SOME GEOMETRY OF NODAL CURVES

TRISTRAM DE PIRO

Abstract. We find a geometrical method of analysing the singularities of a plane nodal curve. The main results will be used in a forthcoming paper on geometric Plucker formulas for such curves. Plane nodal curves, that is plane curves having at most nodes as singularities, form an important class of curves, as *any* projective algebraic curve is birational to a plane nodal curve.

1. An Analysis of the Nodes of an Algebraic Curve

The purpose of this section is to develop the theory of plane nodal curves, using the Weierstrass preparation theorem. We use this theorem to analyse the nodes or ordinary double points of an algebraic curve. In this section, we will use the algebraic definition of a plane algebraic curve as defined by a single homogenous polynomial F in the coordinates $\{X, Y, Z\}$ of P^2. Hence, such a curve may not be irreducible, or may be considered to have "non-reduced" factors. By a nodal curve, we mean any plane algebraic curve, having at most nodes as singularities, see the more precise statement below. It is extremely important to allow for non irreducible curves in the definition of a nodal curve. An example of such a curve is a union of n lines in general position, that is no three of the lines intersect in a point.

We first restate a result from [5], in the special case of plane algebraic curves;

Lemma 1.1. *Weierstrass Preparation for Plane Algebraic Curves*

Let $F(X, Y)$ be a polynomial in $L[X, Y]$ of the form;

$F(X, Y) = \sum a_{ij} X^i Y^j = 0$

with $F(0, Y) \neq 0$ and $d = ord_Y F(0, Y)$. Then there exist unique elements $U(X, Y)$ and $G(X, Y)$ in $L[[X, Y]]$ such that;

$F(X,Y) = U(X,Y)G(X,Y)$

with $U(0,0) \neq 0$ *and*

$G(X,Y) = Y^d + c_1(X)Y^{d-1} + \ldots + c_d(X)$ $(*)$

with $c_i(X) \in L[[X]]$ *and* $c_i(0) = 0$ *for* $1 \leq i \leq d$.

We now give a characterisation of "nodes" for a plane algebraic curve (possibly not irreducible), which is a special case of this theorem. The following definition can be found in [2];

Definition 1.2. *"Node" or Ordinary Double Point of A Plane Algebraic Curve*

Let $F(X,Y)$ *define a plane algebraic curve of degree* d*,with* $F(0,0) = 0$*. We say that* $(0,0)$ *defines a "node" or ordinary double point of* F*, if* $F = F_2 + \ldots + F_d$*,* F_j *is a homogeneous polynomial of degree* j *in* X *and* Y*, for* $2 \leq j \leq d$*,* $F_2 \neq 0$ *and the linear factors of* F_2 *are* distinct.

We then claim;

Lemma 1.3. *Let* $(0,0)$ *define an ordinary double point of* F*, with the linear factors of* F_2 *given by* $(aX + bY)$ *and* $(cX + dY)$*. Then, if* l_{ab} *defines the line* $aX + bY = 0$ *and* l_{cd} *defines the line* $cX + dY = 0$*, we have that, for a line* l *passing through* $(0,0)$*;*

$I_{(0,0)}(F,l) = 2$ *iff* l *is distinct from* l_{ab} *and* l_{cd}

Moreover, this condition characterises an ordinary double point. That is, if C *is a plane algebraic curve and* $p \in C$ *has the property that there exists exactly two distinct lines* $\{l_1, l_2\}$ *passing through* p *such that;*

$I_p(C, l_i) > 2$ *for* $i \in \{1,2\}$

and

$I_p(C,l) = 2$

for any other line l *passing through* p*, then* p *defines an ordinary double point of* C*.*

Proof. The proof is a straightforward algebraic calculation. For the first part of the lemma, suppose that $(0,0)$ defines an ordinary double point of F and let l be defined by $eX + fY = 0$. Without loss of generality, assume that $e \neq 0$. We have to calculate;

$$length(\frac{L[X,Y]}{<eX+fY,F(X,Y)>}) = length(\frac{L[Y]}{<F(gY,Y)>}),\ (g = \frac{-f}{e})$$

We have that;

$$F(gY,Y) = (ag+b)(cg+d)Y^2 + O(Y^3)$$

Then, the result follows from the fact that $(ag+b)(cg+d) \neq 0$ iff l is distinct from l_{ab} and l_{cd}.

For the converse direction, let F be the defining equation for C, and, without loss of generality, assume that p corresponds to the origin $(0,0)$. By writing F in the form $F = F_1 + \ldots + F_d$, and using the same calculation as above, one deduces easily that $F_1 = 0$ and the polynomial F_2 splits into the distinct linear factors given by the equations of the lines l_1 and l_2.

□

We apply this result to obtain;

Lemma 1.4. *Let $(0,0)$ define an ordinary double point of F, such that the Y-axis is distinct from the tangent directions of the ordinary double point. Then, we can find $U(X,Y)$ and $G(X,Y)$, as in Lemma 1.1, such that G has degree 2 in $L((X))[Y]$.*

Proof. By the assumption on the Y-axis and Lemma 1.3, we have that $ord_Y F(0,Y) = 2$. Hence, the result follows immediately from Lemma 1.1. □

We then have;

Lemma 1.5. *Let G be given by the previous Lemma 1.4 and suppose that $char(L) \neq 2$, then we can find $\eta_1(X), \eta_2(X)$ in $L[[X]]$ such that;*

$$G(X,Y) = (Y - \eta_1(X))(Y - \eta_2(X)) \text{ and } \eta_1'(0) \neq \eta_2'(0),$$
$$\eta_1(0) = \eta_2(0) = 0$$

as a formal identity in the ring $L[[X,Y]]$*, where* $\eta_1'(X), \eta_2'(X)$ *denote the formal derivatives of* $\eta_1(X)$ *and* $\eta_2(X)$ *in* $L[[X]]$*, and the tangent directions of the node are given by* $(Y - \eta_1'(0)X)$ *and* $(Y - \eta_2'(0)X)$*.*

