu_\eta(x)| \leq \frac{E}{(n-1)}$, $(*)$

where $E \in \mathcal{R}$.

We have that;

$|\int_{(|x| \geq \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi i x y) d\mu_\eta(x)|$

$\leq \frac{1}{\sqrt{\eta}} {}^* \sum_{|k|=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f_\eta(\frac{k}{\sqrt{\eta}})|$

$= \frac{1}{\sqrt{\eta}} {}^* \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f^*|(\frac{k}{\sqrt{\eta}})$

As $f \in S(\mathcal{R})$, we have that, $|f^*|(x) \leq \frac{C}{|x|^2}$, for $|x| \geq 1$, $C \in \mathcal{R}$. It follows that;

$|f^*|(\frac{k}{\sqrt{\eta}}) \leq \frac{C}{|\frac{k}{\sqrt{\eta}}|^2} = \frac{C\eta}{|k|^2}$, for $|k| \geq [n\sqrt{\eta}]$

Then;

$\frac{1}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f^*|(\frac{k}{\sqrt{\eta}})$

$\leq \frac{1}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} \frac{C\eta}{|k|^2}$

$= \frac{2\eta C}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} \frac{1}{k^2}$

$\leq \frac{2\eta C}{\sqrt{\eta}} \int_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}} \frac{dx}{x^2}$ (by transfer)

$= 2C\sqrt{\eta}[\frac{-1}{x}]_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}}$

$= 2C\sqrt{\eta}(\frac{1}{[n\sqrt{\eta}]-1} - \frac{2}{\eta-1})$

$\leq 2C\sqrt{\eta}\frac{1}{n\sqrt{\eta}-2}$

$\leq 2C\sqrt{\eta}\frac{1}{(n-1)\sqrt{\eta}}$

$= \frac{2C}{(n-1)}$

which gives the result $(*)$, taking $E = 2C$. Let $\overline{V}_{\eta,n} = (-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{R}_\eta$.

Let $\{a, b\} \subset \overline{V_{\eta,n}}$, and let $\chi_{[a,b],\eta} \in V(\overline{V_{\eta,n}})$ be defined by;

$\chi_{[a,b],\eta}(x) = 1$ if $\frac{[a\sqrt{\eta}]}{\sqrt{\eta}} \leq x < \frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}$

$\chi_{[a,b],\eta}(x) = 0$ otherwise

We claim that, for infinite y, $\mathcal{F}_\eta(\chi_{[a,b]})(y) \simeq 0$, $(**)$

We have, as in [5], that for $y \in \overline{R}_\eta$, $x \in \overline{V_\eta, n}$, that;

$(exp_\eta(-2\pi iyx))' = \chi_\eta(y) exp_\eta(-2\pi iyx)$

where;

$\chi_\eta(y) = \sqrt{\eta}(exp_\eta(\frac{-2\pi iy}{\sqrt{\eta}}) - 1)$

and;

$exp_\eta(-\pi iyx) = \frac{(exp_\eta(-\pi iyx))'}{\chi_\eta(y)}$

It follows that;

$= \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) exp_\eta(-2\pi iyx) d\mu_\eta(x)$

$= \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) \frac{(exp_\eta(-2\pi iyx))'}{\chi_\eta(y)} d\mu_\eta(x)$

$= \frac{1}{\chi_\eta(y)} \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x)(exp_\eta(-2\pi iyx))' d\mu_\eta(x)$

$= \frac{1}{\sqrt{\eta}\chi_\eta(y)} * \sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}} \chi_{[a,b]}(\frac{i}{\sqrt{\eta}})(exp_\eta(-2\pi iy\frac{i}{\eta}))'$

$= \frac{1}{\sqrt{\eta}\chi_\eta(y)} * \sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}} \sqrt{\eta}(exp_\eta(-2\pi iy\frac{i+1}{\sqrt{\eta}}) - (exp_\eta(-2\pi iy\frac{i}{\sqrt{\eta}})))$

$= \frac{1}{\chi_\eta(y)}(exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}})) - (exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}))))$

As in [5], Lemma 0.20, for $y \in \overline{\mathcal{R}_\eta}$, we have that;

$4|y| \leq |\chi_\eta(y)|$

It follows that, for y infinite;

$|\int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) exp_\eta(-2\pi iyx) d\mu_\eta(x)|$

$\leq \frac{1}{4|y|}|(exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}})) - (exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}))))|$

$\leq \frac{2}{4|y|} = \frac{1}{2|y|} \simeq 0$

Hence, $(**)$ is proved. As $f \in C([-n,n])$, by Darboux's Theorem, there exists a sequence of step functions $\{g_r : r \in \mathcal{N}\}$, such that;

$\int_{-n}^{n} |f - g_r| d\mu < \frac{1}{r}$

where μ denotes Lebesgue measure. We have that;

$g_{r,\eta} = (\sum_{k=1}^{m(r)-1} c_k \chi_{[b_{kr}, b_{(k+1)r}]})_\eta$

$= (\sum_{k=1}^{m(r)-1} c_k (\chi_{[b_{kr}, b_{(k+1)r}]})_\eta$

$= (\sum_{k=1}^{m(r)-1} c_k \chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]}$, $(***)$

for a partition $-n \leq b_{1r} \leq \ldots \leq b_{m(r)r} \leq n$ and $c_k \in \mathcal{R}$, for $1 \leq k \leq m(r)-1$. We have that, for infinite $y \in \overline{\mathcal{R}_\eta}$, using $(**), (***)$, that;

$\int_{\overline{V}_{\eta,n}} g_{r,\eta}(x) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$= \int_{\overline{V}_{\eta,n}} ((\sum_{k=1}^{m(r)-1} c_k \chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]})) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$= \sum_{k=1}^{m(r)-1} c_k \int_{\overline{V}_{\eta,n}} (\chi_{[\frac{[b_{kr}\sqrt{\eta}]+1}{\sqrt{\eta}}, \frac{[b_{(k+1)r}\sqrt{\eta}]}{\sqrt{\eta}}]}) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$\simeq 0$

Then, for infinite y, and $r \in \mathcal{N}$;

$|\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$= |\int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta} + g_{r,\eta}) exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$\leq |\int_{\overline{V}_{\eta,n}} g_{r,\eta} exp_\eta(-2\pi i y x) d\mu_\eta(x)| + |\int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta}) exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$\leq \frac{1}{n} + \int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta$, $(****)$

As $f \in C[-n, n]$, we have that, f_η is S-continuous, bounded and S-integrable, and $^\circ f_\eta = st^* f$, where $st : \overline{V}_{\eta,n} \to [-n, n]$ is the standard part mapping. $g_{r,\eta}$ is bounded and S-integrable, $|(f_\eta - g_{r,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

$^\circ(\int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta)$

$= \int_{\overline{V}_{\eta,n}} |^\circ(f_\eta - g_{r,\eta})| dL(\mu_\eta)$

$= \int_{\overline{V}_{\eta,n}} |(st^*(f) - (^\circ g_{r,\eta}))| dL(\mu_\eta)$

$\leq \int_{\overline{V}_{\eta,n}} |(st^*(f) - st^* g_r)| dL(\mu_\eta)$

$+ \int_{\overline{V}_{\eta,n}} |(st^*(g_r) - (^\circ g_{r,\eta}))| dL(\mu_\eta)$

$\simeq \int_{-1}^{1} |f - g_r| d\mu < \frac{1}{r}$

Therefore, using $(***)$;

$| \int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi i y x) d\mu_\eta(x)|$

$< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{r,\eta})| d\mu_\eta(x) < \frac{1}{n} + \frac{2}{r}$

Taking $r \geq n$, we obtain that, for infinite y, that;

$|\mathcal{F}_\eta(f_\eta)(y)$

$\leq | \int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi i y x) d\mu_\eta(x)| + | \int_{(|x| > \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi i x y) d\mu_\eta(x)|$

$< \frac{3}{n} + \frac{2C}{(n-1)}$

As this holds for all $n \in \mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(y) \simeq 0$

□

Remarks 1.6. *We let $\mathcal{S}(\mathcal{R})$ denote the Schwartz space. If $h \in \mathcal{S}(\mathcal{R})$, we define its Fourier transform by;*

$\mathcal{F}(h)(y) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i y x} dx$

for $y \in \mathcal{R}$.

