THE CONTINUITY EQUATION AND PARTICLE MOTION

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ABSTRACT. We define a diffusion using nonstandard analysis and the formula, used in [3];

$\rho\overline{v} = \overline{J}$ $(*)$

where ρ is charge density and $\overline{J}$ is current, to define the individual particle velocities, with the initial charge distribution given by ρ_0. We prove that given $\overline{J}$, with ρ' defined by the diffusion, that the continuity equation;

$\frac{\partial\rho'}{\partial t} + div(\overline{J}) = 0$

is satisfied. It follows inductively, as $\rho'_0 = \rho_0$, that $\rho = \rho'$. The result is useful for answering questions arising from [6].

1. The Continuity Equation

Definition 1.1. *We let* $S(\mathcal{R}^3 \times \mathcal{R}_{>0}) = \{f \in C^\infty(\mathcal{R}^3 \times \mathcal{R}_{>0}) :$ *there exist constants* $C_{\overline{I}}, D_{\overline{I}} \geq 0$ *with* $\overline{I} \in \mathcal{Z}^3_{\geq 0}$ *and* $|\frac{\partial f}{\partial x_{\overline{I}}}| \leq \frac{C_{\overline{I}}}{|\overline{x}|^2}, \quad for |\overline{x}| \geq D_{\overline{I}}\}$

Lemma 1.2. *Let* $\{\rho, j_1, j_2, j_3\} \subset S(\mathcal{R}^3 \times \mathcal{R}_{>0})$*, with* $\overline{J} = (j_1, j_2, j_3)$*, such that the pair* $(\rho, \overline{J})$ *satisfies the continuity equation;*

$\frac{\partial\rho}{\partial t} = -div(\overline{J})$

Then, for $t_0 \in \mathcal{R}_{>0}$*, there exists a constant* $c_0 \geq 0$*, such that* $\rho + c_0 > 1$ *and* $(\rho + c_0, \overline{J})$ *satisfies the continuity equation on* $\mathcal{R}^3 \times (0, t_0)$*.*

Proof. As $\rho \in S(\mathcal{R}^3 \times \mathcal{R}_{>0})$, taking $\overline{I} = \overline{0}$, there exists a ball $B(0, r_0)$, $r_0 > 0$, and $C_{\overline{0}}$ with $|\rho| \leq \frac{C_{\overline{0}}}{|\overline{x}|^2}$. As $\rho \in C^\infty(\mathcal{R}^3 \times \mathcal{R}_{>0})$, ρ is bounded on $B(0, r_0) \times (0, t_0)$ and therefore on $\mathcal{R}^3 \times (0, t_0)$. In particularly, there exists a constant $d > 0$ with $|\rho| < d$ on $\mathcal{R}^3 \times (0, t_0)$. Then $\rho + d > 0$,

$\rho+d+1>1$, and clearly $\frac{\partial(\rho+c)}{\partial t}=\frac{\partial\rho}{\partial t}=-div(\overline{J})$ on $\mathcal{R}^3\times(0,t_0)$, where $c=d+1$, as required.

□

Definition 1.3. *Let $(\rho,\overline{J})$ satisfy the continuity equation on $\mathcal{R}^3\times(0,t_0)$, with $\rho>1$. Fix $\{\eta,\nu\}$ positive infinite, with η odd, ν an integer, ϵ infinitesimal. We let;*

$\mathcal{T}_\nu=\bigcup_{0\leq j\leq\nu^2-1}[\frac{j}{\nu},\frac{j+1}{\nu})$

$\mathcal{T}_{\nu,0}=\bigcup_{0\leq j\leq[t_0\nu]}[\frac{j}{\nu},\frac{j+1}{\nu})$

$\mathcal{R}_\eta=\bigcup_{-\frac{(\eta-1)}{2}\leq i\leq\frac{(\eta-1)}{2}}[\frac{i}{\sqrt{\eta}},\frac{i+1}{\sqrt{\eta}})\cup(-\frac{(\eta+1)}{2},-\frac{(\eta-1)}{2})$

We define the integer part $[,]$ for $x\in\mathcal{R}$, by;

$[x]=max(y:y\in\mathcal{Z},y\leq x)$ *if* $x\geq 0$

$[x]=min(y:y\in\mathcal{Z},y\geq x)$ *if* $x<0$

and extend the definition to ${}^*\mathcal{R}$ by transfer. If $\overline{z}\in{}^*\mathcal{R}^3$, we define, $(\overline{z})_\eta=\overline{y}$, where $y_i=\frac{[z_i\sqrt{\eta}]}{\sqrt{\eta}}$. Note that $|(\overline{z})_\eta|\leq|\overline{z}|$, as $|\frac{[z_i\sqrt{\eta}]}{\sqrt{\eta}}|\leq|z_i|$, for $1\leq i\leq 3$.

We define

$\overline{J}_\eta(x_1,x_2,x_3,t)=\overline{J}^*(\frac{[x_1\sqrt{\eta}]}{\sqrt{\eta}},\frac{[x_2\sqrt{\eta}]}{\sqrt{\eta}},\frac{[x_3\sqrt{\eta}]}{\sqrt{\eta}},\frac{[t\nu]}{\nu})$

$\rho_\eta(x_1,x_2,x_3,t)=\rho^*(\frac{[x_1\sqrt{\eta}]}{\sqrt{\eta}},\frac{[x_2\sqrt{\eta}]}{\sqrt{\eta}},\frac{[x_3\sqrt{\eta}]}{\sqrt{\eta}},\frac{[t\nu]}{\nu})$

where $\{\rho^*,\overline{J}^*\}$ are the transfers of $\{\rho,\overline{J}\}$ to $(\mathcal{R}^*)^3\times\mathcal{T}^*_{\geq 0}$

For $\overline{x}\in\mathcal{R}^3_\eta$, with $\frac{[x_i\sqrt{\eta}]}{\sqrt{\eta}}=x_i$, for $1\leq i\leq 3$, we let;

$N_{\epsilon,\overline{x}}=[\frac{\rho_\eta(\overline{x},0)}{\epsilon}]$

$M_\epsilon=\sum^*_{\overline{x}\in\mathcal{R}^3_\eta,\frac{[x_i\sqrt{\eta}]}{\sqrt{\eta}}=x_i,1\leq i\leq 3}N_{\epsilon,\overline{x}}$

For $\overline{x}\in\mathcal{R}^3_\eta$ finite, with $\frac{[x_i\sqrt{\eta}]}{\sqrt{\eta}}=x_i$, for $1\leq i\leq 3$, define a trajectory $\overline{s}_{\overline{x}}$ inductively on $\mathcal{T}_{\nu,0}$ by;

$\overline{s}_{\overline{x}}(\frac{j+1}{\nu})-\overline{s}_{\overline{x}}(\frac{j}{\nu})=\frac{1}{\nu}\frac{\overline{J}_{\eta}(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}{\rho_{\eta}(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}$, *for* $0\leq j\leq[t\nu]-1$

$\overline{s}_{\overline{x}}(0)=\overline{x}$

Lemma 1.4. *For t_0 finite, there exists a constant $G\geq 0$ with $|\frac{\overline{J}_{\eta}}{\rho_{\eta}}|\leq G$ on $\mathcal{R}^3_{\eta}\times[0,t_0]$.*

Proof. As $\rho>1$, and $\overline{J}\in S(\mathcal{R}^3\times\mathcal{R}_{>0})$, for $B(\overline{0},1)$, using smoothness of $\overline{J}$ on $B(\overline{0},1)\times[0,t_0]$, and the fact that $|\frac{1}{\rho}|\leq 1$, that there exists a constant G_1 with $|\frac{\overline{J}}{\rho}|\leq G_1$ on $B(\overline{0},1)\times[0,t_0]$. Moreover, as $\overline{J}\in S(\mathcal{R}^3\times\mathcal{R}_{>0})$ and $|\frac{1}{\rho}|\leq 1$, we can find a constant G_2, with $|\frac{\overline{J}}{\rho}|\leq\frac{G_1}{|\overline{x}|^2}\leq G_1$ on $B(\overline{0},1)^c\times[0,t_0]$. Then the result follows by transfer.