Proof. We can write $G(X,Y)$ in the form;

$$Y^2 + c_1(X)Y + c_2(X) \ (*)$$

with $c_i(X) \in L[[X]]$ and $c_i(0) = 0$ for $1 \leq i \leq 2$, $(**)$. Suppose first that G is reducible in $L((X))[Y]$. Then we have that;

$$G(X,Y) = (Y - \eta_1(X))(Y - \eta_2(X))$$

with $\{\eta_1(X), \eta_2(X)\}$ in $L((X))$. Substituting $\eta_1(X)$ in $(*)$, and using $(**)$, it follows immediately that $\eta_1(X) \in L[[X]]$ and $\eta_1(0) = 0$. The same argument holds for $\eta_2(X)$ as well. Now, let l_{ef} denote the line $eX + fY = 0$, we may assume that $f \neq 0$ by the assumptions of Lemma 1.4. By a straightforward algebraic calculation, as above, we have that;

$$I_{(0,0)}(F, l_{ef}) = ord_X F(X, \tfrac{-e}{f}X)$$

where ord_X may be taken either in $L[X]$ or $L[[X]]$. Now, using the fact that $U(X,Y)$, from Lemma 1.1, is a unit in $L[[X,Y]]$, we have that $ord_X U(X, \frac{-e}{f}X) = 0$. By the elementary property of ord_X, that $ord_X(gh) = ord_X(g) + ord_X(h)$, for g, h power series in $L[[X]]$, we must then have have;

$$ord_X F(X, \tfrac{-e}{f}X) = ord_X G(X, \tfrac{-e}{f}X)$$

$$= ord_X(eX + f\eta_1(X))(eX + f\eta_2(X)) \ (***)$$

It then follows from $(***)$, that;

$I_{(0,0)}(F, l_{ef}) = 2$ iff $\frac{-e}{f}$ is distinct from $\{\eta_1'(0), \eta_2'(0)\}$.

By Lemma 1.3, the Definition 1.2 of an ordinary double point and the assumption in Lemma 1.4 on the tangent directions of the ordinary double point, we must then have that $\eta_1'(0) \neq \eta_2'(0) \neq 0$ and that the tangent directions are given by $(Y - \eta_1'(0)X)$ and $(Y - \eta_2'(0)X)$, as required. Now, suppose that G is irreducible in $L((X))[Y]$, $(****)$, we will argue for a contradiction. Using the method of completing the square, which is valid with the assumption that $char(L) \neq 2$, $(****)$

can only occur if;

$Disc_X(G) = c_1(X)^2 - 4c_2(X)$ is *not* a square in $L[[X]]$, (†)

Let $c_1(X) = X^m U(X)$ and $c_2(X) = X^n V(X)$, with $U(X)$ and $V(X)$ units in $L[[X]]$. Then, a straightforward algebraic calculation shows that;

(†) holds iff either $n < 2m$ and n is odd
or $n = 2m$ and $U(X) - 4V(X)$ is *not* a square in $L[[X]]$

Let $G_1(X,Y) = G(X^2, Y) = Y^2 + c_1(X^2)Y + c_2(X^2)$.

We now claim that the discriminant;

$Disc_X(G_1) = c_1(X^2)^2 - 4c_2(X^2)$ *is* a square in $L[[X]]$, (††)

In order to see this, first observe that $c_1(X^2) = X^{2m}U(X^2)$ and $c_2(X^2) = X^{2n}V(X^2)$, with $U(X^2)$ and $V(X^2)$ units in $L[[X]]$. Now, by the fact that $2n$ is even and $U(X^2) - 4V(X^2)$ is always a square in $L[[X]]$, we see that the condition (†) *cannot* hold, hence (††) holds as required. Now, if $n < 2m$ and n is odd, we have that $2n < 4m$ and;

$Disc_X(G_1) = X^{2n}W(X)$ (1)

where $W(X)$ is the unit in $L[[X]]$ given by $X^{4m-2n}U(X^2)^2 - 4V(X^2)$. If $n = 2m$ and $U(X) - 4V(X)$ is not a square in $L[[X]]$, then $2n = 4m$ and $ord_X(U(X) - 4V(X)) = r$ with r odd. We then have that $ord_X(U(X^2) - 4V(X^2) = 2r$ and;

$Disc_X(G_1) = X^{2n+2r}T(X)$ (2)

where $U(X^2) - 4V(X^2) = X^{2r}T(X)$ and $T(X)$ is a unit in $L[[X]]$. In case (1), we have that n is odd, while in case (2), we have that $n+r$ is odd. It follows that we can always find s odd, a unit $R(X)$ in $L[[X]]$ and $Z(X) = X^s R(X)$ such that;

$Disc_X(G_1) = Z(X)^2$

Let;

$\eta(X) = \frac{-c_1(X^2)}{2} + \frac{Z(X)}{2}$

By the fact that $R(X) = R(-X)$, we have $Z(X) = -Z(-X)$, hence, from the method of completing the square;

$G_1(X,Y) = G(X^2,Y) = (Y - \eta(X))(Y - \eta(-X))$ $(\dagger\dagger\dagger)$

It follows, from the construction of $\eta(X)$, that $\eta(0) = 0$. Now, by making the formal substitution of $X^{1/2}$ for X in $(\dagger\dagger\dagger)$, we obtain that;

$G(X,Y) = (Y - \eta(X^{1/2}))(Y - \eta(-X^{1/2}))$ and $\eta(0) = 0$, $(\dagger\dagger\dagger\dagger)$

as a formal identity in the ring $L[[X^{1/2},Y]]$. Now, by the fact that $\eta(0) = 0$, we have;

$\eta(X) = a_1X + a_2X^2 + a_3X^3 + O(X^4)$, $(\dagger\dagger\dagger\dagger\dagger)$

Let l be the line given by $Y - \lambda X = 0$, for $\lambda \in L$. By a similar argument to the above, and using $(\dagger\dagger\dagger\dagger)$, we have that;

$I_{0,0}(F,l) = ord_X G(X,\lambda X) = ord_X(\lambda X - \eta(X^{1/2}))(\lambda X - \eta(-X^{1/2}))$

We have that;

$$\begin{aligned}(\lambda X-\eta(X^{1/2}))(\lambda X-\eta(-X^{1/2})) &= \lambda^2X^2-\lambda X[\eta(X^{1/2})+\eta(-X^{1/2})]+ \\ &\quad \eta(X^{1/2})\eta(-X^{1/2}) \\ &= \lambda^2X^2 - 2\lambda X[a_2X + O(X^2)]+ \\ &\quad [-a_1^2X + (a_2^2 - 2a_1a_3)X^2 + O(X^3)]\end{aligned}$$

As $I_{0,0}(F,l) \geq 2$, we must have that $a_1 = 0$, and then;

$(\lambda X - \eta(X^{1/2}))(\lambda X - \eta(-X^{1/2})) = (\lambda - a_2)^2X^2 + O(X^3)$

It then follows that $I_{0,0}(F,l) = 2$ iff $\lambda \neq a_2$. In particular, this implies that there can only be *one* tangent direction to the ordinary double point of F, given by $Y - a_2X = 0$, which is a contradiction. This implies that $(****)$ cannot hold, hence the lemma is proved.