It is straightforward to deduce the standard Riemann-Lebesgue Lemma;

If $f \in S(\mathcal{R})$, then;

$lim_{|y| \to \infty} \mathcal{F}(f)(y) = 0$

It is sufficient to show that, given $\epsilon > 0$, there exists $M(\epsilon)$, such that;

$|\mathcal{F}(f)(y)| < \epsilon$, *for all* $y \in \mathcal{R}$, $y \geq M(\epsilon)$.

As, for all infinite $y \in \overline{\mathcal{R}_\eta}$*;*

$$|(\mathcal{F}_\eta(f_\eta))(y)| < \epsilon$$

it follows by underflow, that, for all $|y| \geq M(\epsilon)$, $y \in \overline{\mathcal{R}_\eta}$*;*

$$|(\mathcal{F}_\eta(f_\eta))(y)| < \epsilon$$

The result then follows from the fact, that, for finite $y \in \mathcal{R}$*;*

$$^\circ(\mathcal{F}_{f_\eta}(y)) = \mathcal{F}(f)(^\circ y) = \mathcal{F}(f)(y)$$

as $f_\eta exp_\eta(-\pi i m x)$ *is S-continuous and S-integrable on* $\overline{V_\eta}$*, see* [4].

Lemma 1.7. *If* f *is Riemann integrable on* $[-1,1]$ *and bounded, with corresponding* $f_\eta \in V(\overline{V_\eta})$*, then* f_η *is S-integrable on* $\overline{V_\eta}$ *and there exist* $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$*, such that* $\mu_\eta(W_n) < \frac{1}{n}$ *and* $^\circ(f_\eta) = st^*(f)$ *on* $\overline{V_\eta} \setminus W_n$*. Moreover, for* m *infinite with* $m \in \mathfrak{Z}_\eta$*, we have that;*

$$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$$

Proof. Without loss of generality, assume f is real valued. As f_n is uniformly bounded, it follows that f_η is bounded, by transfer, and, in particular, S-integrable. By Lebesgue's Theorem, there exists $D \subset [-1,1]$ with $\mu(D) = 0$, such that f is continuous on $\{[-1,1] \setminus D\}$. Let $C = st^*(D)$, then $L(\mu_\eta)(C) = 0$, let $E = st^*([-1,1] \setminus D)$, then E and C are disjoint and $L(\mu_\eta)(E) = 2$. Let $x \in E$, with corresponding $^\circ x \in [-1,1] \setminus D$, then we claim that $f_\eta(x) \simeq f(^\circ x) = st^*(f)(x)$, $(*)$. As f is continuous at $^\circ x$, we have, for $n \in \mathcal{N}$, that there exists intervals $[^\circ x - \frac{1}{n}, ^\circ x + \frac{1}{n}]$, with $a(n) \in \mathcal{N}$ increasing, such that $|f(y) - f(^\circ x)| < \frac{1}{a(n)}$, for all $y \in [^\circ x - \frac{1}{n}, ^\circ x + \frac{1}{n}]$. It follows, by transfer, as $|x - ^\circ x| < \frac{1}{n}$, for all $n \in \mathcal{N}$, that $|f_\eta(x) - f(^\circ x)| < \frac{1}{a(\eta)} \simeq 0$, hence $(*)$ is shown. It follows that $^\circ(f_\eta) = st^*(f)$ on E. Using Theorem 3.4 of [**?**], there exist $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$, with $W_n \supset C$, such that $\mu_\eta(W_n) < \frac{1}{n}$. Clearly $^\circ(f_\eta) = st^*(f)$ on $\overline{V_\eta} \setminus W_n$ for $n \in \mathcal{N}$. This gives the first claim. Then, for infinite m, and $n \in \mathcal{N}$. Using the fact that f is Lebesgue integrable, we can find a sequence $\{g_n : n \in \mathcal{N}\}$ as in Lemma 1.2. Then;

$$|\mathcal{F}_\eta(f_\eta)(m)|$$

$= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)|$

$\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)|$

$\leq \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta$, $(***)$

We have that, f_η is bounded and S-integrable, and ${}^\circ f_\eta = st^* f$, on $\overline{V_\eta} \setminus W_n$ where $st : \overline{V}_\eta \to [-1,1]$ is the standard part mapping. $g_{n,\eta}$ is bounded and S-integrable, $|(f_\eta - g_{n,\eta})|$ is bounded and S-integrable. It follows, using the S-integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

${}^\circ(\int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta)$

$= \int_{\overline{V}_\eta} |({}^\circ f_\eta - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$= \int_{\overline{V}_\eta \setminus W_n} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta) + \int_{W_n} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta)$

$\leq \int_{\overline{V}_\eta \setminus W_n} |(st^*(f) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) + R\mu_\eta(V_n)$

$\leq \int_{\overline{V}_\eta \setminus W_n} |(st^*(f) - st^* g_n)| dL(\mu_\eta) + \frac{R}{n}$

$+ \int_{\overline{V}_\eta \setminus W_n} |(st^*(g_n) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta)$

$\simeq \int_{-1}^{1} |f - g_n| d\mu + \frac{R}{n} < \frac{R+2}{n}$

Therefore, using $(***)$;

$|\mathcal{F}_\eta(f_\eta)(m)|$

$< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| < \frac{1}{n} + \frac{R+1}{n} = \frac{R+2}{n}$

As this holds for all $n \in \mathcal{N}$, using countable comprehension and overflow, see [1], $(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

□

Remarks 1.8. *If $h \in C([-1,1])$, we define its Fourier coefficient by;*

$\mathcal{F}(h)(m) = \int_{-1}^{1} h(x) e^{-\pi i m x} dx$

for $m \in \mathcal{Z}$.

As above, it is straightforward to deduce the standard Riemann-Lebesgue Lemma, in the form;

If f is Riemann integrable on $[-1,1]$ and bounded, then;

$lim_{|m|\to\infty}\mathcal{F}(f)(m) = 0$

It is sufficient to show that, given $\epsilon > 0$, there exists $M(\epsilon)$, such that;

$|\mathcal{F}(f)(m)| < \epsilon$, *for all* $m \in \mathcal{Z}$, $m \geq M(\epsilon)$.

As, for all infinite $m \in \mathcal{Z}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

it follows by underflow, that, for all $|m| \geq M(\epsilon)$, $m \in \mathcal{Z}_\eta$;

$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$

The result then follows from the fact, that, for finite $m \in \mathcal{Z}$;

$^\circ(\mathcal{F}_{f_\eta}(m)) = \mathcal{F}(f)(m)$

as $f_\eta exp_\eta(-\pi imx)$ is S-continuous and S-integrable on $\overline{V_\eta}$, see [4].

We now give a different, but more geometric proof of Lemma 1.5, but we require an extra assumption;

Lemma 1.9. *If $f \in S(\mathcal{R})$, with $Re(f)$ and $Im(f)$ analytic, with corresponding $f_\eta \in V(\overline{V_\eta})$, η prime, then, there exists $N \in \overline{R_\eta}$ infinite, $N = \frac{\sqrt{\eta}}{4}$, such that for infinite $y \in \overline{\mathcal{R}_\eta}$, with $|y| \leq N$, we have;*

$(\mathcal{F}_\eta(f_\eta))(y) \simeq 0$

Proof. The first part is the same as Lemma 1.5, for $n \in \mathcal{N}$;

$|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy)d\mu_\eta(x)| \leq \frac{E}{(n-1)}$, $(*)$

where $E \in \mathcal{R}$.

We claim that there exists $N \in \overline{\mathcal{R}_\eta}$ infinite, such that for y infinite, with $|y| \leq N$, and $n \in \mathcal{N}$;

$\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx) d\mu_\eta(x) \simeq 0$, $(*)$

where $\overline{V}_{\eta,n} = (-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{R}_\eta$.