□

Definition 1.5. *Let $R=\{\overline{x}\in\mathcal{R}^3_{\eta}:\frac{[x_i\sqrt{\eta}]}{\sqrt{\eta}}=x_i, for 1\leq i\leq 3, with \overline{x} finite\}$.*
Let $T_{\nu}=\{t\in\mathcal{T}_{\nu}:[t\nu]=t\nu\}$

Lemma 1.6. *Trajectories are bounded*

Let $\overline{x}\in R$, and let $s_{\overline{x}}$ be a trajectory, then there exists a constant $D\in\mathcal{R}$, with $D\geq 0$, such that;

$|\overline{s}_{\overline{x}}(\frac{j}{\nu})|\leq D$, *for* $0\leq j\leq[t\nu]$.

Proof. We have, using Lemma 1.4, that;

$|\overline{s}_{\overline{x}}(\frac{j}{\nu})-\overline{s}_{\overline{x}}(0)|=|\sum^{*}_{0\leq k\leq j-1}(\overline{s}_{\overline{x}}(\frac{k+1}{\nu})-\overline{s}_{\overline{x}}(\frac{k}{\nu}))|$

$\leq\sum^{*}_{0\leq k\leq j-1}|\overline{s}_{\overline{x}}(\frac{k+1}{\nu})-\overline{s}_{\overline{x}}(\frac{k}{\nu})|$

$=\frac{1}{\nu}\sum^{*}_{0\leq k\leq j-1}|\frac{\overline{J}_{\eta}(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}{\rho_{\eta}(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}|$

$\leq\frac{Gj}{\nu}\leq Gt_0$

so that;

$|\overline{s}_{\overline{x}}(\frac{j}{\nu})|\leq|\overline{s}_{\overline{x}}(0)|+Gt_0$

$=\overline{x}+Gt_0=D$

where $|\frac{\overline{J}_{\eta}}{\rho_{\eta}}| \leq G$ on $\mathcal{R}^3_{\eta} \times \mathcal{T}_{\nu,0}$.

$\square$

Lemma 1.7. *We have, for t_0 finite and fixed $\{\overline{x},\overline{y}\} \subset \mathcal{R}^3$, $1 \leq i \leq 3$, that $\frac{j_{i,0}}{\rho_0}(\overline{y}) = \frac{j_{i,0}}{\rho_0}(\overline{x}) + \epsilon_i + \delta_i$, where $|\epsilon_i| \leq D_i|\overline{z}|$, $|\delta_i| \leq F_i|\overline{z}|^2$, $\overline{z} = \overline{y} - \overline{x}$, and $D_i, F_i \geq 0$, and $(\frac{j_{1,0}}{\rho_0}, \frac{j_{2,0}}{\rho_0}, \frac{j_{3,0}}{\rho_0}) = \frac{\overline{J}}{\rho}|_{t_0}$. Moreover, varying $\{\overline{x},\overline{y}\}$ in a ball $B(\overline{0}, r)$, we have that $\frac{j_{i,0}}{\rho_0}(\overline{y}) = \frac{j_{i,0}}{\rho_0}(\overline{x}) + \epsilon_{i,\overline{x},\overline{y}} + \delta_{i,\overline{x},\overline{y}}$, where $|\epsilon_{i,\overline{x},\overline{y}}| \leq H_r|\overline{z}|$, $|\delta_{i,\overline{x},\overline{y}}| \leq K_r|\overline{z}|^2$, and $H_r, K_r \geq 0$. For $\{\overline{x},\overline{y}\} \subset B^*(\overline{0}, r)$, $(\frac{j_{i,0}}{\rho_0})_{\eta}(\overline{y}) = (\frac{j_{i,0}}{\rho_0})_{\eta}(\overline{x}) + \epsilon_{\eta,i,\overline{x},\overline{y}} + \delta_{\eta,i,\overline{x},\overline{y}}$, where $|\epsilon_{\eta,i,\overline{x},\overline{y}}| \leq H_r|\overline{z}_{\eta}|$, $|\delta_{\eta,i,\overline{x},\overline{y}}| \leq K_r|\overline{z}_{\eta}|^2$. We have, for t_0 finite and fixed $\{\overline{x},\overline{y}\} \subset \mathcal{R}^3$, $1 \leq i \leq 3$, that $|\frac{j_{i,0}}{\rho_0}(\overline{y}) - \frac{j_{i,0}}{\rho_0}(\overline{x})| \leq M_i|\overline{z}|$, $M_i \geq 0$. Moreover, varying $\{\overline{x},\overline{y}\}$ in a ball $B(\overline{0}, r)$, we have that $M_i \leq L_r$, $L_r \geq 0$. For $\{\overline{x},\overline{y}\} \subset B^*(\overline{0}, r)$, $|(\frac{j_{i,0}}{\rho_0})_{\eta}(\overline{y}) - (\frac{j_{i,0}}{\rho_0})_{\eta}(\overline{x})| \leq L_r|\overline{z}_{\eta}|$. Finally, there exists a constant $A_r \geq 0$, with $|(\frac{j_i}{\rho})(\overline{y}, t) - (\frac{j_i}{\rho})(\overline{x}, t)| \leq A_r|\overline{z}|$, for $\{\overline{x},\overline{y}\} \subset B(\overline{0}, r)$ and $0 \leq t \leq t_0$, and with $|(\frac{j_i}{\rho})_{\eta}(\overline{y}, t) - (\frac{j_i}{\rho})_{\eta}(\overline{x}, t)| \leq A_r|\overline{z}_{\eta}|$, for $\{\overline{x},\overline{y}\} \subset B^*(\overline{0}, r)$ and $0 \leq t \leq t_0$.*

Proof. For the first part, let $f_i(t) = \frac{j_{i,0}}{\rho_0}(\overline{x} + t(\overline{y} - \overline{x}))$ and $g_i(t) = f_i(t) - f_i(0) - f_i'(t)t$. Then $g_i(0) = 0$, $g_i(1) - g_i(0) = g_i(1) = f_i(1) - f_i(0) - f_i'(1) = \frac{j_i}{\rho_0}(\overline{y}) - \frac{j_i}{\rho_0}(\overline{x}) - \epsilon_i$. By the mean value theorem, we have that $\delta_i = g_i(1) - g_i(0) = g_i'(c)$, for some $c \in (0,1)$. We have that $g_i'(t) = f_i'(t) - f_i''(t)t - f_i'(t) = -f_i''(t)t$, so that $|g_i'(c)| \leq max_{c \in (0,1)}|f_i''(c)c| \leq max_{c \in (0,1)}|f_i''(c)|$. By the chain rule;

$$|\epsilon_i| = |f_i'(1)| = |\bigtriangledown(\frac{j_{i,0}}{\rho_0})|_{\overline{y}} \cdot (\overline{y} - \overline{x})| \leq |\bigtriangledown(\frac{j_{i,0}}{\rho_0})|_{\overline{y}}||\overline{z}| = D_i|\overline{z}|$$

where $D_i = |\bigtriangledown(\frac{j_i}{\rho_0})|_{\overline{y}}|$

$$|\delta_i|| \leq max_{c \in (0,1)}|f_i''(c)|$$

$$= max_{c \in (0,1)}|(\bigtriangledown(\frac{j_{i,0}}{\rho_0})|_{\overline{x}+t(\overline{y}-\overline{x})} \cdot (\overline{y} - \overline{x}))'(c)|$$

$$= max_{c \in (0,1)}|\sum_{j=1}^{3}(\bigtriangledown(\frac{\partial(\frac{j_{i,0}}{\rho_0})}{\partial x_j})|_{\overline{x}+c(\overline{y}-\overline{x})} \cdot \overline{z})z_j|$$

$$\leq |\overline{z}| max_{c \in (0,1)} \sum_{j=1}^{3} |\bigtriangledown(\frac{\partial(\frac{j_{i,0}}{\rho_0})}{\partial x_j})|_{\overline{x}+c(\overline{y}-\overline{x})} \cdot \overline{z}|$$

$$\leq |\overline{z}|^2 max_{c \in (0,1)} \sum_{j=1}^{3} G_{ij}(c)$$

$= F_i|\overline{z}|^2$

where $G_{ij}(c) = | \bigtriangledown (\frac{\partial(\frac{j_{i,0}}{\rho_0})}{\partial x_j})|_{\overline{x}+c(\overline{y}-\overline{x})}|$ and $F_i = max_{c\in(0,1)} \sum_{j=1}^{3} G_{ij}(c)$

For the second part, we can use the proof of the first part, the fact that $B(\overline{0},r)$ is convex, and take;