□

Remarks 1.6. *In Definition 6.3 of the paper* [6]*, we defined a node of a plane algebraic curve to be the origin of two ordinary branches with distinct tangent directions, see that paper for relevant terminology, in* *particular, by a plane algebraic curve, we meant an* irreducible *closed subvariety of P^2, having dimension* 1*. This is slightly different from the definition that we have used here. For future reference and to avoid ambiguity, we will refer to Definition 1.2 as referring to an ordinary double point, reserving the terminology node for its use in* [6]*. It follows immediately from the definition, that a node p of a plane algebraic curve C (irreducible) has the following property;*

That there exist exactly 2 *distinct lines l_1, l_2 passing through p such that;*

$I_p(C, l_i) = 3$, *for* $i \in \{1, 2\}$

and, for any other line l, we have that;

$I_p(C, l) = 2$ $(*)$

By Lemma 1.3, it follows that a node is an ordinary double point. However, the converse need not be true. In the sense of Definition 6.3 in [6]*, we can instead give the following geometric definition of an ordinary double point, for an irreducible plane algebraic curve C, as the origin of two* linear *branches with distinct tangent directions,* (†)*. In the case of an irreducible plane algebraic curve C, this definition is equivalent to Definition 1.2,* $(**)$*. For, suppose that p defines an ordinary double point in the sense of* (†)*, then it follows, see* [6]*;*

That there exist exactly 2 *distinct lines l_1, l_2 passing through p such that;*

$I_p(C, l_i) > 2$, *for* $i \in \{1, 2\}$

and, for any other line l, we have that;

$I_p(C, l) = 2$

By the same reasoning as above, Lemma 1.3, it follows that this implies p corresponds to an ordinary double point in the sense of Definition 1.2. Conversely, suppose that p does not *define an ordinary double point in the sense of* (†)*, then, we have one of the following cases;*

Case 1. There is a single branch centred at p.

Case 2. There are at least three branches centred at p.

Case 3. There are two branches centred at p and at least one of them is non-linear.

Case 4. There are two linear branches centred at p with the same tangent directions.

Using Theorem 5.13 of [6], *the property of tangent lines given in Theorem 6.2 of* [6] *and the general result of the paper* [5], *Lemma 4.16, that Zariski multiplicity coincides with algebraic multiplicity, we have that, in Cases 1 and 4, there is a* single *line l_1 with the property that $I_p(C, l_1) > I_p(C, l)$ for any other line l passing through p while in Cases 2 and 3, we have that $I_p(C, l) \geq 3$, for* any *line l passing through p. By Lemma 1.3 again, it follows that, in all these cases, p* cannot *define an ordinary double point of C. Hence, $(**)$ is shown.*

As an intuitive example, the figure "8", centred at the origin, may be considered to have an ordinary double point which is not a node. The reason being that the two branches centred at the origin are both inflexions, hence are linear but not ordinary.

We now extend the result of Theorem 2.10 in [7] to the case of ordinary double points. We assume that $char(L) \neq 2$.

Theorem 1.7. *Let $F(X, Y) = 0$ define an irreducible plane algebraic curve C, with an ordinary double point at $(0, 0)$. Let $(T, \eta_1(T))$ and $(T, \eta_2(T))$ be the power series representations of this point, as given by Lemma 1.5, and let $\{\gamma_1, \gamma_2\}$ be the branches centred at $(0, 0)$, see* [6]. *Then, for any plane, possibly not irreducible, algebraic curve $H(X, Y)$ passing through $(0, 0)$;*

$H(T, \eta_j(T)) \equiv 0$, *for $j = 1$ or $j = 2$ iff H contains C as a component.*

$ord_T H(T, \eta_j(T)) = I_{\gamma_j}(C, H)$ *otherwise, $j \in \{1, 2\}$*

where I_{γ_j} denotes the branched intersection multiplicity, as defined in [6] *and* [7].

Proof. The proof relies both on the method of Theorem 2.10 in [7] and the methods of the paper [6]. First, observe that we have $F(T, \eta_j(T)) \equiv$

0 for $j \in \{1,2\}$, $(*)$. This follows immediately from Lemmas 1.4 and 1.5. If H contains C as a component, then, by the same argument as Theorem 2.10 in [7], using the Nullstellenstatz, we would have that $H(T,\eta_j(T)) \equiv 0$, for $j=1$ *and* $j=2$. For the converse direction, suppose that $H(T,\eta_1(T)) \equiv 0$. By $(*)$, we have that $\eta_1(X)$ is an algebraic power series. Hence, we can interpret the equation $Y-\eta_1(X)$ as defining a curve C_1 on some etale extension $i:(A^2_{et},(00)^{lift}) \rightarrow (A^2,(00))$ such that $i(C_1) \subset C$. As in Theorem 2.10 of [7], we can then argue to obtain that H vanishes on C. The same argument holds if $H(T,\eta_2(T)) \equiv 0$. For the second part of the theorem, we may therefore assume that H has finite intersection with C and $ord_T H(T,\eta_j(T))$ is finite for $j \in \{1,2\}$. Now, suppose that $deg(H)=e$ and let Σ be a maximal linear system consisting of curves of degree e, having finite intersection with C. As in Theorem 2.10 of [7], we can write $H(X,Y)$ in the form $H(X,Y,\bar{v}_0)$, where $\bar{v} \in Par_\Sigma$ and $F(X,Y)$ in the form $F(X,Y,\bar{u}_0)$, for some non-varying constant $\bar{u}_0$. Similarly to Theorem 2.10 of [7], and using Lemma 1.5, we then have the sequence of maps;

$$L[\bar{v}] \rightarrow \frac{L[X,Y][\bar{v}]}{<F(X,Y,\bar{u}_0),H(X,Y,\bar{v})>} \rightarrow \frac{L[X]^{ext}[Y][\bar{v}]}{<(Y-\eta_1(X))(Y-\eta_2(X)),H(X,Y,\bar{v})>}$$

which corresponds to a sequence of finite covers;

$$F_1 \rightarrow F' \rightarrow Par_\Sigma \ (1)$$

We claim that the left hand morphism is etale at $(\bar{v}_0,(00)^{lift})$, $(\dagger)$. In order to see this, observe that the local rings $\frac{L[X]^{ext}[Y]}{<F(X,Y,\bar{u}_0)>}_{(00)}$ and $\frac{L[X]^{ext}[Y]}{<(Y-\eta_1(X))(Y-\eta_2(X))>}_{(00)}$ are isomorphic, using the factorisations of Lemma 1.5, Lemma 1.1, and the invertibility of the unit U, obtained in Lemma 1.1, in the first local ring. It then follows that the completions of these local rings must be isomorphic as well. The claim $(\dagger)$ then follows by the criteria for etale morphisms given in [3], (Theorem 3, p179). Intuitively, the power series factors $(Y-\eta_1(X))$ and $(Y-\eta_2(X))$ of $F(X,Y)$ together "preserve" the shape of the node at $(0,0)$. The reader should also look at [4]. In this paper, the etale extension i or the ring $L[X]^{ext}[Y]$ can be realised as the conic projection of a surface S in P^3 to the plane P^2. The power series $(Y-\eta_1(X))$ and $(Y-\eta_2(X))$ then correspond to curves C_1 and C_2, passing through $(00)^{lift}$, which project onto $(C,(00))$.