As in Lemma 2.1, we have that;

$\int_{\overline{V}_{\eta,n}} f_\eta exp_\eta(-2\pi iyx) d\mu_\eta(x)$

$= \int_{\overline{V}_{\eta,n}} Re(f_\eta) cos_\eta(2\pi xy) d\mu_\eta(x)$

$+i \int_{\overline{V}_{\eta,n}} Im(f_\eta) cos_\eta(2\pi xy) d\mu_\eta(x)$

$-i \int_{\overline{V}_{\eta,n}} Re(f_\eta) sin_\eta(2\pi xy) d\mu_\eta(x)$

$+ \int_{\overline{V}_{\eta,n}} Im(f_\eta) sin_\eta(2\pi xy) d\mu_\eta(x)$, $(\sharp)$

It is sufficient to prove that $\int_{\overline{V}_{\eta,n}} Re(f_\eta) cos_\eta(2\pi xy) d\mu_\eta(x) \simeq 0$, the remaining cases are left to the reader;

We have that;

$\int_{\overline{V}_{\eta,n}} Re(f_\eta) cos_\eta(2\pi xy) d\mu_\eta(x)$

$= \frac{1}{\sqrt{\eta}}^* \sum_{|l|<[n\sqrt{\eta}]} Re(f^*)(\frac{l}{\sqrt{\eta}}) \theta_k(\frac{l}{\sqrt{\eta}})$, where $y = \frac{k}{\sqrt{\eta}}$

where $\theta_k(\frac{l}{\sqrt{\eta}}) = cos_\eta(\frac{2\pi lk}{\eta})$, $\theta_k(x) = cos^*(\frac{2\pi kx}{\sqrt{\eta}})$

We compute an upper bound, for given $k \in {}^*\mathcal{Z}$, with $-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}$ $n \in \mathcal{N}$, of;

$\frac{1}{\sqrt{\eta}}^* \sum_{|l|<n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}}) \theta_k(\frac{l}{\sqrt{\eta}}))$

by transfer of the result for;

$\frac{1}{m}^* \sum_{|l|<nm} Re(f)(\frac{l}{m}) \theta_r(\frac{l}{m})$

where $m \in \mathcal{R}_{>0}$, $r \in \mathcal{Z}$, $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$ $n \in \mathcal{N}$, and $\theta_r(\frac{l}{m}) = cos_{m^2}(\frac{2\pi lr}{m^2})$

For $r \in \mathcal{Z}$, $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$, $r \neq 0$, $m \in \mathcal{R}_{>0}$, $x \in \mathcal{R}$, we let;

$\theta_r(x) = cos(\frac{2\pi rx}{m})$

For $x \in \mathcal{R}$, we have that;

$\theta_r(x) = cos(\frac{2\pi rx}{m}) = 0$

iff $\frac{2\pi rx}{m} = \frac{\pi}{2} + t\pi$, $(t \in \mathcal{Z})$

iff $x = \frac{\frac{\pi m}{2} + \pi tm}{2\pi r}$

iff $x = (\frac{1}{4r} + \frac{t}{2r})m$, $(\sharp\sharp)$

$\{r, t\} \subset \mathcal{Z}$, $m \in \mathcal{R}_{>0}$

With the assumption that $|\frac{r}{m}| \leq \frac{m}{4}$, we have that $|r| \leq \frac{m^2}{4}$, $\frac{1}{|r|} \geq \frac{4}{m^2}$, $\frac{m}{2|r|} \geq \frac{2}{m} > \frac{1}{m}$, where $\frac{m}{2|r|} = z_2 - z_1$, $\theta_r(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_r(z_2) = 0)$. It follows, by transfer, with m corresponding to $\sqrt{\eta}$ and r corresponding to k, that for $|\frac{k}{\sqrt{\eta}}| \leq \frac{\sqrt{\eta}}{4}$, $\frac{\sqrt{\eta}}{2|k|} \geq \frac{2}{\sqrt{\eta}} > \frac{1}{\sqrt{\eta}}$, where $\frac{\sqrt{\eta}}{2|k|} = z_2 - z_1$, $\theta_k(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_k(z_2) = 0)$. We let $N = \frac{\sqrt{\eta}}{4} < \frac{\sqrt{\eta}}{2}$. Suppose y is infinite, with $|y| \leq N$ and $y = \frac{k}{\sqrt{\eta}}$. Using the above calculation, we have, by transfer, that;

$\frac{\sqrt{\eta}}{2|k|} = \frac{\sqrt{\eta}}{2|y\sqrt{\eta}|} = \frac{1}{2|y|} = z_2 - z_1 \simeq 0$

With the assumption that m^2 is prime, we have that $m^2(1 + 2t)$ is odd, so that, for $r \neq 0$ $\frac{m^2(1+2t)}{4r} \notin \mathcal{Z}$, so if $\theta_r(x_0) = 0$, then $mx_0 \notin \mathcal{Z}$, $(\dagger)$.

As in Lemma 2.1, using the assumption that $Re(f)$ is analytic, we have that $Re(f)$ has finitely many zeroes at $\{x_1, \ldots, x_{a(n)}\}$, with $-n \leq x_1 \leq \ldots \leq x_i \leq \ldots \leq x_{a(n)} \leq n$, and finitely many maxima and minima, $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq x_{i,j} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$, $0 \leq i \leq a(n)$, $1 \leq j \leq b(i) - 1$. As in [5], let $x_{i,0} = x_i$, for $0 \leq i \leq a(n) + 1$, then it follows that $Re(f)|_{[x_{i,j}, x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i) - 1$ and $Re(f)|_{[x_{i,b(i)}, x_{i+1,0}]}$ is monotone for

$0 \leq i \leq a(n)$.

Again, as in Lemma 2.1, we have that, for sufficiently large m, with $m \notin \mathcal{Z}$, $mx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$ and $mx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$, $1 \leq j \leq b(i)-1$, (††).

Let $\{r_{ijs} : 1 \leq s \leq e(i,j)\}$ enumerate the zeroes of θ_r on $[x_{i,j}, x_{i,j+1}]$, $r_{ij0} = x_{i,j}$, $r_{ij(e(i,j)+1))} = x_{i,j+1}$, for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i)-1$, and let $\{r_{is} : 1 \leq s \leq e(i)\}$ enumerate the zeroes of θ_r on $[x_{i,b(i)}, x_{i+1,0}]$, $r_{i0} = x_{i,b(i)}$, $r_{i(e(i)+1)} = x_{i+1}$, $0 \leq i \leq a(n)$. Then, using (†), (††), we have that;

$\frac{1}{m}{}^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

$= \frac{1}{m}{}^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1*} \sum_{s=0}^{e(i,j)*} \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

$+\frac{1}{m}{}^* \sum_{i=0}^{a(n)*} \sum_{s=0}^{e(i)*} \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

We compute $\frac{1}{m}|^* \sum_{s=1}^{e(i,j)-1*} \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$.