$H_r = max_{\overline{w}\in B(\overline{0},r)}| \bigtriangledown (\frac{j_i}{\rho_0})|_{\overline{w}}|$, $K_r = max_{\overline{w}\in B(\overline{0},r)} \sum_{j=1}^{3} | \bigtriangledown (\frac{\partial(\frac{j_{i,0}}{\rho_0})}{\partial x_j})|_{\overline{w}}|$

For the third part, we obtain, by transfer, on $B^*(0,r)$, that;

$(\frac{j_{i,0}}{\rho_0})^*(\overline{y}) = (\frac{j_{i,0}}{\rho_0}(\overline{x}))^* + \epsilon^*_{i,\overline{x},\overline{y}} + \delta^*_{i,\overline{x},\overline{y}}$

where $|\epsilon^*_{i,\overline{x},\overline{y}}| \leq H_r|\overline{z}^*|$, $|\delta^*_{i,\overline{x},\overline{y}}| \leq K_r|\overline{z}^*|^2$

In particular, we have that;

$(\frac{j_{i,0}}{\rho_0})_\eta(\overline{y}) = (\frac{j_{i,0}}{\rho_0})_\eta(\overline{x}) + \epsilon_{\eta,i,\overline{x},\overline{y}} + \delta_{\eta,i,\overline{x},\overline{y}}$

where $|\epsilon_{\eta,i,\overline{x},\overline{y}}| \leq H_r|\overline{z}_\eta|$, $|\delta_{\eta,i,\overline{x},\overline{y}}| \leq K_r|\overline{z}_\eta|^2$

For the fourth part, with notation as above, we have by the intermediate value theorem, that;

$|f_i(1) - f_i(0)| \leq max_{c\in(0,1)}|f'(c)|$

$= max_{c\in(0,1)}| \bigtriangledown (\frac{j_{i,0}}{\rho_0})|_{\overline{x}+c(\overline{y}-\overline{x})} \cdot (\overline{y} - \overline{x})|$

$\leq max_{c\in(0,1)}| \bigtriangledown (\frac{j_{i,0}}{\rho_0})|_{\overline{x}+c(\overline{y}-\overline{x})}||\overline{z}|$

$= M_i|\overline{z}|$

where $M_i = max_{c\in(0,1)}| \bigtriangledown (\frac{j_{i,0}}{\rho_0})|_{\overline{x}+c(\overline{y}-\overline{x})}|$. For the fifth part, using again the fact that $B(0,r)$ is convex, and using the proof of the fourth part, we can take $L_r = max_{\overline{w}\in B(0,r)}| \bigtriangledown (\frac{j_{i,0}}{\rho_0})|_{\overline{w}}|$. For the last part, we have, by transfer, on $B^*(0,r)$, that;

$|(\frac{j_{i,0}}{\rho_0})^*(\overline{y}) - (\frac{j_{i,0}}{\rho_0}(\overline{x}))^*| \leq L_r|\overline{z}^*|$

In particular, we have that;

$|(\frac{j_{i,0}}{\rho_0})_\eta(\overline{y})-(\frac{j_{i,0}}{\rho_0})_\eta(\overline{x})|\leq L_r|\overline{z}_\eta|$

as required. For the final part, let $A_r=max_{\overline{w}\in B(0,r),0\leq t\leq t_0}|\bigtriangledown(\frac{j_i}{\rho})|_{(\overline{w},t)}|$, to deduce, as before, that;

$|(\frac{j_i}{\rho})_\eta(\overline{y},t)-(\frac{j_i}{\rho})_\eta(\overline{x},t)|\leq A_r|\overline{z}_\eta|$, for $0\leq t\leq t_0$

□

Lemma 1.8. *Trajectories are injective on Initial Conditions*

For $\{\overline{x}_1,\overline{x}_2\}\subset R$, *with* $\overline{x}_1\neq\overline{x}_2$, *then we have that;*

$\overline{s}_{\overline{x}_1}(\frac{j}{\nu})\neq\overline{s}_{\overline{x}_2}(\frac{j}{\nu})$, *for* $0\leq j\leq[t\nu]$

Proof. Suppose not, and choose $t_1=\frac{j_1}{\nu}$ least, with $1\leq j_1\leq[t\nu]$, such that $\overline{s}_{\overline{x}_1}(t_1)=\overline{s}_{\overline{x}_2}(t_1)$, so that $\overline{s}_{\overline{x}_1}(t_1-\frac{1}{\nu})\neq\overline{s}_{\overline{x}_2}(t_1-\frac{1}{\nu})$. By the definition of trajectories, we have that;

$\overline{s}_{\overline{x}_1}(t_1)-\overline{s}_{\overline{x}_2}(t_1)=\overline{s}_{\overline{x}_1}(t_1-\frac{1}{\nu})-\overline{s}_{\overline{x}_2}(t_1-\frac{1}{\nu})$

$+\frac{1}{\nu}(\frac{\overline{J}_\eta(\overline{s}_{\overline{x}_1}(t_1-\frac{1}{\nu}),t_1-\frac{1}{\nu})}{\rho_\eta(\overline{s}_{\overline{x}_1}(t_1-\frac{1}{\nu}),t_1-\frac{1}{\nu})}-\frac{\overline{J}_\eta(\overline{s}_{\overline{x}_2}(t_1-\frac{1}{\nu}),t_1-\frac{1}{\nu})}{\rho_\eta(\overline{s}_{\overline{x}_2}(t_1-\frac{1}{\nu}),t_1-\frac{1}{\nu})})$

$=\overline{z}+\frac{1}{\nu}(\overline{f}+\overline{\delta})=\overline{0}$

where, with the constants $\{H_r,K_r,D_{t_1-\frac{1}{\nu}}\}$ determined by Lemmas 1.7 and 1.6;

$\overline{z}=\overline{s}_{\overline{x}_1}(t_1-\frac{1}{\nu})-\overline{s}_{\overline{x}_2}(t_1-\frac{1}{\nu})$, $|\overline{f}|\leq D|\overline{z}_\eta|$, $|\overline{\delta}|\leq F|\overline{z}_\eta|^2$, for some $\{D,F\}\subset\mathcal{R}$ and $D,F\geq 0$, $D=3H_{2D_{t_1-\frac{1}{\nu}}}$, $F=3K_{2D_{t_1-\frac{1}{\nu}}}$. It follows that;

$\overline{z}+\frac{\overline{f}}{\nu}=-\frac{\overline{\delta}}{\nu}$

$||\overline{z}|-|\frac{\overline{f}}{\nu}||\leq\frac{F|\overline{z}_\eta|^2}{\nu}$

$|\overline{z}|\leq\frac{D|\overline{z}_\eta|}{\nu}+\frac{F|\overline{z}_\eta|^2}{\nu}$

$1 \leq \frac{D|\overline{z}_\eta|}{\nu|\overline{z}|} + \frac{F|\overline{z}_\eta|^2}{\nu|\overline{z}|} \simeq 0$

which is a contradiction.

□

Lemma 1.9. *Trajectories are S-continuous on initial conditions Let $\{\overline{x}_1, \overline{x}_2\} \subset R$, with $\overline{x}_1 \simeq \overline{x}_2$, then for finite $t \in \mathcal{T}_\nu$, we have that $s_{\overline{x}_1}(t) \simeq s_{\overline{x}_2}(t)$. Moreover, $|s_{\overline{x}_1}(t) - s_{\overline{x}_2}(t)| \leq (e^{A_{D_t}{}^\circ t} + 1)|\overline{x}_1 - \overline{x}_2|$, where A_{D_t} is the constant from Lemma 1.6.*

Proof. We have, using the definition of trajectories, with constants $\{D_t, A_{D_t}\}$ from Lemmas 1.6, 1.7 and the remark in Definition 1.3 that, for $0 \leq i \leq [t\nu] - 1$;

$|s_{\overline{x}_1}(\frac{i+1}{\nu}) - s_{\overline{x}_2}(\frac{i+1}{\nu})| = |s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu}) + \frac{1}{\nu}(\frac{\overline{J}_\eta}{p_\eta}(\overline{x}_1(\frac{i}{\nu}), \frac{i}{\nu}) - \frac{\overline{J}_\eta}{p_\eta}(\overline{x}_2(\frac{i}{\nu}), \frac{i}{\nu}))|$