We also have the maps;

$\frac{L[X]^{ext}[Y][\bar{v}]}{<(Y-\eta_1(X))(Y-\eta_2(X)),H(X,Y,\bar{v})>} \rightarrow \frac{L[X]^{ext}[Y][\bar{v}]}{<Y-\eta_j(X),H(X,Y,\bar{v})>}$ $(j=1$ or $j=2)$

which correspond to inclusions;

$i_{2,j}: F_{2,j} \rightarrow F_1$, $(j=1$ or $j=2)$ (2)

We will be interested in the covers $F_{2,j} \rightarrow Par_\Sigma$, obtained by combining (1) and (2). Let $d_j = ord_X H(X,\eta_j(X),\bar{v}_0)$. We claim that the Zariski multiplicity of the cover $F_{2,j} \rightarrow Par_\Sigma$ at $(\bar{v}_0,(00)^{lift})$ is d_j as well, $(\dagger\dagger)$. This follows by imitating the corresponding proof in Theorem 2.10 of [7].

We now fix a non-singular model $C^{ns} \subset P^w$ of C, with birational presentation $\Phi_{\Sigma_1}: C^{ns} \rightarrow C$. Let $U_{\Phi_{\Sigma_1}} \subset C$ and $V_{\Phi_{\Sigma_1}} \subset C^{ns}$ be the canonical sets associated to this presentation, see [6]. Corresponding to the family of forms $\{H_{\bar{v}} : \bar{v} \in Par_\Sigma\}$, we obtain a lifted family of forms $\{\overline{H_{\bar{v}}} : \bar{v} \in Par_\Sigma\}$ on C^{ns}. Let the branches $\{\gamma_1,\gamma_2\}$ of the node $(0,0)$ of C correspond to the distinct points $\{p_1,p_2\}$ of C^{ns}. By the methods of [6], we may assume that $Base(\Sigma_1)$ is disjoint from $\{p_1,p_2\}$. From the definition of Σ, considered as a linear system on C^{ns}, we may also assume that $Base(\Sigma)$ is disjoint from $\{p_1,p_2\}$. It then follows, from Definition 5.9 and Remarks 5.10 of the paper [6], that;

$I_{\gamma_j}(C,H) = Card(C^{ns} \cap \overline{H_{\bar{v}'}} \cap \mathcal{V}_{p_j})$, $\bar{v}' \in \mathcal{V}_{\bar{v}_0}$ generic in Par_Σ, $(**)$

Now define $F_3 \subset Par_\Sigma \times P^w$ by;

$F_3(\bar{v},x)$ iff $x \in (C^{ns} \cap \overline{H}_{\bar{v}})$

We have that $F_3 \rightarrow Par_\Sigma$ is a finite cover and we may interpret the result $(**)$ by saying that this cover has Zariski multiplicity d_j at $(\bar{v}_0,p_j)$, $(***)$.

Now let C_j denote the irreducible curves defined by the algebraic power series $Y-\eta_j(X)$, for $j \in \{1,2\}$, and let C_{12} be the reducible curve defined by the product $(Y-\eta_1(X))(Y-\eta_2(X))$. The curves C_j are non-singular at $(0,0)^{lift}$, $(\dagger\dagger\dagger)$, as one can see by calculating directly that the completions of the local rings $\frac{L[X]^{ext}[Y]}{<Y-\eta_j(X)>}_{(00)}$ are in both cases equal to the formal power series ring $L[[X]]$. Let;

$i_j : (C_j, (00)^{lift}) \rightarrow (C_{12}, (00)^{lift})$, $j \in \{1, 2\}$

$\Psi : (C_{12}, (00)^{lift}) \rightarrow (C, (00))$

denote the inclusion morphisms and the locally etale morphism (at $(00)^{lift}$) defined respectively by the covers above. Let $W_j \subset C_j$ be the open sets defined by $(\Psi \circ i_j)^{-1}(U_{\Phi_{\Sigma_1}})$. Then we obtain morphisms;

$\Theta_j = (\Phi_{\Sigma_1}^{-1} \circ \Psi \circ i_j) : W_j \rightarrow C^{ns}$, $j \in \{1, 2\}$

By (†††), the morphisms Θ_j extend to include the point $(0, 0)^{lift}$ of C_j. We now show;

Claim 1. $\Theta_j((00)^{lift}) \in \{p_1, p_2\}, j \in \{1, 2\}$

Claim 2. $\Theta_1((00)^{lift}) \neq \Theta_2((00)^{lift})$ $(****)$

Claim 3. $\Theta_j : (C_j, (00)^{lift}) \rightarrow (C^{ns}, p_j)$ is etale at $(00)^{lift}, j \in \{1, 2\}$

Proof of Claim 1. Suppose, for contradiction, that $\Theta_1((00)^{lift}) = p_3 \notin \{p_1, p_2\}$. Choose $x \in W_1 \cap \mathcal{V}_{(00)^{lift}}$, then, by an elementary specialisation argument, $y = (\Psi \circ i_1)(x) \in U_{\Phi_{\Sigma_1}} \cap \mathcal{V}_{(00)}$ and $\Phi_{\Sigma_1}^{-1}(y) = \Theta_1(x) \in C^{ns} \cap \mathcal{V}_{p_3}$. By elementary properties of specialisations, we would then have that $\Gamma_{\Phi_{\Sigma_1}}(p_3, (00))$ in the correspondence between C and C^{ns}, which is a contradiction. As the same argument holds for Θ_2, the proof is shown.

Proof of Claim 2. Suppose, for contradiction, that $\Theta_1((00)^{lift}) = \Theta_2((00)^{lift}) = p_1$. Let $y \in V_{\Phi_{\Sigma_1}} \cap \mathcal{V}_{p_2}$, then $\Phi_{\Sigma_1}(y) \in U_{\Phi_{\Sigma_1}} \cap \mathcal{V}_{(00)}$. By Lemma 2.7 of [5] (Lifting Lemma for etale covers), there exists a *unique* $x \in C_{12} \cap \mathcal{V}_{(00)^{lift}}$ with $\Psi(x) = \Phi_{\Sigma_1}(y)$, hence, there clearly exists a unique $x' \in C_j \cap \mathcal{V}_{(00)^{lift}}$, with $(\Psi \circ i_j)(x') = \Phi_{\Sigma_1}(y)$, for either $j = 1$ or $j = 2$. In either case, we would then have that $\Theta_j(x') = y$. By an elementary specialisation argument, this implies that $y \in C^{ns} \cap \mathcal{V}_{p_1}$. As the infinitesimal neighborhoods $C \cap \mathcal{V}_{p_1}$ and $C \cap \mathcal{V}_{p_2}$ are disjoint, this gives the required contradiction. As the same argument holds, reversing the roles of p_1 and p_2, the proof is shown.