Let $\theta_{i,j}(s) = \frac{1}{m}{}^* \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}-1]} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})$

Considering Case 1 in Lemma 2.1, we have that;

$0 \leq |^* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)| \leq l_{i,j}$, $(*)$

where;

$l_{i,j} = \frac{1}{m}{}^* \sum_{l=[mr_{ij1}]}^{[mr_{ij2}]} |Re(f)|(\frac{l}{m})\theta_r(\frac{l}{m})$

$\leq \frac{1}{m}{}^* \sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1} D$, where $|Re(f)| \leq D$, and $0 \leq \theta_r|_{[r_{ij1}, r_{ij2}]} \leq 1$

$\leq \frac{D}{m}(([mr_{ij2}]-1) - ([mr_{ij1}]+1)+1)$

$= \frac{D}{m}([mr_{ij2}] - [mr_{ij1}] - 1)$

$\leq \frac{D}{m}((mr_{ij2}+1) - (mr_{ij1}-1) - 1)$

$= D((r_{ij2} - r_{ij1}) + \frac{1}{m})$

so that, using $(*), (\sharp\sharp)$;

$0 \leq \frac{1}{m}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})| \leq l_{i,j}$

$\leq D((r_{ij2} - r_{ij1}) + \frac{1}{m})$

$= D(\frac{m}{2|r|} + \frac{1}{m})$

Similarly, for Case 2 in Lemma **??**. In the other cases, reversing the sequences, we obtain;

$\frac{1}{m}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$\leq D((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{m})$

$= D(\frac{m}{2|r|} + \frac{1}{m})$

Considering all 4 cases, we obtain the same bound;

$\frac{1}{m}|^* \sum_{s=1}^{e(i)-1} {}^* \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})| \leq D(\frac{m}{2|r|} + \frac{1}{m})$

Similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq D(\frac{m}{2|r|} + \frac{1}{m})$

where;

$A_{i,j} = \frac{1}{m}|^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$B_{i,j} = \frac{1}{m}|^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$C_i = \frac{1}{m}|^* \sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$D_i = \frac{1}{m}|^* \sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|)$

As in Lemma **??**, we have that;

$\frac{1}{m}|^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_r(\frac{l}{m})|$

$\leq (w(n)+a(n)+1)3D(\frac{m}{2|r|}+\frac{1}{m})$

where $w(n) = Card(Re(f)'|_{[-n,n]} = 0)$, $a(n) = Card(Re(f)|_{[-n,n]} = 0)$

It follows, by transfer, with the assumption that y is infinite, $y = \frac{k}{\sqrt{\eta}}$ and $y \leq N$, using ($\sharp\sharp$);

$|\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta}}, \frac{n\sqrt{\eta}}{\sqrt{\eta}})} Re(f_\eta) cos_\eta(2\pi xy) d\mu_\eta(x)|$

$= \frac{1}{\sqrt{\eta}}|^* \sum_{|l|<n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_k(\frac{l}{\sqrt{\eta}}))|$

$\leq (w(n)+a(n)+1)3D(\frac{\sqrt{\eta}}{2|k|}+\frac{1}{\sqrt{\eta}}) \simeq 0$, as $w(n), a(n)$ are finite and $\frac{\sqrt{\eta}}{2|k|} = \frac{1}{2|y|} \simeq 0$.

and, similarly, as in Lemma 2.1, considering the additional terms from ($\sharp$), we have that;

$|\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy) d\mu_\eta(x)|$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)6D(\frac{\sqrt{\eta}}{2|k|}+\frac{1}{\sqrt{\eta}}) \simeq 0$

again, as $w(n), a(n), w'(n), a'(n)$ are finite and $\frac{\sqrt{\eta}}{2|k|} = \frac{1}{2|y|} \simeq 0$.

It follows that, for infinite $y \in \overline{\mathcal{R}_\eta}$, with $|y| \leq N$, that;

$|\int_{\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy) d\mu_\eta(x)|$

$\leq |\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy) d\mu_\eta(x)| + |\int_{(|x|\geq\frac{[n\sqrt{\eta}]}{\sqrt{\eta}})\cap\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy) d\mu_\eta(x)|$

$\leq \delta + \frac{E}{n-1} < \frac{E+1}{n-1}$

where $\delta \simeq 0$. As this holds for all $n \in \mathcal{N}$, we obtain that;

$\int_{\overline{\mathcal{R}_\eta}} f_\eta exp_\eta(-2\pi ixy) d\mu_\eta(x) \simeq 0$

□

Remarks 1.10. *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f \in S(\mathcal{R})$, with $Re(f)$ and $Im(f)$ analytic, then;

$lim_{|y|\to\infty}\mathcal{F}(f)(y) = 0$

Just follow the proof in Remark 1.6, the addition restraint that $|y| \leq N$ doesn't effect the application of underflow.

Lemma 1.11. *If $f \in C[-1,1]$, with $Re(f)$ and $Im(f)$ analytic, and corresponding $f_\eta \in V(\overline{V_\eta})$, η prime, then, there exists $N \in \mathcal{Z}_\eta$ infinite, $N = \eta - 1$, such that for infinite $m \in \mathcal{Z}_\eta$, with $|m| \leq N$, we have;*

$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

Proof. We claim that there exists $N \in \mathcal{Z}_\eta$ infinite, such that for m infinite, with $|m| \leq N$;

$\int_{\overline{V}_\eta} f_\eta(x) exp_\eta(-\pi i m x) d\mu_\eta(x) \simeq 0$, $(*)$

As above, we have that;

$\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi i m x) d\mu_\eta(x)$

$= \int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi m x) d\mu_\eta(x)$

$+i \int_{\overline{V}_\eta} Im(f_\eta) cos_\eta(\pi m x) d\mu_\eta(x)$

$-i \int_{\overline{V}_\eta} Re(f_\eta) sin_\eta(\pi m x) d\mu_\eta(x)$

$+ \int_{\overline{V}_\eta} Im(f_\eta) sin_\eta(\pi m x) d\mu_\eta(x)$, $(\sharp)$

It is sufficient to prove that $\int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi m x) d\mu_\eta(x) \simeq 0$, the remaining cases are left to the reader;

As $Re(f_\eta)$ and $cos^*(\pi m \frac{[x\eta]}{\eta})$ are bounded, $\frac{1}{\eta} Re(f_\eta)(-1) cos^*(-\pi m) \simeq 0$, so we have that;

$\int_{\overline{V}_\eta} Re(f_\eta) cos_\eta(\pi m x) d\mu_\eta(x)$

$\simeq \frac{1}{\eta}^* \sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta}) cos_\eta(\frac{\pi m l}{\eta})$

$= \frac{1}{\eta}^* \sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta}) \theta_m(\frac{l}{\eta})$

where $\theta_m(\frac{l}{\eta}) = cos_\eta(\frac{\pi ml}{\eta})$, $\theta_m(x) = cos^*(\pi mx)$

We compute an upper bound, given $m \in {}^*\mathcal{Z}$, $-\eta \leq m \leq \eta - 1$, for;

$\frac{1}{\eta}^* \sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta})\theta_m(\frac{l}{\eta})$

by transfer of the result for;

$\frac{1}{n}^* \sum_{|l|<n-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

where $n \in \mathcal{N}_{>0}$ is prime, $r \in \mathcal{Z}$, $-n \leq r \leq n-1$, and $\theta_r(\frac{l}{n}) = cos(\frac{\pi rl}{n})$

For $r \in \mathcal{Z}$, $-n \leq r \leq n-1$, $r \neq 0$, n prime, $x \in [-1, 1]$, we let;

$\theta_r(x) = cos(\pi rx)$

For $x \in [-1, 1]$, we have that;

$\theta_r(x) = cos(\pi rx) = 0$

iff $\pi rx = \frac{\pi}{2} + t\pi$, $(t \in \mathcal{Z})$

iff $x = \frac{\frac{\pi}{2}+\pi t}{\pi r}$

iff $x = (\frac{1}{2r} + \frac{t}{r})$, $(\sharp\sharp)$

$\{r, t\} \subset \mathcal{Z}$, $-n \leq r \leq n-1$, $-r - \frac{1}{2} \leq t \leq r - \frac{1}{2}$, for $r > 0$, $r - \frac{1}{2} \leq t \leq -r - \frac{1}{2}$, for $r < 0$

With the assumption that $|r| \leq n - 1$, we have that, $\frac{1}{|r|} \geq \frac{1}{n-1} > \frac{1}{n}$, where $\frac{1}{|r|} = z_2 - z_1$, $\theta_r(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_r(z_2) = 0)$. It follows, by transfer, with n corresponding to η and r corresponding to m, that for $|m| \leq \eta - 1$, $\frac{1}{|m|} \geq \frac{1}{\eta-1} > \frac{1}{\eta}$, where $\frac{1}{|m|} = z_2 - z_1$, $\theta_m(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_m(z_2) = 0)$. We let $N = \eta - 1$. Suppose m is infinite, with $|m| \leq \eta - 1$. Clearly, we have that;

$\frac{1}{|m|} = z_2 - z_1 \simeq 0$

With the assumption that n is prime, we have that $n(1 + 2t)$ is odd, so that, for $r \neq 0$ $\frac{n(1+2t)}{2r} \notin \mathcal{Z}$, so if $\theta_r(x_0) = 0$, then $nx_0 \notin \mathcal{Z}$, $(\dagger)$.