$\leq |s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu})| + \frac{1}{\nu}|(\frac{\overline{J}_\eta}{p_\eta}(\overline{x}_1(\frac{i}{\nu}), \frac{i}{\nu}) - \frac{\overline{J}_\eta}{p_\eta}(\overline{x}_2(\frac{i}{\nu}), \frac{i}{\nu}))|$

$\leq |s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu})| + \frac{A_{D_t}}{\nu}|(s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu}))_\eta|$

$= (1 + \frac{A_{D_t}}{\nu})|(s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu}))_\eta|$

$\leq (1 + \frac{A_{D_t}}{\nu})|s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu})|$

Suppose that $|s_{\overline{x}_1}(0) - s_{\overline{x}_2}(0)| = |\overline{x}_1 - \overline{x}_2| = \epsilon \simeq 0$, then, by a simple induction, we obtain that, for $0 \leq i \leq [t\nu]$;

$|s_{\overline{x}_1}(\frac{i}{\nu}) - s_{\overline{x}_2}(\frac{i}{\nu})| \leq (1 + \frac{A_{D_t}}{\nu})^{i-1}\epsilon$, $(*)$

We have that $((1+\frac{A_{D_t}}{\nu})^{i-1}) = ((1+\frac{A_{D_t}}{\nu})^\nu)^{\frac{i-1}{\nu}}$, with $(1+\frac{A_{D_t}}{\nu})^\nu \simeq e^{A_{D_t}}$, so that, as $\frac{i-1}{\nu}$ is finite, $(1 + \frac{A_{D_t}}{\nu})^{i-1} \simeq e^{A_{D_t}(\frac{i-1}{\nu})^\circ}$, which is finite, so that, as ϵ is infinitesimal, $|\overline{x}_1(\frac{i}{\nu}) - \overline{x}_2(\frac{i}{\nu})| \simeq 0$, as required.

For the final claim, we have by $(*)$, that;

$|s_{\overline{x}_1}(t) - s_{\overline{x}_2}(t)| = |s_{\overline{x}_1}(\frac{[t\nu]}{\nu}) - s_{\overline{x}_2}(\frac{[t\nu]}{\nu})|$

$\leq (1 + \frac{A_{D_t}}{\nu})^{[t\nu]-1}|\overline{x}_1 - \overline{x}_2|$

$\simeq e^{A_{D_t}\frac{[t\nu]-1}{\nu}}|\overline{x}_1 - \overline{x}_2|$

$\simeq e^{A_{D_t}{}^\circ t}|\overline{x}_1-\overline{x}_2|$

It follows that;

$|s_{\overline{x}_1}(t)-s_{\overline{x}_2}(t)|\leq(e^{A_{D_t}{}^\circ t}+1)|\overline{x}_1-\overline{x}_2|$ as required.

□

Definition 1.10. *For $\overline{x}\in\mathcal{R}^3_\eta$ finite, with $\frac{[x_i\sqrt{\eta}]}{\sqrt{\eta}}=x_i$, for $1\leq i\leq 3$, $t_0\in\mathcal{T}_\nu$ finite, define a reverse trajectory $\overline{s}_{\overline{x},t_0}$ inductively on $[0,\frac{[t_0\nu]}{\nu}]$ by;*

$\overline{s}_{\overline{x},t_0}(\frac{j+1}{\nu})-\overline{s}_{\overline{x},t_0}(\frac{j}{\nu})=\frac{1}{\nu}\frac{\overline{J}_\eta(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})}{\rho_\eta(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})}$, *for* $0\leq j\leq[t_0\nu]-1$

$\overline{s}_{\overline{x},t_0}(0)=\overline{x}$

Lemma 1.11. *Trajectories are essentially onto For $t_0\in\mathcal{T}_\nu$ finite, $\overline{y}\in\mathcal{R}^3_\eta$ finite, there exists $\overline{x}\in\mathcal{R}^3_\eta$, with $\overline{x}\simeq\overline{y}$, and $\overline{x}_1\in\mathcal{R}^3_\eta$ finite, with $s_{\overline{x}_1}(t_0)=\overline{x}$. Moreover;*

$|\overline{y}-\overline{x}|\leq(e^{\circ t_0}+2)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})+\frac{Fe^{\circ t_0}}{\nu}$

Proof. Let $\overline{s}_{\overline{y}',t_0}$ be the reverse trajectory defined by $\overline{y}'$ and t_0, where $\frac{[y_i\sqrt{\eta}]}{\sqrt{\eta}}=y'_i$, for $1\leq i\leq 3$, so that $\overline{y}'\simeq\overline{y}$ and $\overline{y}'$ is finite. Let $\overline{x}_1=\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu})$. By the proof of Lemma 1.6, $\overline{x}_1$ is finite. Let $\overline{x}_0$ be defined by $\frac{[x_{1,i}\sqrt{\eta}]}{\sqrt{\eta}}=x_{0,i}$, so that $\overline{x}_1\simeq\overline{x}_0$, and $\overline{x}_0$ is finite. Let $\overline{x}=\overline{s}_{\overline{x}_0}(\frac{[t_0\nu]}{\nu})$, where $\overline{s}_{\overline{x}_0}$ is the forward trajectory defined by $\overline{x}_0$, see Definition 1.3, so that, again by Lemma 1.6, $\overline{x}$ is finite. We claim that $\overline{x}\simeq\overline{y}'$, (†), so that $\overline{x}\simeq\overline{y}$ by transitivity. We have, using Definitions 1.3, 1.10 and Lemma 1.4 that;

$|\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu})-\overline{s}_{\overline{x}_0}(\frac{1}{\nu})|$

$\leq|\overline{x}_1-\overline{x}_0|+|\overline{s}_{\overline{x}_0}(0)-\overline{s}_{\overline{x}_0}(\frac{1}{\nu})|$

$\leq\frac{3}{\sqrt{\eta}}+\frac{F}{\nu}\simeq 0$ $(*)$

Then, by Definitions 1.3 and 1.10, using the bound $D_{t_0,\overline{y}',\overline{x}_0}$ from Lemma 1.6 and the constant $N=A_{D_{t_0,\overline{y}',\overline{x}_0}}$ from Lemma 1.7, we have, for $0\leq j\leq[t_0\nu]-2$;

$|\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+2}{\nu})|$

$=|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))+\frac{1}{\nu}(\frac{\overline{J}_\eta(\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}),\frac{j+1}{\nu})}{\rho_\eta(\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}),\frac{j+1}{\nu})}-\frac{\overline{J}_\eta(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu},\frac{j+1}{\nu})}{\rho_\eta(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu}),\frac{j+1}{\nu}))})|$

$\leq |(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|+\frac{N}{\nu}|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))_\eta|$

$\leq |(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|+\frac{N}{\nu}|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|$

$\leq |(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|+\frac{N}{\nu}(|\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu})$

$-\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})|+|\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu})|)$

$=(1+\frac{N}{\nu})|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|+\frac{N}{\nu}|\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j+1}{\nu})$

$-\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})|$

$\leq (1+\frac{N}{\nu})|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu}-\frac{j}{\nu})-\overline{s}_{\overline{x}_0}(\frac{j+1}{\nu}))|+\frac{NF}{\nu^2}$

It follows, by a simple induction, and the formula for a geometric progression, that;

$|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{[t_0\nu]}{\nu})|$

$\leq (1+\frac{N}{\nu})^{[t_0\nu]-2}|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu})-\overline{s}_{\overline{x}_0}(\frac{1}{\nu}))|+\delta({}^*\sum_{k=0}^{[t_0\nu]-3}(1+\frac{N}{\nu})^k)$

$=(1+\frac{N}{\nu})^{[t_0\nu]-2}|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu})-\overline{s}_{\overline{x}_0}(\frac{1}{\nu}))|+\frac{\delta\nu}{N}((1+\frac{N}{\nu})^{[t_0\nu]-2}-1)\ (**)$

where $\delta=\frac{NF}{\nu^2}$. We have that $\frac{\delta\nu}{N}=\frac{F}{\nu}\simeq 0$, and by the proof of Lemma 1.9, $(1+\frac{N}{\nu})^{[t_0\nu]-2}\simeq e^{\circ\frac{[t_0\nu]}{\nu}-\frac{2}{\nu}}=e^{\circ t_0}$. It follows, using $(*)$ and $(**)$, that;