Proof of Claim 3. We may assume that $\Theta_1((00)^{lift}) = p_1$ and $\Theta_2((00)^{lift}) = p_2$. Let $y \in V_{\Phi_{\Sigma_1}} \cap \mathcal{V}_{p_1}$, then, by a similar argument to the previous proof, we can find a *unique* $x \in C_1 \cap \mathcal{V}_{(00)^{lift}}$ with $\Theta_1(x) = y$. This implies that Θ_1 is Zariski unramified at $(00)^{lift}$. By Theorems 2.7 and 2.8 of [7] and the fact that C^{ns} is smooth, if Θ_1 fails to be etale, then it follows that it cannot be seperable either. In this case, the restriction of Θ_1 to W_1 would also be inseperable, and, hence, either i_1, Ψ or Φ_{Σ_1} would be inseperable. As this is not the case, the proof is shown.

We have, therefore, shown $(****)$. Now observe that we can lift the family of forms $\{H_{\bar{v}} : \bar{v} \in Par_\Sigma\}$ to a family of forms on the etale cover $i : (A^2_{et}, (00)^{lift}) \rightarrow (A^2, (00))$, which we will denote by $\{\overline{\overline{H_{\bar{v}}}} : \bar{v} \in Par_\Sigma\}$. We may then rewrite the cover $F_{2,j} \rightarrow Par_\Sigma$, using the more suggestive notation;

$F_{2,j}(\bar{v}, x)$ iff $x \in C_j \cap \overline{\overline{H_{\bar{v}}}}$

Moreover, observe that, if $x \in (W_j \cup (00)^{lift}) \cap \overline{\overline{H_{\bar{v}}}}$, then $\Theta_j(x) \in C^{ns} \cap \overline{H_{\bar{v}}}$. Hence, restricting the covers if necessary, we can obtain a factorisation;

$$(F_{2,j}, ((00)^{lift}, \bar{v}_0)) \rightarrow (F_3, (p_j, \bar{v}_0)) \rightarrow (Par_\Sigma, \bar{v}_0)$$

$$(x, \bar{v}) \mapsto (\Theta_j(x), \bar{v}) \mapsto (\bar{v})$$

Using $(****)$, Claim 3, it is easy to check that the left hand cover is Zariski unramified at $((00)^{lift}, \bar{v}_0)$. It, therefore, follows that the Zariski multiplicity of the covers $F_{2,j} \rightarrow Par_\Sigma$ and $F_3 \rightarrow Par_\Sigma$, at $((00)^{lift}, \bar{v}_0)$ and $(p_j, \bar{v}_0)$ respectively, is the same. Now, the result of the Theorem follows from $(\dagger\dagger)$ and $(***)$.

□

Remarks 1.8. *The geometric idea behind this proof is quite straightforward. The reader should have in mind the following hierarchy of images; a node, a line, a cross and a circle. The relationship between these images is simply expressed in many Gothic churches and cathedrals, in which a large circular window is placed above a series of "Gothic" nodal or cruciform arches, The Abbazia di San Galgano in Italy is a particularly good example. In the language of Christianity, it expresses a relationship between the image of the Crucifixion and the image of The Lamp of Heaven. I hope to make this clearer in a book I am currently* *writing, entitled "Christian Geometry".*

In more algebraic terms, this result may be expressed, by saying that the power series $(T, \eta_j(T))$*, found in Lemma 1.5, define parametristions of the branches* γ_j *of the node, in the sense of* [6]*. It follows, from calculations in* [8]*, that, if another parametrisation (in the sense of* [6]*) of the branch* γ_j *is given, of the form* $(T, \lambda_j(T))$*, then* $\eta_j(T) = \lambda_j(T)$*. We will, therefore, refer to the power series, given by Lemma 1.5, as defining* the *parametrisations of the ordinary double point (or node).*

We now observe the following useful corollaries of Theorem 1.7;

Lemma 1.9. *Let* $F(X, Y) = 0$ *define an irreducible plane algebraic curve* C*, with an ordinary double point at* $(0, 0)$*. Let* l_{γ_1} *and* l_{γ_2} *be the tangent lines to the two branches, centred at* $(0, 0)$*, as defined in* [6]*, and let* $\eta_1(X)$ *and* $\eta_2(X)$ *be the power series given by Lemma 1.5. Then, the equations of* l_{γ_1} *and* l_{γ_2} *are given by* $(Y - \eta_1'(0)X) = 0$ *and* $(Y - \eta_2'(0)X) = 0$ *respectively.*

Proof. By Definition 6.3 of [6], the tangent lines l_{γ_j} are characterised uniquely by the property that;

$I_{\gamma_j}(C, l_{\gamma_j}) \geq 2$, $(j \in \{1, 2\})$ (1)

By Theorem 1.7, we have that;

$I_{\gamma_j}(C, Y - \lambda X) = ord_T(\eta_j(T) - \lambda T)$ (2)

Combining (1) and (2), we then obtain immediately that the tangent l_{γ_j} is given by $(Y - \eta_j'(0)X) = 0$ as required.

□

Remarks 1.10. *This result is an improvement on Lemma 1.5, as this lemma does* not *specify the correspondence between the power series* $\{\eta_1(X), \eta_2(X)\}$ *and the branches* $\{\gamma_1, \gamma_2\}$*.*

Lemma 1.11. *Let hypotheses be as in the Theorem 1.7, with the additional assumption that* $H(X, Y)$ *is smooth at the point of intersection* $(0, 0)$ *and has finite intersection with* C*. Let* l_H *be the tangent line to* H *at* $(0, 0)$ *and let* l_{γ_1} *and* l_{γ_2} *be the tangent lines to the branches* $\{\gamma_1, \gamma_2\}$ *of* C*. Then;*

$I_{(0,0)}(C, H) = 2$*, if* l_H *is distinct from* l_{γ_1} *and* l_{γ_2}

$I_{(0,0)}(C,H) > 2$, *otherwise.*

Even without the assumption that $H(X,Y)$ *is smooth at the point of intersection, we always have that;*

$I_{(0,0)}(C,H) \geq 2$

Proof. By the main result of [5], Lemma 4.16, and Theorem 5.13 of [6], Branched Version of Bezout's Theorem, we have that;

$I_{(0,0)}(C,H) = I_{\gamma_1}(C,H) + I_{\gamma_2}(C,H)$ (1)

By Theorem 1.7, we have that;

$I_{\gamma_j}(C,H) = ord_T H(T,\eta_j(T))$, $j \in \{1,2\}$ (2)

where $\{\eta_1(T), \eta_2(T)\}$ are given by Lemma 1.5. By an application of the chain rule for differentiating algebraic power series, see the proof of Lemma 2.10 in [6], and the previous Lemma 1.9, we have that;

$ord_T H(T,\eta_j(T)) > 1$ iff $H_X|_{(0,0)} + H_Y|_{(0,0)}\eta_j'(0) = 0$

iff $dH_{(0,0)} \cdot l_{\gamma_j} = 0$. (3)

Now, the first part of the result follows immediately by combining $(1),(2)$ and (3). The final part is clear, just using (1).

