ASYMPTOTES OF A HYPERBOLA

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We have the following definitions of an asymptote l;

(i). For a parametric representation $(x(t), y(t))$ of the curve, if $A(t) = (x(t), y(t))$;

$lim_{t\rightarrow b}d(A(t), l) = 0$, $lim_{t\rightarrow b}(x(t)^2 + y(t)^2) = \infty$

where d is the shortest distance between a point and a line.

(ii). l is tangent to the curve at a "point at infinity".

We can check these two "equivalent" definitions hold for the hyperbola $y = \frac{1}{x}$ and the asymptotes $x = 0$ and $y = 0$.

We use the parametrisation $(t, \frac{1}{t})$, $t \neq 0$, for $y = \frac{1}{x}$. Then, letting l be $x = 0$;

$lim_{t\rightarrow 0}d(A(t), l) = lim_{t\rightarrow 0}t = 0$

$lim_{t\rightarrow 0}(t^2 + \frac{1}{t^2}) = lim_{t\rightarrow 0}\frac{1}{t^2} = 0$

so $x = 0$ is an asymptote in the sense of Definition (i). Intuitively, the "point at infinity", should be the line $x = 0$, so that the curve eventually passes through this additional point. We can make this precise by defining the projective plane $P^2(\mathcal{R})$ to be;

$\{[a : b : c] : (a, b, c) \in \mathcal{R}^3 \setminus (0, 0, 0)\}$

where $[a : b : c]$ and $[d : e; f]$ represent the same point if there exists $\lambda \in \mathcal{R}_{\neq 0}$ such that $\lambda(a, b, c) = (d, e, f)$. Informally, this is just the set of lines passing through the origin in $\mathcal{R}^3$. Note this contains a copy of $\mathcal{R}^2$ as;

$\{[a:b:1]:(a,b)\in\mathcal{R}^2\}$

together with the addition "points at infinity", consisting of;

$\{[a:b:0]:(a,b)\in\mathcal{R}^2\}$

which are just lines passing through the origin in $\mathcal{R}^2$. With this definition, and coordinates X,Y,Z, $x=\frac{X}{Z}$, $y=\frac{Y}{Z}$, $Z\neq 0$, $y=\frac{1}{x}$ takes the form;

$\frac{Y}{Z}=\frac{1}{\frac{X}{Z}}$

$\frac{Y}{Z}=\frac{Z}{X}$

$YX-Z^2=0$ $(*)$

The point at infinity for $x=0$ is;

$lim_{\epsilon\rightarrow 0}[0:\frac{1}{\epsilon}:1]$

$=lim_{\epsilon\rightarrow 0}[0:1:\epsilon]$

$=[0:1:0]$

Noting that $Y([0:1:0])\neq 0$, and dividing $(*)$ by Y^2;

$(\frac{X}{Y})-(\frac{Z}{Y})^2=0$

we can write this as $x_1-z_1^2=0$ and the line $x=0$ as $\frac{X}{Y}=0$, $x_1=0$. As we have seen $x_1=0$ is tangent to the curve $z_1^2=x_1$, in coordinates (x_1,z_1), as it intersects it with multiplicity 2. So $x=0$ is an asymptote in the sense of Definition (ii). The case $y=0$ is similar and left to the reader. Note that as we vary the line $x=0$ in a family, we never get more than 2 points of intersection.

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