$\frac{\delta\nu}{N}((1+\frac{N}{\nu})^{[t_0\nu]-2}-1)\simeq\frac{\delta\nu}{N}(e^{\circ t_0}-1)\simeq 0$

$(1+\frac{N}{\nu})^{[t_0\nu]-2}|(\overline{s}_{\overline{y}',t_0}(\frac{[t_0\nu]}{\nu})-\overline{s}_{\overline{x}_0}(\frac{1}{\nu}))|\leq(e^{\circ t_0}+1)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})\simeq 0$

$|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{[t_0\nu]}{\nu})|\simeq 0\ (***)$

We have, using Lemma 1.6 and Definition 1.10, that $\overline{y}'=\overline{s}_{\overline{y}',t_0}(0)\simeq\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})$ and $\overline{x}=\overline{s}_{\overline{x}_0}(\frac{[t_0\nu]}{\nu})$, so that, by $(***)$, $\overline{y}'\simeq\overline{x}$, proving (†). For the final claim, using the above calculation;

$|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{x}|=|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{s}_{\overline{x}_0}(\frac{[t_0\nu]}{\nu})|$

$\leq(e^{\circ t_0}+1)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})+\frac{Fe^{\circ t_0}}{\nu}$

$|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{y}'|=|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{s}_{\overline{y}',t_0}(0)|\leq\frac{F}{\nu}$

$|\overline{y}'-\overline{y}|\leq\frac{3}{\sqrt{\eta}}$

so that;

$|\overline{y}-\overline{x}|\leq|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{x}|+|\overline{s}_{\overline{y}',t_0}(\frac{1}{\nu})-\overline{y}'|+|\overline{y}'-\overline{y}|$

$\leq(e^{\circ t_0}+1)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})+\frac{Fe^{\circ t_0}}{\nu}+\frac{F}{\nu}+\frac{3}{\sqrt{\eta}}$

$=(e^{\circ t_0}+2)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})+\frac{Fe^{\circ t_0}}{\nu}$

□

Lemma 1.12. *Let* $\{\overline{x},\overline{y}\}\subset R\cap{}^*B(\overline{0},r)$, $\overline{x}\neq\overline{y}$, *with* $\nu\geq\eta$ *and* $r\in\mathcal{R}_{>0}$, *then for finite* $t\in\mathcal{T}_\nu$, *there exists a constant* $C_{t,r}\in\mathcal{R}_{>0}$, *uniform in* $\{\overline{x},\overline{y}\}$, *such that* $|s_{\overline{x}}(t)-s_{\overline{y}}(t)|\geq\frac{C_{t,r}}{\sqrt{\eta}}$.

Proof. First, fix $\{\overline{x},\overline{y}\}\subset R\cap{}^*B(\overline{0},r)$, $\overline{x}\neq\overline{y}$, we claim that for any infinitesimal $\epsilon>0$, $|s_{\overline{x}}(t)-s_{\overline{y}}(t)|\geq\frac{\epsilon}{\sqrt{\eta}}$, $(*)$. Given this, we have that $|s_{\overline{x}}(t)-s_{\overline{y}}(t)|\sqrt{\eta}\geq\frac{1}{n}$ for all infinite $n\in{}^*\mathcal{N}$. By underflow, the internal set $\{n\in{}^*\mathcal{N}:|s_{\overline{x}}(t)-s_{\overline{y}}(t)|\sqrt{\eta}\geq\frac{1}{n}\}$, must contain a finite element $C_{t,\overline{x},\overline{y}}\in\mathcal{N}$, for which $|s_{\overline{x}}(t)-s_{\overline{y}}(t)|\geq\frac{C_{t,\overline{x},\overline{y}}}{\sqrt{\eta}}$. To prove $(*)$, suppose that $|s_{\overline{x}}(t)-s_{\overline{y}}(t)|<\frac{\epsilon}{\sqrt{\eta}}$, $(\dagger)$, for some $\epsilon>0$ infinitesimal. Let $\overline{x}_1=s_{\overline{x}}(t)$ and $\overline{x}_2=s_{\overline{x}}(t)$. By Lemma 1.6, $\overline{x}_1$ and $\overline{x}_2$ are finite. We extend the definition of a reverse trajectory in Definition 1.10 to include arbitrary finite starting points in $\mathcal{R}^3_\eta$. By a simple adaptation of Lemma 1.9;

$|s_{\overline{x}_1,t}(t)-s_{\overline{x}_2,t}(t)|\leq(e^{A_{D_t}{}^\circ t}+1)|\overline{x}_1-\overline{x}_2|$

$<(e^{A_{D_t}{}^\circ t}+1)\frac{\epsilon}{\sqrt{\eta}}$

By the definition of a trajectory, a reverse trajectory and $\overline{x}_1$;

$|\overline{x}-s_{\overline{x}_1,t}(t)|=|s_{\overline{x}}(0)-s_{\overline{x}_1,t}(t)|$

$|s_{\overline{x}}(t)-s_{\overline{x}_1,t}(0)|=|\overline{x}_1-\overline{x}_1|=0$

By a straightforward adaptation of the calculation in Lemma 1.11, replacing trajectories by reversed trajectories, we have that that;

$|s_{\overline{x}}(\frac{1}{\nu})-s_{\overline{x}_1,t}(t)|$

$\leq (1+\frac{N}{\nu})^{[t_0\nu]-2}|s_{\overline{x}}(t)-s_{\overline{x}_1,t}(\frac{1}{\nu})|+\frac{F}{N}((1+\frac{N}{\nu})^{[t_0\nu]-2}-1)$

$\leq (e^{N^\circ t}+1)|s_{\overline{x}}(t)-s_{\overline{x}_1,t}(\frac{1}{\nu})|+\frac{F}{\nu}(e^{N^\circ t}+1)$

where $N=A_{D_{t,\overline{x},\overline{x}_1}}$

$|s_{\overline{x}}(0)-s_{\overline{x}_1,t}(t)|\leq |s_{\overline{x}}(0)-s_{\overline{x}}(\frac{1}{\nu})|+|s_{\overline{x}}(\frac{1}{\nu})-s_{\overline{x}_1,t}(t)|$

$\leq \frac{2F}{\nu}+(e^{N^\circ t}+1)|s_{\overline{x}}(t)-s_{\overline{x}_1,t}(\frac{1}{\nu})|+\frac{F}{\nu}(e^{N^\circ t}+1)$

$\leq \frac{2F}{\nu}+(e^{N^\circ t}+1)(|s_{\overline{x}}(t)-s_{\overline{x}_1,t}(0)|+|s_{\overline{x}_1,t}(0)-s_{\overline{x}_1,t}(\frac{1}{\nu})|)+\frac{F}{\nu}(e^{N^\circ t}+1)$

$\leq \frac{2F}{\nu}+(e^{N^\circ t}+1)(\frac{2F}{\nu})+\frac{F}{\nu}(e^{N^\circ t}+1)$

$=\frac{F}{\nu}(5+3e^{N^\circ t})$

to obtain that;

$|\overline{x}-s_{\overline{x}_1,t}(t)|\leq \frac{F}{\nu}(5+3e^{N^\circ t})$ (††)

Similarly, we obtain that;

$|\overline{y}-s_{\overline{x}_2,t}(t)|\leq \frac{F}{\nu}(5+3e^{N^\circ t})$ (†††)

so that, using $(\dagger),(\dagger\dagger),(\dagger\dagger\dagger)$, using the fact that $2F(5+3e^{N^\circ t})\leq \eta^{\frac{1}{3}}$, and with the choice $\nu\geq\eta$;

$|\overline{x}-\overline{y}|\leq |\overline{x}-s_{\overline{x}_1,t}(t)|+|\overline{y}-s_{\overline{x}_2,t}(t)|+|s_{\overline{x}_1,t}(t)-s_{\overline{x}_2,t}(t)|$

$\leq 2\frac{F}{\nu}(5+3e^{N^\circ t})+\frac{\epsilon}{\sqrt{\eta}}$

$\leq 2\frac{F}{\eta}(5+3e^{N^\circ t})+\frac{\epsilon}{\sqrt{\eta}}$

$\leq \eta^{\frac{-2}{3}}+\frac{\epsilon}{\sqrt{\eta}}$

$\leq \frac{1}{2\sqrt{\eta}}$

which contradicts the fact that $\{\overline{x},\overline{y}\}\subset R$, with $\overline{x}\neq\overline{y}$, so that $|\overline{x}-\overline{y}|\geq\frac{1}{\sqrt{\eta}}$, proving $(*)$. It follows that for the internal function $f_{t,r}:(R\cap{}^*B(\overline{0},r))\times(R\cap{}^*B(\overline{0},r))\setminus\Delta\rightarrow{}^*\mathcal{R}$, defined by;