□

Remarks 1.12. *It seems difficult to establish this type of result by purely algebraic methods, except in the simplest cases. In general, one would have to show that, for polynomials of the form* $F(X,Y) = (aX+bY)(cX+dY)+F_1(X,Y)$ *and* $H(X,Y) = (eX+fY)+H_1(X,Y)$, *with* F_1 *and* H_1 *having first term in their homogeneous expansion of orders at least* 3 *and* 2 *respectively, and* $\{l_{ab}, l_{cd}\}$ *distinct, that;*

$length(\frac{L[X,Y]}{<F(X,Y),H(X,Y)>})_{(0,0)} > 2$ *iff* $l_{ef} \notin \{l_{ab}, l_{cd}\}$

I would be very interested to know how this can be done.

We finish this section by proving the following useful result concerning the effect of translations on ordinary double point (or nodes).

Theorem 1.13. *Let $F(X,Y)=0$ define an irreducible algebraic curve C, with an ordinary double point at $(0,0)$. Let l_{γ_1} and l_{γ_2} be the tangent lines to the branches $\{\gamma_1,\gamma_2\}$, centred at $(0,0)$, given in affine coordinates by $aX+bY=0$ and $cX+dY=0$. Let (u,v) be* any *choice of non-zero vector, with the property that the line l defined by $vX-uY$ is* distinct *from l_{γ_1} and l_{γ_2}. Let $\{C_t\}_{t\in A^1}$ be the family of irreducible curves defined by;*

$$F_t(X,Y)=F(X-tu,Y-tv)=0 \quad (t\in A^1)$$

Then, for generic $t\in A^1\cap\mathcal{V}_0$, there exist exactly two points $\{q_1,q_2\}=C\cap C_t\cap\mathcal{V}_{(0,0)}$. Moreover, in the terminology of [6]*, these points lie on the branches $\{\gamma_1,\gamma_2\}$ respectively. Finally, the intersections are transverse.*

Remarks 1.14. *This property is a peculiar feature of nodal curves. For a general curve, with a smooth point at $(0,0)$, one would not expect to find* any *such points of intersection, as is easily seen by direct calculation, in the simplest case of lines. The reader is strongly encouraged, by drawing a picture of a node, to see why, intuitively, the result should be true in this case. Our proof follows this intuitive idea. It seems clear geometrically that the result also holds for any deformation of a nodal curve C, which preserves the nodes, and for which the given node is not a base point of the deformation. Such deformations were studied extensively by Severi in* [10].

Proof. (Theorem 1.13)

By making a linear change of coordinates, we may assume that the line l corresponds to the Y-axis, which is distinct from the tangent lines to the ordinary double point $(0,0)$. By Lemmas 1.4 and 1.5, we can find a factorisation;

$$F(X,Y)=(Y-\eta_1(X))(Y-\eta_2(X))U(X,Y)$$

as a formal identity in $L[[X,Y]]$, with $\eta_1(X)$ and $\eta_2(X)$ defining the parametrisations of the ordinary double point and $U(X,Y)$ a unit in $L[[X,Y]]$. We then have a corresponding factorisation of the translated curve;

$$F_t(X,Y)=F(X,Y-t)=(Y-t-\eta_1(X))(Y-t-\eta_2(X))U(X,Y-t) \quad (*)$$

By the remarks at the beginning of Section 3 of [5], we can find an etale cover $i:(U,(00)^{lift})\rightarrow(A^{2},(00))$, such that the algebraic power series $\{\eta_{1}(X),\eta_{2}(X),U(X,Y)\}$ belong to the coordinate ring $R(U)$. Without loss of generality, we can assume that U is irreducible. As $(Y-\eta_{1}(X))$ and $(Y-\eta_{2}(X))$ both vanish at $(00)^{lift}$, and are clearly irreducible in the power series ring $L[[X,Y]]$, they define irreducible algebraic curves C_1 and C_2 passing through $(00)^{lift}$. We can consider $Y-\eta_{j}(X)$ as defining a morphism from the algebraic variety U to A^1. As U is irreducible and $Y-\eta_{j}(X)$ is not identically zero in $R(U)$, the image of this morphism consists of an open subset $V_j\subset A^1$ containing 0. By elementary dimension considerations, for $t\in V_j$, the corresponding fibre $(Y-t-\eta_{j}(X))=0$ defines an irreducible algebraic curve $C_{j,t}$ in U. We let $V=V_1\cap V_2$. It follows immediately that, for $t\in V$, the function $U(X,Y-t)$ also belongs to the fraction field $Frac(R(U))$. Moreover, if $S=\{t_1,\ldots,t_r\}$ denotes the finitely many elements of A^1 for which the Y-axis intersects the algebraic curve C, then, a straightforward calculation, using $(*)$, shows that, for $t\in(V\setminus S)$, $(Y-t-\eta_{j}(X))|_{(00)^{lift}}\neq 0$ and, hence, $U(X,Y-t)|_{(00)^{lift}}\neq\{0,\infty\}$, $(\dagger)$, in particular $U(X,Y-t)$ defines a unit in $L[[X,Y]]$. We let $V'=(V\setminus S)\cup\{0\}$. For $t\in V'$, let $R_t:=(U(X,Y-t)=\infty)$ be the infinite locus of $U(X,Y-t)$. By $(*)$ and the fact that the $G_t(X,Y)$ is finite on the affine plane A^2, $R_t\subset C_{1,t}\cup C_{2,t}$. If R_t is non-empty, by elementary dimension considerations, R_t would contain at least one of the components $C_{j,t}$. Hence, $(00)^{lift}\in R_t$, which contradicts $(\dagger)$. This shows that $R_t=\emptyset$ and $U(X,Y-t)$ belongs to $R(U)$ for $t\in V'$. The above calculation shows that the liftings C_t^{lift} of the irreducible translated curves C_t to U, for $t\in V'$, have the following decomposition;

$C_t^{lift}=C_{1,t}\cup C_{2,t}\cup W_t$ $(**)$

where $C_{1,t}$ and $C_{2,t}$ are the irreducible curves defined above, and W_t is a (possibly empty) union of irreducible curves defined by $U(X,Y-t)=0$, disjoint from $(00)^{lift}$. We now show;

Claim 1. For $t\in V'\cap\mathcal{V}_0$, $q\in C\cap C_t\cap\mathcal{V}_{(0,0)}$ iff there exists a *unique* $q^{lift}\in C^{lift}\cap C_t^{lift}\cap\mathcal{V}_{(0,0)^{lift}}$, with $i(q^{lift})=q$.