As in Lemma 1.9, using the assumption that $Re(f)$ is analytic, we have that $Re(f)$ has finitely many zeroes at $\{x_1, \ldots, x_{a(1)}\}$, with $-1 \leq x_1 \leq \ldots \leq x_i \leq \ldots \leq x_{a(n)} \leq 1$, and finitely many maxima and minima, $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq x_{i,j} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$, $0 \leq i \leq a(1)$, $1 \leq j \leq b(i) - 1$. As in Lemma 1.9 , let $x_{i,0} = x_i$, for $0 \leq i \leq a(1) + 1$, then it follows that $Re(f)|_{[x_{i,j}, x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(1)$, $0 \leq j \leq b(i) - 1$ and $Re(f)|_{[x_{i,b(i)}, x_{i+1,0}]}$ is monotone for $0 \leq i \leq a(1)$.

If $w \in \mathcal{Q}$, then, for any $d \in \mathcal{R} \setminus \mathcal{Q}$, we have that $w + d \in \mathcal{R} \setminus \mathcal{Q}$. Given a finite set $\{w_1, \ldots, w_i, \ldots, w_m\} \subset \mathcal{R} \setminus \mathcal{Q}$, let $U_i = \{u \in \mathcal{R} \setminus \mathcal{Q} : w_i + u \in \mathcal{R} \setminus \mathcal{Q}\}$, then U_i is dense in $\mathcal{R}$, hence, the finite intersection $\bigcap_{1 \leq i \leq m} U_i$ is dense in $\mathcal{R}$, in particular nonempty. It follows that there exists $d \in \mathcal{R}$, with $x_i + d \in \mathcal{R} \setminus \mathcal{Q}$, $x_{i,j} + d \in \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(1) - 1$, $1 \leq i \leq a(1) - 1$, $1 \leq j \leq b(i) - 1$. By considering $g(x) = f(x - \epsilon)$, for sufficiently small ϵ, with $\epsilon \in \mathcal{R} \setminus \mathcal{Q}$, we can assume that $\{x_1, \ldots, x_{a(1)}\} \subset \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(n) - 1$, and $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\} \subset \mathcal{R} \setminus \mathcal{Q}$, for $1 \leq i \leq a(n) - 1$, $1 \leq j \leq b(i) - 1$. In particular, it follows that for n prime, $nx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$ and $nx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$, $1 \leq j \leq b(i) - 1$, (††).

Let $\{r_{ijs} : 1 \leq s \leq e(i,j)\}$ enumerate the zeroes of θ_r on $[x_{i,j}, x_{i,j+1}]$, $r_{ij0} = x_{i,j}$, $r_{ij(e(i,j)+1))} = x_{i,j+1}$, for $0 \leq i \leq a(1)$, $0 \leq j \leq b(i) - 1$, and let $\{r_{is} : 1 \leq s \leq e(i)\}$ enumerate the zeroes of θ_r on $[x_{i,b(i)}, x_{i+1,0}]$, $r_{i0} = x_{i,b(i)}$, $r_{i(e(i)+1)} = x_{i+1}$, $0 \leq i \leq a(1)$. Then, using (†), (††), we have that;

$\frac{1}{n}^* \sum_{|l|<n-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

$= \frac{1}{n}^* \sum_{i=0}^{a(1)*} \sum_{j=0}^{b(i)-1*} \sum_{s=0}^{e(i,j)*} \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

$+ \frac{1}{n}^* \sum_{i=0}^{a(1)*} \sum_{s=0}^{e(i)*} \sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

We compute $\frac{1}{n}|^* \sum_{s=1}^{e(i,j)-1*} \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$.

Let $\theta_{i,j}(s) = \frac{1}{n}^* \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}-1]} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})$

Considering Case 1 as above, we have that;

$0 \leq |^* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)| \leq l_{i,j}$, $(*)$

where;

$l_{i,j} = \frac{1}{n} * \sum_{l=[nr_{ij1}]}^{[nr_{ij2}]} |Re(f)|(\frac{l}{n})\theta_r(\frac{l}{n})$

$\leq \frac{1}{n} * \sum_{l=[nr_{ij1}]+1}^{[nr_{ij2}]-1} D$, where $|Re(f)| \leq D$, and $0 \leq \theta_r|_{[r_{ij1}, r_{ij2}]} \leq 1$

$\leq \frac{D}{n}(([nr_{ij2}] - 1) - ([nr_{ij1}] + 1) + 1)$

$= \frac{D}{n}([nr_{ij2}] - [nr_{ij1}] - 1)$

$\leq \frac{D}{n}((nr_{ij2} + 1) - (nr_{ij1} - 1) - 1)$

$= D((r_{ij2} - r_{ij1}) + \frac{1}{n})$

so that, using $(*), (\sharp\sharp)$;

$0 \leq \frac{1}{n}| * \sum_{s=1}^{e(i,j)-1} * \sum_{l=[nx_{ij(s+1)}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})| \leq l_{i,j}$

$\leq D((r_{ij2} - r_{ij1}) + \frac{1}{n})$

$= D(\frac{1}{|r|} + \frac{1}{n})$

Similarly, for Case 2, as in Lemma 2.1. In the other cases, reversing the sequences, we obtain;

$\frac{1}{n}| * \sum_{s=1}^{e(i,j)-1} * \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$\leq D((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{n})$

$= D(\frac{1}{|r|} + \frac{1}{n})$

Considering all 4 cases, we obtain the same bound;

$\frac{1}{n}| * \sum_{s=1}^{e(i)-1} * \sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})| \leq D(\frac{1}{|r|} + \frac{1}{n})$

Similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq D(\frac{1}{|r|} + \frac{1}{n})$

where;

$A_{i,j} = \frac{1}{n}|^* \sum_{l=[nr_{ij1}]+1}^{[nr_{ij1}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$B_{i,j} = \frac{1}{n}|^* \sum_{l=[nr_{ije(i,j)}]+1}^{[nr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$C_i = \frac{1}{n}|^* \sum_{l=[nr_{i0}]+1}^{[nr_{i1}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$D_i = \frac{1}{n}|^* \sum_{l=[nr_{ie(i)}]+1}^{[nr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|)$

As in Lemma **??**, we have that;

$\frac{1}{n}|^* \sum_{|l|<n-1} Re(f)(\frac{l}{n})\theta_r(\frac{l}{n})|$

$\leq (w(1) + a(1) + 1)3D(\frac{1}{|r|} + \frac{1}{n})$

where $w(1) = Card(Re(f)'|_{[-1,1]} = 0)$, $a(1) = Card(Re(f)|_{[-1,1]} = 0)$

It follows, by transfer, with the assumption that m is infinite and $m \leq N$, using ($\sharp\sharp$);

$|\int_{\overline{V}_\eta} Re(f_\eta)cos_\eta(\pi mx)d\mu_\eta(x)|$

$\simeq \frac{1}{\eta}|^* \sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta})\theta_m(\frac{l}{\eta})|$

$\leq (w(1) + a(1) + 1)3D(\frac{1}{|m|} + \frac{1}{\eta}) \simeq 0$, as $w(1), a(1)$ are finite and $\frac{1}{|m|} \simeq 0$, $\frac{1}{\eta} \simeq 0$.

and, similarly, as in Lemma 2.1, considering the additional terms from ($\sharp$), we have that;

$|\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi imx)d\mu_\eta(x)|$

$\simeq \frac{1}{\eta}|^* \sum_{|l|<\eta-1} Re(f^*)(\frac{l}{\eta})\theta_m(\frac{l}{\eta})|$

$\leq (w(1) + a(1) + w'(1) + a'(1) + 2)6D(\frac{1}{|r|} + \frac{1}{n}) \simeq 0$

again, as $w(1), a(1), w'(1), a'(1)$ are finite and $\frac{1}{|r|} \simeq 0$, $\frac{1}{\eta} \simeq 0$

It follows that, for infinite $m \in \mathcal{Z}_\eta$, with $|m| \leq N$, that;

$\int_{\overline{V}_\eta} f_\eta exp_\eta(-\pi imx) d\mu_\eta(x) \simeq 0$

□

Remarks 1.12. *Again, it is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If $f \in C[-1,1]$, with $Re(f)$ and $Im(f)$ analytic, then;

$lim_{|m|\to\infty}\mathcal{F}(f)(m) = 0$

Just follow the proof in Remark 1.6, the addition restraint that $|m| \leq N$ doesn't effect the application of underflow.