$f_{t,r}(\overline{x},\overline{y})=\sqrt{\eta}|s_{\overline{x}}(t)-s_{\overline{y}}(t)|$

has the property that;

$f_{t,r}\geq\epsilon$ for all infinitesimals ϵ. By underflow, similarly to the above argument, we can find a constant $C_{t,r}\in\mathcal{R}_{>0}$, uniform in $\{\overline{x},\overline{y}\}$, with $f_{t,r}\geq C_{t,r}$ as required. □

Definition 1.13. *We define;*

$R_\eta=\{\overline{x}\in\mathcal{R}^3_\eta:[x_i\sqrt{\eta}]=x_i\sqrt{\eta},\ for\ 1\leq i\leq 3\}$

$\mathcal{T}_\nu=\bigcup_{0\leq j\leq\nu^2-1}[\frac{j}{\nu},\frac{j+1}{\nu})$

We extend the definition of trajectories, For $\overline{x}\in\mathcal{R}_\eta$, by defining;

$s_{\overline{x}}(\frac{j+1}{\nu})=s_{\overline{x}}(\frac{j}{\nu})+\frac{1}{\nu}\frac{\overline{J}_\eta(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}{\rho_\eta(\overline{s}_{\overline{x}}(\frac{j}{\nu}),\frac{j}{\nu})}$ *(mod $\mathcal{R}^3_\eta$), for $0\leq j\leq\nu^2-2$*

where for $x\in\mathcal{R}^*$, we define $x(mod\mathcal{R}_\eta)$, to be the unique $y\in\mathcal{R}_\eta$, for which $x-y=n\sqrt{\eta}$, with $n\in{}^*\mathcal{Z}$, and for $\overline{x}\in(\mathcal{R}^*)^3$, we extend the definition coordinate wise. For $t\in\mathcal{T}_\nu$, we define $s_{\overline{x}}(t)=s_{\overline{x}}(\frac{[t\nu]}{\nu})$.

For $t\in\mathcal{T}_\nu\setminus[\frac{\nu^2-1}{\nu},\nu)$, we define a point $s_{\overline{x}}(t)$ to be critical, if;

$|s_{\overline{x}}(t)-s_{\overline{x}}(t+\frac{1}{\nu})|>1$

Lemma 1.14. *For $\overline{x}\in\mathcal{R}^3_\eta$ infinite and finite $t_0\in\mathcal{T}_\nu$, we have that $s_{\overline{x}}(t_0)$ is infinite.*

Proof. Enumerate the $*$-finite set of critical points in the trajectory $s_{\overline{x}}$, by $\{s_i:1\leq i\leq\kappa\}$. For any such critical point s_{i_0}, we must have, using Definition 1.13 and the bound G in Lemma 1.6, that $|(s_{i_0})_i|>\frac{\eta-1}{2\sqrt{\eta}}-\frac{G}{\nu}$, for some $1\leq i\leq 3$. In particularly, such critical points in the trajectory are infinite. If $s_{\overline{x}}(\frac{[t_0\nu]}{\nu})$ is a critical point, the proof is complete. Otherwise, let s_{i_1} be the greatest element in the $*$-finite set consisting of critical points with the corresponding time $t_{i_1} \leq \frac{[t_0\nu]}{\nu}$, and use the proof of Lemma 1.7, to conclude that $|s_{i_1} - s_{\overline{x}}(\frac{[t_0\nu]}{\nu})| \leq \frac{Gt_0}{\nu}$. In particularly, as $\frac{Gt_0}{\nu}$ is finite, $s_{\overline{x}}(t_0)$ is infinite.

□

Definition 1.15. *For $t \in \mathcal{T}_\nu$, we define the range of the trajectories R_t by;*

$$R_t = \{s_{\overline{x}}(t) : \overline{x} \in \mathcal{R}^3_\eta\}$$

It is easy to see, as R_η is $*$-finite, and the definition of the trajectories is internal, that R_t is $*$-finite. Moreover, as $\{t \in \mathcal{T}_\nu : [t\nu] = t\nu\}$ is $*$-finite, we have that $\bigcup_{t\in\mathcal{T}_\nu} R_t$ is $*$-finite.

Lemma 1.16. *For all $S \subset {}^*\mathcal{R}^3$, with S $*$-finite and S infinite, there exists $\kappa_S \in {}^*\mathcal{N}$, κ_S odd and infinite, and $\overline{x}_S \in {}^*\mathcal{R}^3$, with $|\overline{x}_S| < \frac{1}{\kappa_S}$, such that if;*

$$V_{\kappa_S,\overline{x}_S} = \{\overline{x}_S + \overline{w} : w_i = \tfrac{j}{\sqrt{\kappa_S}}, -\tfrac{(\kappa_S-1)}{2} \leq j \leq \tfrac{(\kappa_S-1)}{2}, 1 \leq i \leq 3\}$$

then for all $\overline{v} \in V_{\kappa_S,\overline{x}_S}$, and $\{\overline{x}_1, \overline{x}_2\} \subset S$, with $\overline{x}_1 \neq \overline{x}_2$, we have that $|\overline{v} - \overline{x}_1| \neq |\overline{v} - \overline{x}_2|$, $(*)$.

Proof. For $S = \{s_1, \ldots s_n\}$ finite, then $\{\overline{w} \in \mathcal{R}^3 : |\overline{w} - s_i| = |\overline{w} - s_j|, \ for\ some s_i \neq s_j\}$ forms a finite union of hyperplanes $H_j \subset \mathcal{R}^3$, $1 \leq j \leq r$. Choose $m \in \mathcal{N}$, with m even, $m > n$, and let;

$$W_m = \{\overline{z} : z_i = \tfrac{j}{\sqrt{m}}, -\tfrac{(m-1)}{2} \leq j \leq \tfrac{(m-1)}{2}, 1 \leq i \leq 3\}$$

We have that $W_m \cap \bigcup_{1\leq j\leq r} H_j$ is a finite set of points $\{p_1, \ldots, p_s\}$. Let $t = min\{d(\overline{z}, \bigcup_{1\leq j\leq r} H_j) : \overline{z} \in W_m \setminus \{p_1, \ldots, p_s\}\}$, and choose $\overline{y} \in B(\overline{0}, \lambda)$, with $\lambda < min(t, \frac{1}{m}$, such that $\overline{y} - p_i \notin \bigcup_{1\leq j\leq r} H_j$, for $1 \leq j \leq s$. Then, it is clear that $(\overline{y} + W_m) \cap \bigcup_{1\leq j\leq r} H_j = \emptyset$. By construction, $\{m, \overline{y}, \overline{y} + W_m\}$ has the property $(*)$, relative to S. We can formulate the above property $(*)$ for finite sets in the language $L_{\mathcal{R}}$, using second order logic, developed in [8]. By transfer to ${}^*\mathcal{R}$, for any given $*$-finite set S, we can find $\{\kappa_S, \overline{x}_S, V_{\kappa_S,\overline{x}_S}\}$, with the same property $(*)$ relative to S. As $m > n$, with $m \in \mathcal{N}$ odd, where $n = Card(S)$, in the property $(*)$ for finite sets, $\kappa_S > {}^*CardS$, with $\kappa_S \in {}^*\mathcal{N}$ odd and infinite.

□

Lemma 1.17. *Let $S=\bigcup_{t\in\mathcal{T}_\nu}R_t$, then there exists $\kappa\in {}^*\mathcal{N}$, κ odd and infinite, with $\overline{x}_\kappa\in {}^*\mathcal{R}^3$, $|\overline{x}_\kappa|<\frac{1}{\kappa}\simeq 0$, such that if, as in Definitions 1.3 and 1.13;*

$\mathcal{R}_\kappa=\bigcup_{-\frac{(\kappa-1)}{2}\leq i\leq\frac{(\kappa-1)}{2}}[\frac{i}{\sqrt{\kappa}},\frac{i+1}{\sqrt{\kappa}})$

$R_\kappa=\{\overline{x}\in\mathcal{R}_\kappa^3:[x_i\sqrt{\kappa}]=x_i\sqrt{\kappa},\ for\ 1\leq i\leq 3\}$

then every $\overline{y}\in\overline{x}_\kappa+R_\kappa$ has a nearest neighbour in R_t, that is there exists $\overline{w}_t\in R_t$, with the property that;

$|\overline{y}-\overline{w}_t|<|\overline{y}-\overline{z}_t|$ *for every* $\overline{z}_t\in R_t$.