By Theorem 6.3 of [9] or even Lemma 2.7 of [5], the finite cover (U/A^2) (possibly localised) is Zariski unramified at $((00),(00)^{lift})$. Hence, if $q\in A^2\cap\mathcal{V}_{(0,0)}$, there exists a *unique* $q^{lift}\in U\cap\mathcal{V}_{(0,0)^{lift}}$ with $i(q^{lift})=q$, $(***)$. In particular, if $q\in C\cap C_t\cap\mathcal{V}_{(0,0)}$, q^{lift} is given by $(***)$, and as, by definition of C^{lift} and C_t^{lift}, $q^{lift} \in C^{lift} \cap C_t^{lift}$, we have shown one direction of the claim. The other direction follows easily from definitions and the fact that, if $q^{lift} \in U \cap \mathcal{V}_{(0,0)^{lift}}$, then $i(q^{lift}) \in A^2 \cap \mathcal{V}_{(0,0)}$.

Claim 2. For $t \in (V' \setminus \{0\}) \cap \mathcal{V}_0$;

$$C^{lift} \cap C_t^{lift} \cap \mathcal{V}_{(0,0)^{lift}} = (C_1 \cap C_2^t \cap \mathcal{V}_{(0,0)^{lift}}) \cup (C_2 \cap C_1^t \cap \mathcal{V}_{(0,0)^{lift}})$$

We use the decomposition given in $(**)$. Suppose that $q^{lift} \in C^{lift} \cap C_t^{lift} \cap \mathcal{V}_{(0,0)^{lift}}$. First, we show that q^{lift} cannot belong to W_0 or W_t. For, if either $W_0(q^{lift})$ or $W_t(q^{lift})$ holds, then, by specialisation, $W_0((00)^{lift})$ holds as well. This contradicts the fact that W_0 is disjoint from $(00)^{lift}$. The reader should compare the proof of Lemma 4.15 (Unit Removal) in [5], where a similar argument was used. Secondly, we show that q^{lift} cannot belong to $C_1 \cap C_{1,t}$ or $C_2 \cap C_{2,t}$. This follows from an easy algebraic calculation. Namely, we would have that either the pair of functions $\{Y - \eta_1(X), Y - t - \eta_1(X)\}$ vanished at q^{lift}, or the pair of functions $\{Y - \eta_2(X), Y - t - \eta_2(X)\}$ vanished at q^{lift}. In either case, this implies the constant t vanishes at q^{lift}, contradicting the assumption that $t \neq 0$. This shows the claim.

Claim 3. For $t \in (V' \setminus \{0\}) \cap \mathcal{V}_0$;

There exists a *unique* $q_1^{lift} \in C_1 \cap C_2^t \cap \mathcal{V}_{(0,0)^{lift}}$ and a *unique* $q_2^{lift} \in C_2 \cap C_1^t \cap \mathcal{V}_{(0,0)^{lift}}$.

We show the first part of the claim, the proof of the second part is the same. The proof follows the methods of Section 2 in [7], which the reader is recommended to revise. We denote the coordinate ring of V' by $L[t]_h$, for a polynomial $h(t)$ vanishing exactly at $(A^1 \setminus V')$ We have the map;

$$L[t]_h \rightarrow \frac{L[X]^{ext}[Y][t]_h}{<Y-\eta_1(X), Y-t-\eta_2(X)>}$$

which corresponds to a finite cover;

$F \rightarrow V'$, where $F \subset V' \times U$ is defined by $F(t,x)$ iff $x \in C_1 \cap C_2^t$. We compute the Zariski multiplicity of the cover $F \rightarrow V'$ at $(0,(00)^{lift})$, $(\dagger)$. First, observe that by Lemma 1.5, we have that;

$\eta_1(X) - \eta_2(X) = XU(X)$, for a unit $U(X) \in L[[X]] \cap L(X)^{alg}$

Hence, without loss of generality, it is sufficient to compute the Zariski multiplicity at $(0, 0^{lift})$ of the cover ϕ determined by;

$L[t] \rightarrow \frac{L[X]^{ext}[t]}{<XU(X)-t>}$ (††)

By the inverse function theorem, or explicit calculation using the method of determining coefficients, we can find an algebraic power series $c(t) \in L[[t]] \cap L(t)^{alg}$, with $c(0) = 0$ and $c'(0) \neq 0$, such that $c(t)U(c(t)) = t$. We then have;

$$XU(X) - t = XU(X) - c(t)U(c(t))$$

$$= (X - c(t))U(X) + c(t)(U(X) - U(c(t)))$$

$$= (X - c(t))(U(X) + c(t)V(X, t))$$

$$= (X-c(t))W(X, t) \text{ for a unit } W(X, t) \in L[[X, t]] \cap L(X, t)^{alg}$$

where, in the last step, we used the fact that $U(0) \neq 0$ and $c(0) = 0$. We can now show directly that the cover ϕ determined by (††) is etale at $(0, 0^{lift})$. This follows by observing that the map on formal power series;

$\frac{L[[X,t]]}{<(X-c(t))W(X,t)>} \rightarrow L[[t]]$, $f(X, t) \mapsto f(c(t), t)$

is an isomorphism, and applying the local criteria for etale morphisms, given in [3] (p 179) (in this case ϕ induces an isomorphism on the formal power series ring $L[[t]]$ given by $\phi^* : t \mapsto t$) Then, one can use Theorems 2.7 and 2.8 of [7], to deduce that the cover ϕ determined by (††) is Zariski unramified at $(0, 0^{lift})$. Hence, the claim follows.

We now complete the proof of the Theorem. By combining Claims 1,2 and 3, for generic $t \in A^1 \cap \mathcal{V}_0$, (even more generally for $t \in V' \setminus \{0\} \cap \mathcal{V}_0$), the intersection $C \cap C_t \cap \mathcal{V}_{(00)}$ consists of at most two points $\{q_1, q_2\} = \{i(q_1^{lift}), i(q_2^{lift})\}$, where;

$q_1^{lift} = C_1 \cap C_2^t \cap \mathcal{V}_{(0,0)^{lift}}$, $q_2^{lift} = C_2 \cap C_1^t \cap \mathcal{V}_{(0,0)^{lift}}$