2. Appendix

We include a Lemma which was originally Lemma 84 in the unfinished paper [5] (Lemma 82 also uses a similar method but is unfinished). The Lemma might clarify some of the methods used in Lemmas 1.9 and 1.11. There is some nonstandard rescaling in the statement of the Lemma, but it is the method of the transfer argument which is important and relevant to the rest of the paper.

Lemma 2.1. *If $f \in S(\mathcal{R})$, with the extra assumption that $Re(f)$ and $Im(f)$ and are real analytic. For $0 < \gamma < \frac{1}{2}$, $\gamma \in \mathcal{R}$, $\sqrt{c}(\eta^{\frac{1}{2}-\gamma}) \leq |y| \leq \frac{\sqrt{\eta c}}{4}$;*

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq (w(n) + a(n) + w'(n) + a'(n) + 2)\frac{6D}{\sqrt{c}}(\frac{\eta^{\gamma-\frac{1}{2}}}{2} + \frac{1}{\sqrt{\eta}}) + \frac{E}{(n-1)\sqrt{c}}$

where;

$w(n) = Card(Re(f)'|_{[-n,n]} = 0)$, $a(n) = Card(Re(f)|_{[-n,n]} = 0)$, $w'(n) = Card(Im(f)'|_{[-n,n]} = 0)$, $a'(n) = Card(Im(f)|_{[-n,n]} = 0)$ $E = 2C$, $|f| \leq \frac{C}{|x|^2}$, $|x| > 1$.

Proof. We can assume that $Re(f)$ and $Im(f) \neq 0$. Let $n \in \mathcal{N}$, we first prove that;

$|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)| \leq \frac{E}{(n-1)\sqrt{c}} \simeq 0$, $(*)$

where $E \in \mathcal{R}$.

We have, using the definition of $f^{\sqrt{c}}$ in Definition **??**, that;

$|\int_{(|x| \geq \frac{[n\sqrt{\eta}]}{\sqrt{\eta c}}) \cap \overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi i x y) d\mu_{\eta c}(x)|$

$\leq \frac{1}{\sqrt{\eta c}} * \sum_{|k|=[n\sqrt{\eta}]}^{\eta c - 1} |f^{\sqrt{c}}(\frac{k}{\sqrt{\eta c}})|$

$= \frac{1}{\sqrt{\eta c}} * \sum_{k=[n\sqrt{\eta}]}^{\eta c - 1} |f^*|(\frac{k}{\sqrt{\eta}})$

As $f \in S(\mathcal{R})$, we have that, $|f^*|(x) \leq \frac{C}{|x|^2}$, for $|x| \geq 1$, $C \in \mathcal{R}$. It follows that;

$|f^*|(\frac{k}{\sqrt{\eta}}) \leq \frac{C}{|\frac{k}{\sqrt{\eta}}|^2} = \frac{C\eta}{|k|^2}$, for $|k| \geq [n\sqrt{\eta}]$

Then;

$\frac{1}{\sqrt{\eta c}} * \sum_{k=[n\sqrt{\eta}]}^{\eta c - 1} |f^*|(\frac{k}{\sqrt{\eta}})$

$\leq \frac{1}{\sqrt{\eta c}} * \sum_{k=[n\sqrt{\eta}]}^{\eta c - 1} \frac{C\eta}{|k|^2}$

$= \frac{2\eta C}{\sqrt{\eta c}} * \sum_{k=[n\sqrt{\eta}]}^{\eta c - 1} \frac{1}{k^2}$

$\leq \frac{2\eta C}{\sqrt{\eta c}} \int_{[n\sqrt{\eta}]-1}^{\eta c - 1} \frac{dx}{x^2}$ (by transfer)

$= \frac{2C\sqrt{\eta}}{\sqrt{c}} \left[\frac{-1}{x}\right]_{[n\sqrt{\eta}]-1}^{\eta c - 1}$

$= \frac{2C\sqrt{\eta}}{\sqrt{c}} (\frac{1}{[n\sqrt{\eta}]-1} - \frac{1}{\eta c - 1})$

$\leq \frac{2C\sqrt{\eta}}{\sqrt{c}} \frac{1}{n\sqrt{\eta} - 2}$

$\leq \frac{2C\sqrt{\eta}}{\sqrt{c}} \frac{1}{(n-1)\sqrt{\eta}}$

$= \frac{2C}{(n-1)\sqrt{c}}$

which gives the result $(*)$, taking $E = 2C$.

We have that;

$\int_{(|x| < \frac{[n\sqrt{\eta}]}{\sqrt{\eta c}}) \cap \overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi i x y) d\mu_{\eta c}(x)$

$\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} exp_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$

$= \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+i \int \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+ \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

We have;

$\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \frac{1}{\sqrt{\eta c}}^* \sum_{|\frac{l}{\sqrt{\eta c}}| < \frac{n\sqrt{\eta}}{\sqrt{\eta c}}} Re(f^{\sqrt{c}})(\frac{l}{\sqrt{\eta c}}) cos_{\eta c}(\frac{2\pi lk}{\eta c}),\ (y = \frac{k}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{\eta c}}^* \sum_{|l| < n\sqrt{\eta}} Re(f^{\sqrt{c}})(\frac{l}{\sqrt{\eta c}}) cos_{\eta c}(\frac{2\pi lk}{\eta c})$

$= \frac{1}{\sqrt{\eta c}}^* \sum_{|l| < n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}}) cos_{\eta c}(\frac{2\pi lk}{\eta c})$

$= \frac{1}{\sqrt{\eta c}}^* \sum_{|l| < n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}}) \theta_{c,k}(\frac{l}{\sqrt{\eta}})$

where $\theta_{c,k}(\frac{l}{\sqrt{\eta}}) = cos_{\eta c}(\frac{2\pi lk}{\eta c})$

We compute an upper bound, for given $k \in {}^*\mathcal{Z}$, $n \in \mathcal{N}$, of;

$\frac{1}{\sqrt{\eta c}}^* \sum_{|l| < n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}}) \theta_{c,k}(\frac{l}{\sqrt{\eta}}))$

by transfer of the result for;

$\frac{1}{mv}^* \sum_{|l| < nm} Re(f)(\frac{l}{m}) \theta_{v^2,r}(\frac{l}{m})$

where $\{v, m\} \subset \mathcal{R}_{>0}$, $r \in \mathcal{Z}$, $n \in \mathcal{N}$, and $\theta_{v^2,r}(\frac{l}{m}) = cos_{v^2m^2}(\frac{2\pi lr}{v^2m^2})$, ($v$ corresponds to $\sqrt{c}$,m to $\sqrt{\eta}$, r to k)

For $x \in \mathcal{R}$, we have that;

$\theta_{v^2,r}(x) = cos_{v^2m^2}(\frac{2\pi rx}{v^2m}) = 0$

iff $\frac{2\pi rx}{v^2m} = \frac{\pi}{2} + t\pi$, $(t \in \mathcal{Z}$

iff $x = \frac{\frac{\pi v^2m}{2} + \pi t v^2m}{2\pi r}$

iff $x = (\frac{1}{4r} + \frac{t}{2r})v^2m$, $(\sharp)$

$\{r, t\} \subset \mathcal{Z}$, $\{v, m\} \subset \mathcal{R}_{>0}$

With the assumption that $|\frac{r}{vm}| \leq \frac{vm}{4}$, we have that $|r| \leq \frac{v^2m^2}{4}$, $\frac{1}{|r|} \geq \frac{4}{v^2m^2}$, $\frac{v^2m}{2|r|} \geq \frac{2}{m} > \frac{1}{m}$, where $\frac{v^2m}{2|r|} = z_2 - z_1$, $\theta_{v^2,r}(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_{v^2,r}(z_2) = 0)$. Fix $\gamma_0 \in \mathcal{R}$, with $0 < \gamma_0 < \frac{1}{2}$, then $\frac{vm}{m^{2\gamma_0}} \leq \frac{|r|}{vm}$ iff $|r| \geq \frac{v^2}{\gamma_0}$, so we require that, $\frac{v^2}{\gamma_0} \leq |r| \leq \frac{v^2m^2}{4}$, $(*)$, $m^2 \geq \frac{4}{\gamma_0}$. With the assumption that v is odd and m^2 is prime, we have that $v^2m^2(1 + 2t)$ is odd, so that $\frac{v^2m^2(1+2t)}{4r} \notin \mathcal{Z}$, so if $\theta_{v^2,r}(x_0) = 0$, then $mx_0 \notin \mathcal{Z}$, $(\dagger)$.