Proof. By Lemma 1.15, $S=\bigcup_{t\in\mathcal{T}_\nu}R_t$ is $*$-finite, and by Lemma 1.16, we can find $\kappa\in {}^*\mathcal{N}$, κ odd and infinite, with $\overline{x}_\kappa\in {}^*\mathcal{R}^3$, $|\overline{x}_\kappa|<\frac{1}{\kappa}$, and the property that there do not exist $\{\overline{x}_1,\overline{x}_2\}\subset S$, with $|\overline{y}-\overline{x}_1|=|\overline{y}-\overline{x}_2|$, for a given $\overline{y}\in\overline{x}_\kappa+R_\kappa$, $(*)$. In particular, the property $(*)$ holds for R_t, when $t\in\mathcal{T}_\nu$. Let $D_{t,\overline{y}}=\{|\overline{y}-\overline{z}_t|:\overline{z}_t\in R_t\}$, then D_t is $*$-finite, hence has a least element μ. Choosing $\overline{w}_t\in R_t$, with $|\overline{y}-\overline{w}_t|=\mu$, it is clear, by the property $(*)$, that $\overline{w}_t$ has the required property. □

Definition 1.18. *Let notation be as in the previous lemma. Let $\overline{w}\in R_t$, then we define the $*$-finite sets $I_{\overline{w},t}$ and $\overline{p}_{\overline{w},t}$, by;*

$I_{\overline{w},t}=\{\overline{x}\in R:s_{\overline{x}}(t)=\overline{w}\}$

$\overline{p}_{\overline{w},t}\in I_{\overline{w}}$, $\overline{p}_{\overline{w},t}<<\overline{x}$, $\overline{x}\in I_{\overline{w},t}$

$J_{\overline{w},t}=\{\overline{z}\in\overline{x}_\kappa+R_\kappa,\overline{z}\ is\ a\ nearest\ neighbour\ to\ \overline{w}\}$

where $<<$ is the lexographic total well order on $I_{\overline{w},t}$.

For $\overline{w}\in R_t$, we define $\lambda_{\overline{w},t}={}^*Card(J_{\overline{w},t})$. We define ρ_κ on $\overline{x}_\kappa+\mathcal{R}_\kappa^3\times\mathcal{T}_\nu$ by;

$\rho_\kappa(\overline{x}_\kappa+\overline{x},\frac{j}{\nu})=\frac{\kappa^{\frac{3}{2}}\rho_\eta(\overline{p}_{\overline{w_{\overline{x}}},t},0)}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}}$

where $\overline{w}_{\overline{x}}$ is the nearest neighbour to $\overline{x}_\kappa+\overline{x}$ in R_t, and $0\leq j\leq\nu^2-1$, $[x_i\sqrt{\kappa}]=x_i\sqrt{\kappa}$, for $1\leq i\leq 3$.

$\rho_\kappa(\overline{x}_\kappa+\overline{x},t)=\rho_\kappa(\overline{x}_\kappa+\overline{y},\frac{[t\nu]}{\nu})$

where $y_i=\frac{[x_i\sqrt{\kappa}]}{\sqrt{\eta}}$, $\overline{x}\in\overline{R}^3_\kappa$.

Lemma 1.19. *For* $\overline{y}\in\overline{R}^3_\kappa$ *and* $t\in\mathcal{T}_\nu$ *finite, with* $\nu\geq\eta$, *then* $|I_{\overline{w}_{\overline{x}},t}|=1$, *where* $x_i=\frac{[y_i\sqrt{\kappa}]}{\sqrt{\kappa}}$, *for* $1\leq i\leq 3$, *and* $\rho_\kappa(\overline{x}_\kappa+\overline{y},t)$ *is finite and not infinitesimal.*

Proof. Using Definition 1.18, if $\overline{z}\in I_{\overline{w}_{\overline{x}},t}$, then $\overline{z}\in R$ and $s_{\overline{z}}(t)=\overline{w}_{\overline{x}}$. If $\overline{z}$ is infinite, then, by Lemma 1.14, $s_{\overline{z}}(t)$ is infinite, so that $\overline{w}_{\overline{x}}$ is infinite, $(*)$. By Lemma 1.11, as $\overline{x}_\kappa+\overline{x}$ is finite, there exists $\overline{x}_1\in R$, with $s_{\overline{x}_1}(t)\simeq\overline{x}_\kappa+\overline{x}$, in particularly $s_{\overline{x}_1}(t)\in R_t$, and the nearest neighbour $\overline{w}_{\overline{x}}$ to $\overline{x}_\kappa+\overline{x}$ must be finite, contradicting $(*)$. It follows that $\overline{z}$ is finite. Now suppose that $I_{\overline{w}_{\overline{x}},t}$ contains two distinct finite elements $\{\overline{z}_1,\overline{z}_2\}\subset R$, then, using Lemma 1.8, $s_{\overline{z}_1}(t)\neq s_{\overline{z}_2}(t)$ and define distinct elements of R_t, so that, by the definition of $I_{\overline{w}_{\overline{x}},t}$, $\overline{w}_{\overline{x}}=s_{\overline{z}_1}(t)\neq s_{\overline{z}_2}(t)=\overline{w}_{\overline{x}}$, a contradiction. It follows that $|I_{\overline{w}_{\overline{x}},t}|=1$, and, using the definition of R_t, and the first argument again, that $\overline{p}_{\overline{w}_{\overline{x}},t}$ is finite, $(\sharp)$. It follows, by Definitions 1.3 and 1.18, that $\rho_\eta(\overline{p}_{\overline{w}_{\overline{x}},t},0)$ is finite. Let $\overline{w}\in R_t$, and let $C=(3+2F)e^{^\circ t_0}+6$. We claim that if $\overline{x}\in\overline{x}_\kappa+R_\kappa$, and $\overline{x}$ is a nearest neighbour to $\overline{w}$, then $\overline{x}\in{}^*B(\overline{w},\frac{C}{\sqrt{\eta}})$, $(\dagger)$. Suppose not, then there exists $\overline{x}\in\overline{x}_\kappa+R_\kappa$, such that $|\overline{x}-\overline{w}_{\overline{x}}|>\frac{C}{\sqrt{\eta}}$, where $\overline{w}_{\overline{x}}$ is the nearest neighbour to $\overline{x}$. By Lemma 1.11, using the fact that $\nu\geq\sqrt{\eta}$, we can find $\overline{x}'\in R_t$ with;

$|\overline{x}-\overline{x}'|\leq(e^{^\circ t_0}+2)(\frac{3}{\sqrt{\eta}}+\frac{F}{\nu})+\frac{Fe^{^\circ t_0}}{\nu}$

$\leq\frac{1}{\sqrt{\eta}}((3+2F)e^{^\circ t_0}+6)$

which implies that $|\overline{x}-\overline{x}'|<|\overline{x}-\overline{w}_{\overline{x}}|$, contradicting the fact that $\overline{w}_{\overline{x}}$ is the nearest neighbour to $\overline{x}$, proving $(\dagger)$. Using Lemmas 1.6 and 1.12, we can find $C_{t,r}\in\mathcal{R}_{>0}$, such that if $\overline{w}'\in R_t$, and $|\overline{w}-\overline{w}'|<1$, then $|\overline{w}-\overline{w}'|\geq\frac{C_{t,r}}{\sqrt{\eta}}$. It follow that if $\overline{x}\in\overline{x}_\kappa+R_\kappa$, and $\overline{x}\in{}^*B(\overline{w},\frac{C_{t,r}}{2\sqrt{\eta}})$, then $\overline{x}$ is a nearest neighbour to $\overline{w}$, $(\dagger\dagger)$. Suppose that $C_{t,r}\geq C$, then;

${}^*Card(\{\overline{x}\in\overline{x}_\kappa+R_\kappa:\overline{w}\ is\ a\ nearest\ neighbour\ to\ \overline{x}\})$

$={}^*Card((\overline{x}_\kappa+R_\kappa)\cap{}^*B(\overline{w},\frac{C}{\sqrt{\eta}}))$ $(\dagger\dagger\dagger)$

If $C_{t,r}<C$, then;

$^*Card(\overline{x}_\kappa + R_\kappa \cap {}^*B(\overline{w}, \frac{C_{t,r}}{\sqrt{\eta}}))$

$\leq {}^*Card(\{\overline{x} \in \overline{x}_\kappa + R_\kappa : \overline{w}\ is\ a\ nearest\ neighbour\ to\ \overline{x}\})$

$\leq {}^*Card((\overline{x}_\kappa + R_\kappa) \cap {}^*B(\overline{w}, \frac{C}{\sqrt{\eta}}))$ $(\dagger\dagger\dagger\dagger)$

Observe that ${}^*D(\overline{w}, \frac{r}{\sqrt{2}}) \subset {}^*B(\overline{w}, r) \subset {}^*D(\overline{w}, r)$, where ${}^*D(\overline{w}, r)$ is the $*$-cube of diameter $2r$ centred at $\overline{w}$.