It is straightforward to see that q_1^{lift} and q_2^{lift} are distinct. If not, $q_1^{lift} = q_2^{lift}$ belongs to $C_1 \cap C_2 \cap \mathcal{V}_{(0,0)^{lift}}$, hence $q_1^{lift} = q_2^{lift} = (00)^{lift}$. This contradicts the fact we observed earlier, that $(00)^{lift}$ does not belong to C_1^t or C_2^t, for $t \in V' \setminus \{0\}$. It follows that q_1 and q_2 are also distinct. If not, we would have that $Card(U \cap \mathcal{V}_{(00)^{lift}} \cap i^{-1}(q_1)) = 2$, contradicting the fact that the cover (U/A^2) is Zariski unramified at $((00),(00)^{lift})$. Hence, the intersection $C \cap C_t \cap \mathcal{V}_{(00)}$ consists of exactly two points $\{q_1, q_2\}$, as required. In order to show that these points belong to the branches $\{\gamma_1, \gamma_2\}$, we use the method of Theorem 1.7. Using the notation there, it is sufficient to check that the points $\{q_1, q_2\}$ belong to the open sets $\{W_1, W_2\}$ respectively and that the images $\{\Theta_1(q_1), \Theta_2(q_2)\}$ belong to the infinitesimal neighborhoods $\{C^{ns} \cap \mathcal{V}_{p_1}, C^{ns} \cap \mathcal{V}_{p_2}\}$ respectively. This is a straightforward exercise which we leave to the reader.

Finally, we show the transversality result. It is clear that both the intersections $\{q_1, q_2\}$ define nonsingular points of both C and its translation C_t. It is, therefore, sufficient to show that the pairs of tangent lines $\{l_{q_1,C}, l_{q_1,C_t}\}$ and $\{l_{q_2,C}, l_{q_2,C_t}\}$ are distinct, (†). The proofs of the remaining parts of the theorem show that the points $\{q_1, q_2\}$ also lie on the translated branches $\{\gamma_2^t, \gamma_1^t\}$ *respectively* of C_t. It follows that we can find a pair $\{p_2, p_1\}$, lying on the branches $\{\gamma_2, \gamma_1\}$ of C respectively, such that $l_{p_2,C}$ is parallel to l_{q_2,C^t} and $l_{p_1,C}$ is parallel to l_{q_1,C_t}. In order to show (†), it is, therefore, sufficient to prove that both the pairs $\{grad(l_{p_2}), grad(l_{q_1})\}$ and $\{grad(l_{p_1}), grad(l_{q_2})\}$ are distinct, (††). In order to show (††), we require the methods of [8]. We recall the definition of the gradient function, *grad*, see the remarks before Lemma 3.9 of [8], given in the coordinate system (X, Y) by;

$$grad = -\frac{F_X}{F_Y}$$

It follows easily from the explanation in [8], see specifically the power series calculation given immediately before Lemma 3.9 of [8], that *grad* defines a rational function on C with the following property;

If $U \subset C$ denotes the open subset of nonsingular points of C in finite position, whose tangent directions are *not* parallel to the y-axis, then, for $x \in U$, $grad(x)$ is equal to the gradient of the tangent line l_x in the coordinate system (X, Y).

We may, without loss of generality, assume that the pairs $\{p_2, q_1\}$ and $\{p_1, q_2\}$ belongs to U. Hence, it is sufficient to show that $grad(p_2) \neq grad(q_1)$ and $grad(p_1) \neq grad(q_2)$, $(\dagger\dagger\dagger)$. In order to show this last claim, fix a nonsingular model C^{ns} of C, with birational morphism $\Phi : C^{ns} \leftrightsquigarrow C$, such that the branches $\{\gamma_1, \gamma_2\}$ of the node centred at (00) of C, correspond to infinitesimal neighborhoods $\{\mathcal{V}_{O_1}, \mathcal{V}_{O_2}\}$ of $\{O_1, O_2\} \subset C^{ns}$ in the fibre $\{O_1, O_2\} = \Gamma_{[\Phi]}(y, (00))$, see Section 5 of [6]. The function $grad$ lifts to a rational function $grad^{lift} = grad \circ \Phi$ on C^{ns}. Using the fact that C^{ns} is nonsingular, it extends uniquely to a morphism $grad^{lift} : C^{ns} \rightarrow P^1$. We claim that $grad^{lift}(O_1)$ defines the gradient of the tangent line l_{γ_1} of the node, centred at (00) of C, with a corresponding statement for $grad^{lift}(O_2)$, $(\dagger\dagger\dagger\dagger)$. In order to see this, use Lemma 2.2 of [8], to show that one can unambigiously assign a value $val_{\gamma_1}(grad)$ at the branch γ_1 of C. By the construction of val_γ, given before Lemma 2.1 of [8], and the power series calculation, given before Lemma 3.9 of [8], $val_{\gamma_1}(grad)$ gives the gradient of the tangent line l_{γ_1}. By Lemma 2.3 of [8], which shows that val_γ is birationally invariant, $val_{\gamma_1}(grad) = grad^{lift}(O_1)$. Hence, $(\dagger\dagger\dagger\dagger)$ is shown. By the definition of a node, we obtain immediately that $grad^{lift}(O_1) \neq grad^{lift}(O_2)$. Now, using the fact that the pair $\{p_2, q_1\}$ corresponds to points $\{p_2^{lift}, q_1^{lift}\}$ in the infinitesimal neighborhoods $\{\mathcal{V}_{O_2}, \mathcal{V}_{O_1}\}$ of C^{ns} respectively, the result $(\dagger\dagger\dagger)$ follows immediately from elementary properties of specialisations. This gives the result.

□

References

[1] S.S. Abhyankar, Algebraic Geometry for Scientists and Engineers, AMS Mathematical Surveys 35, (1990)

[2] R. Hartshorne, Algebraic Geometry, Springer (1977)

[3] D. Mumford. Red Book of Varieties and Schemes, Springer (1999)

[4] T. de Piro, Flash Geometry of Algebraic Curves, available on my website http://www.magneticstrix.net, (2010)

[5] T. de Piro, A Non-Standard Bezout Theorem for Curves, available on my website http://www.magneticstrix.net, (2010), old version "A Non Standard Bezout Theorem" on AG/LO ArXiv(0406176) (2004)

[6] T. de Piro, A Theory of Branches for Algebraic Curves, available on my website http://www.magneticstrix.net, (2010), old version on Modnet preprint server (2006).

[7] T. de Piro, A Theory of Divisors for Algebraic Curves, available on my website http://www.magneticstrix.net, (2010), old version on AG Arxiv (0703290) (2007)

[8] T. de Piro, A Theory of Duality for Algebraic Curves, available on my website http://www.magneticstrix.net, (2010)

[9] T. de Piro, Zariski Structures and Algebraic Curves, available on my website http://www.magneticstrix.net, (2010), old version "Zariski Structures and Algebraic Geometry" on AG arXiv, (0402301) (2004)

[10] F. Severi, Vorlesungen uber Algebraische Geometrie, (Anhang F), Liepzig: Teubner 1921.

55b, Via Ludovico Albertoni, Rome, 00152
E-mail address: `tristam.depiro@unicam.it`