We claim that $Re(f)$ has finitely many zeroes at $\{x_1, \ldots, x_{a(n)}\}$, with $-n \leq x_1 \leq \ldots \leq x_i \leq \ldots \leq x_{a(n)} \leq n$, $(**)$. Suppose not, then, choose an infinite set of zeroes $\{x_i : i \geq 1\} \subset [-n, n]$, with a convergent subsequence $\{x_{i(j)} : j \geq 1\} \subset [-n, n]$. As $[-n, n]$ is closed, the limit $b \in [-n, n]$, and, as $Re(f)$ is continuous, $f(b) = 0$. As $Re(f)$ is analytic, it is identically zero on an open neighborhood $[b - \epsilon, b + \epsilon]$, then, using the fact that $[-n, n]$ is connected, and, repeating the argument, $Re(f) = 0$ on $[-n, n]$, and, similarly, $f \equiv 0$ on $\mathcal{R}$. Let $x_0 = -n$ and $x_{a(n)+1} = n$. Since, $Re(f)|_{[x_i,x_{i+1}]}$ is differentiable, for $0 \leq i \leq a(n)$, as above, it has finitely many maxima and minima, $\{x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq x_{i,j} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$, $0 \leq i \leq a(n)$, $1 \leq j \leq b(i) - 1$. Let $x_{i,0} = x_i$, for $0 \leq i \leq a(n) + 1$, then it follows that $Re(f)|_{[x_{i,j},x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i) - 1$ and $Re(f)|_{[x_{i,b(i)},x_{i+1,0}]}$ is monotone for $0 \leq i \leq a(n)$.

Without loss of generality, there are four cases to consider, $Re(f) \geq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) < Re(f)(x_{i,j+1})$, $Re(f) \geq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) > Re(f)(x_{i,j+1})$, $Re(f) \leq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) < Re(f)(x_{i,j+1})$, $Re(f) \leq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) > Re(f)(x_{i,j+1})$, $0 \leq i \leq i \leq a(n)$, $0 \leq j \leq b(i) - 1$, and, similarly, for $[x_{i,b(i)}, x_{i+1,0}]$, $0 \leq i \leq a(n)$.

If $x_0 \in \mathcal{R}$, with $mx_0 \in \mathcal{Z}$, then $m^2x_0^2 \in \mathcal{Z}$. If $\{m_1^2, m_2^2\}$ are prime, $\{n_1, n_2\} \subset \mathcal{Z}$, with $x_0^2 = \frac{n_1}{m_1^2} = \frac{n_2}{m_2^2}$, then $n_1m_2^2 = n_2m_1^2$, $n_1 = m_1^2$, $n_2 = m_2^2$, and $|x_0| = 1$. As $Card(\{x_{i,j} : 1 \leq i \leq a(n) - 1, 1 \leq j \leq b(i) - 1\})$ and $Card(\{x_i : 1 \leq i \leq a(n) - 1\})$ are finite, we have that for sufficiently large, and $m \notin \mathcal{Z}$, $mx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$ and $mx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n) - 1$, $1 \leq j \leq b(i) - 1$, (††).

Let $\{r_{ijs} : 1 \leq s \leq e(i,j)\}$ enumerate the zeroes of $\theta_{v^2,r}$ on $[x_{i,j}, x_{i,j+1}]$, $r_{ij0} = x_{i,j}$, $r_{ij(e(i,j)+1))} = x_{i,j+1}$, for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i) - 1$, and let $\{r_{is} : 1 \leq s \leq e(i)\}$ enumerate the zeroes of $\theta_{v^2,r}$ on $[x_{i,b(i)}, x_{i+1,0}]$, $r_{i0} = x_{i,b(i)}$, $r_{i(e(i)+1)} = x_{i+1}$, $0 \leq i \leq a(n)$. Then, using (†), (††), we have that;

$\frac{1}{mv}{}^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$= \frac{1}{mv}{}^* \sum_{i=0}^{a(n)*} \sum_{l=[mx_i]+1}^{[mx_{i+1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$= \frac{1}{mv}{}^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1*} \sum_{l=[mx_{i,j}]+1}^{[mx_{i,j+1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$+\frac{1}{mv}{}^* \sum_{i=0}^{a(n)*} \sum_{l=[mx_{i,b(i)}]+1}^{[mx_{i+1,0}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$= \frac{1}{mv}{}^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1*} \sum_{s=0}^{e(i,j)*} \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$+\frac{1}{mv}{}^* \sum_{i=0}^{a(n)*} \sum_{s=0}^{e(i)*} \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

We compute $\frac{1}{mv}{}^* \sum_{s=1}^{e(i,j)-1*} \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

Let $\theta_{i,j}(s) = \frac{1}{mv}{}^* \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}-1]} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

We consider Case 2, $Re(f) \geq 0$ on $[x_{i,j}, x_{i,(j+1)}]$, with $Re(f)(x_{i,j}) > Re(f)(x_{i,(j+1)})$. Assume, without loss of generality, that $\theta_{v^2,r}|_{[r_{ij1},r_{ij2}]} > 0$, (†††), then, as $Re(f)|_{[x_{i,j},x_{i,(j+1)}]} \geq 0$, and;

$\frac{\theta_{v^2,r}|_{[r_{ijs},r_{ij(s+1)}]}}{|\theta_{v^2,r}|_{[r_{ijs},r_{ij(s+1)}]}|} = -\frac{\theta_{v^2,r}|_{[r_{ij(s+1)},r_{ij(s+2)}]}}{|\theta_{v^2,r}|_{[r_{ij(s+1)},r_{ij(s+2)}]}|}$, $1 \leq s \leq e(i,j) - 2$

the sequence $\{\theta_{i,j}(s) : 1 \leq s \leq e(i,j) - 1\}$ is alternating. As $Re(f)|_{[x_{i,j},x_{i,(j+1)}]}$ is decreasing, we have that;

$|\theta_{i,j}(s)| \geq |\theta_{i,j}(s+1)|$, for $1 \leq s \leq e(i,j) - 2$

so that sequence $\{\theta_{i,j}(s) : 1 \leq s \leq e(i,j) - 1\}$ is decreasing in magnitude.

We show that $0 \leq \theta_{i,j}(1) \leq l_{i,j}$, $l_{i,j} \in \mathcal{R}$, where $l_{i,j} = \theta_{i,j}(1) > 0$.

Assume that $e(i,j)$ is odd. We claim that for all sequences $\{\theta_{i,j}(s) : 1 \leq s \leq e(i,j) - 1\}$, decreasing in magnitude, with $l_{i,j} > 0$, and $e(i,j)$ odd, that ${}^*\sum_{s=1}^{e(i,j)} -1\theta_{i,j}(s) \leq l_{i,j}$. We can prove this by induction, the base case is trivial. Assume true for $e(i,j)$, and consider the sequence $\{\theta_{i,j}(s) : 1 \leq s \leq e(i,j) + 1\}$. We have that $\theta_{i,j}(3) > 0$, and the sequence $\{\theta_{i,j}(s) : 3 \leq s \leq e(i,j) + 1\}$ is alternating and decreasing in magnitude. By the induction hypothesis, we have that;

${}^*\sum_{s=3}^{e(i,j)} +1\theta_{i,j}(s) \leq \theta_{i,j}(3)$

Hence, ${}^*\sum_{s=1}^{e(i,j)} +1\theta_{i,j}(s)$

$= \theta_{i,j}(1) + \theta_{i,j}(2) + {}^*\sum_{s=3}^{e(i,j)} +1\theta_{i,j}(s)$

$\leq \theta_{i,j}(1) + \theta_{i,j}(2) + \theta_{i,j}(3)$

$\leq \theta_{i,j}(1)$

as $\theta_{i,j}(2) + \theta_{i,j}(3) \leq 0$

We have that $0 \leq \theta_{i,j}(1) \leq l_{i,j}$, $l_{i,j} \in \mathcal{R}$.

As;

$\theta_{i,j}(2p-1) + \theta_{i,j}(2p) \geq 0$, for $1 \leq p \leq \frac{e(i,j)-1}{2}$

$0 \leq {}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)$

therefore, $0 \leq {}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$, $(***)$

Assume that $e(i,j)$ is even, then as;

$\theta_{i,j}(2p) + \theta_{i,j}(2p+1) \leq 0$, for $1 \leq p \leq \frac{e(i,j)-2}{2}$

so that;

$^* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$

and, as;

$\theta_{i,j}(2p-1) + \theta_{i,j}(2p) \geq 0$, for $1 \leq p \leq \frac{e(i,j)-2}{2}$

$\theta_{i,j}(e(i,j)-1) \geq 0$

therefore;

$0 \leq {}^* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$, $(***)$

We compute;

$l_{i,j} = \frac{1}{mv} {}^* \sum_{l=[mr_{ij1}]}^{[mr_{ij2}]} Re(f)(\frac{l}{m}) \theta_{v^2,r}(\frac{l}{m})$

$\leq \frac{1}{mv} {}^* \sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1} D$, where $|Re(f)| \leq D$, and $0 \leq \theta_{v^2,r}|_{[r_{ij1},r_{ij2}]} \leq 1$

$\leq \frac{D}{mv}(([mr_{ij2}]-1) - ([mr_{ij1}]+1)+1)$

$= \frac{D}{mv}([mr_{ij2}] - [mr_{ij1}] - 1)$

$\leq \frac{D}{mv}((mr_{ij2}+1) - (mr_{ij1}-1) - 1)$

$= \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

so that, using $(***), (****)$;

$0 \leq \frac{1}{mv} {}^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m}) \theta_{v^2,r}(\frac{l}{m}) \leq l_{i,j}$

$\leq \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

$= \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$, using $(\sharp)$

We can remove the assumption $(\dagger\dagger\dagger)$, to obtain;

$\frac{1}{mv}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m}) \theta_{v^2,r}(\frac{l}{m})| \leq \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

Cases 4 is similar, with the same bound. For Cases 1 and 3, reversing the sequences, we obtain;

$\frac{1}{mv}|^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{D}{v}((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{m})$

$\leq \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

Similarly, considering all 4 cases,we obtain the same bound;

$\frac{1}{mv}|^* \sum_{s=1}^{e(i)-1} {}^* \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})| \leq \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

We have that;

$\frac{1}{mv}|^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv} {}^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} D$

$\leq \frac{D}{mv}(([mr_{ij1}] - 1) - ([mr_{ij0}] + 1) + 1)$

$= \frac{D}{mv}([mr_{ij1}] - [mr_{ij0}] - 1)$

$\leq \frac{D}{mv}((mr_{ij1} + 1) - (mr_{ij0} - 1) - 1)$

$= \frac{D}{v}((r_{ij1} - r_{ij0}) + \frac{1}{m})$

$= \frac{D}{v}((r_{ij1} - x_{ij}) + \frac{1}{m})$

$\leq \frac{D}{v}((r_{ij1} - r_{i(j-1)b(j-1)}) + \frac{1}{m})$

$= \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

and, similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

where;

$A_{i,j} = \frac{1}{mv}|^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$B_{i,j} = \frac{1}{mv}|^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$C_i = \frac{1}{mv}|^* \sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$D_i = \frac{1}{mv}|^* \sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|)$

It follows that;

$\frac{1}{mv}|^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv}|^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1*} \sum_{s=0}^{e(i,j)*} \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)*} \sum_{s=0}^{e(i)*} \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv}|^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1*} \sum_{s=1}^{e(i,j)-1*} \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1} (^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}) +^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}))|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)*} \sum_{s=1}^{e(i)-1*} \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)} (^* \sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}) +^* \sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}))|$

$\leq {}^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1} (\frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$+^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1} (\frac{2D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$+^* \sum_{i=0}^{a(n)} (\frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$+^* \sum_{i=0}^{a(n)} (\frac{2D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$= {}^* \sum_{i=0}^{a(n)*} \sum_{j=0}^{b(i)-1} (\frac{3D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$+^* \sum_{i=0}^{a(n)} (\frac{3D}{v}(\frac{v^2 m}{2r} + \frac{1}{m}))$

$(w(n) + a(n) + 1)\frac{3D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

where $w(n) = Card(Re(f)'|_{[-n,n]} = 0)$, $a(n) = Card(Re(f)|_{[-n,n]} = 0)$

It follows, by transfer, that;

$|\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}})cos_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$= \frac{1}{\sqrt{\eta c}}|^* \sum_{|l|<n\sqrt{\eta}} Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_{c,k}(\frac{l}{\sqrt{\eta}}))|$

$\leq (w(n)+a(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

and, similarly;

Similarly;

$B_n) \leq (w(n)+a(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

and;

$max(A_n, C_n) \leq (w'(n)+a'(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

where;

$w'(n) = Card(Im(f)'|_{[-n,n]} = 0)$ and $a'(n) = Card(Im(f)|_{[-n,n]} = 0)$

$A_n = |i\int\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}})cos_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$B_n = |-i\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}})sin_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$C_n = |\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}})sin_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

It follows that;

$|\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2}\frac{1}{c\eta^{1-\gamma}}+\frac{1}{\sqrt{\eta}})$

$= (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{\sqrt{\eta}}{2\eta^{1-\gamma}}+\frac{1}{\sqrt{\eta}})$

as;

$|\frac{k}{\sqrt{\eta c}}| \geq \sqrt{c}(\eta^{\frac{1}{2}-\gamma})$

$|k| \geq c\eta^{1-\gamma}$

$\frac{1}{|k|} \leq \frac{1}{c\eta^{1-\gamma}}$

so that;

$|\int_{\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)|$

$\leq |\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)| + |\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)|$

$\leq (w(n) + a(n) + w'(n) + a'(n) + 2)\frac{6D}{\sqrt{c}}(\frac{\sqrt{\eta}}{2\eta^{1-\gamma}} + \frac{1}{\sqrt{\eta}}) + \frac{E}{(n-1)\sqrt{c}}$

$= (w(n) + a(n) + w'(n) + a'(n) + 2)\frac{6D}{\sqrt{c}}(\frac{\eta^{\gamma-\frac{1}{2}}}{2} + \frac{1}{\sqrt{\eta}}) + \frac{E}{(n-1)\sqrt{c}}$

□

References

[1] Loeb Measures in Practice: Recent Advances, Nigel Cutland, Springer, (1999).

[2] A Simple Proof of the Pointwise Convergence of Tent Series using Nonstandard Analysis, Tristram de Piro, ,(2019).

[3] Advances in Nonstandard Analysis, Tristram de Piro, (2018).

[4] A Simple Proof of the Inversion Theorem for Fourier Transforms using Nonstandard Analysis, Tristram de Piro, ,(2019).

[5] Nonstandard Methods for Solving Schrodinger's Equation, Tristram de Piro, (2019).

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: t.depiro@curvalinea.net