We have, with the usual definition of $[,]$ the binomial theorem, and $r^2\kappa$ infinite,that;

$^*Card((\overline{x}_\kappa + R_\kappa) \cap {}^*D(\overline{w}, r))$

$= {}^*Card((\overline{v} + R_\kappa) \cap {}^*D(\overline{0}, r))$

$= (2[r\sqrt{\kappa}] + 1 - \delta_1)(2[r\sqrt{\kappa}] + 1 - \delta_2)(2[r\sqrt{\kappa}] + 1 - \delta_3)$

$= (2r\sqrt{\kappa} + \gamma_1)(2r\sqrt{\kappa} + \gamma_2)(2r\sqrt{\kappa} + \gamma_3)$

$= 8r^3\kappa^{\frac{3}{2}} + \theta_r$

where $v_i = x_{i,\kappa} - \frac{[x_{i,\kappa}\sqrt{\kappa}]}{\sqrt{\kappa}}$, for $1 \leq i \leq 3$, so that $0 \leq v_i < \frac{1}{\sqrt{\kappa}}$, and $\delta_i \in \{0, 1\}$, $1 \leq i \leq 3$, $0 \leq \gamma_i < 2$, $|\theta_r| \leq 50r^2\kappa$. It follows, from $(\dagger\dagger\dagger)$, that;

$\frac{8C^3\kappa^{\frac{3}{2}}}{(2\eta)^{\frac{3}{2}}} + \theta_{\frac{C}{\sqrt{2\eta}}} \leq \lambda_{\overline{w},t} \leq 8\frac{C^3\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}} + \theta_{\frac{C}{\sqrt{\eta}}}$ $(*)$

and, from $(\dagger\dagger\dagger\dagger)$, that;

$\frac{8C_{t,r}^3\kappa^{\frac{3}{2}}}{(2\eta)^{\frac{3}{2}}} + \theta_{\frac{C_{t,r}}{\sqrt{2\eta}}} \leq \lambda_{\overline{w},t} \leq 8\frac{C^3\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}} + \theta_{\frac{C}{\sqrt{\eta}}}$ $(**)$

It follows from $(*)$, that;

$$\frac{1}{8C^3 + \frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C}{\sqrt{\eta}}}} = \frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}(8\frac{C^3\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}} + \theta_{\frac{C}{\sqrt{\eta}}})} \leq \frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}} \leq \frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}(\frac{8C^3\kappa^{\frac{3}{2}}}{(2\eta)^{\frac{3}{2}}} + \theta_{\frac{C}{\sqrt{2\eta}}})} = \frac{1}{8\frac{C^3}{2^{\frac{3}{2}}} + \frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C}{\sqrt{2\eta}}}}$$

$$\frac{1}{8C^3 + \alpha} \leq \frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}} \leq \frac{1}{8\frac{C^3}{2^{\frac{3}{2}}} + \beta}$$

where $|\alpha| = |\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C}{\sqrt{\eta}}}| \leq 50C^2\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\frac{\kappa}{\eta} = 50C^2\frac{\eta^{\frac{1}{2}}}{\kappa^{\frac{1}{2}}} \simeq 0$

$|\beta|=|\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C}{\sqrt{2\eta}}}|\leq 25C^2\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\frac{\kappa}{\eta}=25C^2\frac{\eta^{\frac{1}{2}}}{\kappa^{\frac{1}{2}}}\simeq 0$

,

and, from (∗∗), that;

$$\frac{1}{8C^3+\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C}{\sqrt{\eta}}}}=\frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}(8\frac{C^3\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}+\theta_{\frac{C}{\sqrt{\eta}}})}\leq\frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}}\leq\frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}(\frac{8C_{t,r}^3\kappa^{\frac{3}{2}}}{(2\eta)^{\frac{3}{2}}}+\theta_{\frac{C_{t,r}}{\sqrt{2\eta}}})}=\frac{1}{8\frac{C_{t,r}^3}{2^{\frac{3}{2}}}+\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C_{t,r}}{\sqrt{2\eta}}}}$$

$$\frac{1}{8C^3+\alpha}\leq\frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}}\leq\frac{1}{8\frac{C_{t,r}^3}{2^{\frac{3}{2}}}+\gamma}$$

where $|\gamma|=|\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\theta_{\frac{C_{t,r}}{\sqrt{2\eta}}}|\leq 25C_{t,r}^2\frac{\eta^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\frac{\kappa}{\eta}=25C_{t,r}^2\frac{\eta^{\frac{1}{2}}}{\kappa^{\frac{1}{2}}}\simeq 0$

As $\{C,C_{t,r}\}\subset\mathcal{R}_{>0}$, and $\{\alpha,\beta,\gamma\}$ are infinitesimal, in both cases, we have that $\frac{\kappa^{\frac{3}{2}}}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t}}$ is finite and not infinitesimal. It follows, using (♯), Definition 1.18 and the fact that $1<\rho_{\eta,0}\leq H$, for some $H\in\mathcal{R}_{>0}$, that $\rho_\kappa(\overline{x}_\kappa+\overline{y},t)$ is finite and not infinitesimal as well. □

Definition 1.20. *We say that an internal set $V\subset\overline{x}_\kappa+\mathcal{R}^3_\kappa$ is finitely bounded, if there exists a constant $r\in\mathcal{R}_{>0}$ such that $V\subset\overline{x}_\kappa+{}^*B(\overline{0},r)$. We say that f is measurable on $\overline{x}_\kappa+\mathcal{R}^3_\kappa$ if;*

$f(\overline{x}_\kappa+\overline{z})=f(\overline{x}_\kappa+\overline{y})$

for $\overline{z}\in\mathcal{R}^3_\kappa$ and $y_i=\frac{[z_i\sqrt{\kappa}]}{\sqrt{\kappa}}$, for $1\leq i\leq 3$. We let μ_κ be the internal counting measure on $\mathcal{R}^3_\kappa$. By Definition 1.18, we have that $\rho_{\kappa,t}$ is measurable on $\overline{x}_\kappa+\mathcal{R}^3_\kappa$ for $t\in\mathcal{T}_\nu$. For f measurable on $\overline{x}_\kappa+\mathcal{R}^3_\kappa$, and $V\subset\overline{x}_\kappa+\mathcal{R}^3_\kappa$ finitely bounded, we define the internal integral;

$\int_V fd\mu_\kappa=\frac{1}{\kappa^{\frac{3}{2}}}{}^*\sum_{\overline{x}\in(\overline{x}_\kappa+R_\kappa)\cap V}f(\overline{x})$

We let $\mathfrak{C}_\kappa$ be the internal ∗-sigma algebra generated by the internal sets;

$\{V_{\overline{x}}:\overline{x}\in\overline{x}_\kappa+R_\kappa\}$

where $V_{\overline{x}}=\{\overline{y}\in\overline{x}_\kappa+\mathcal{R}^3_\kappa:0\leq y_i-x_i<\frac{1}{\sqrt{\kappa}},\ 1\leq i\leq 3\}$, and let $L(\mathfrak{C}_\kappa)$ be the σ-algebra generated by $\mathfrak{C}_\kappa$. We let $L(\mu_\kappa)$ be the Loeb measure on $L(\mathfrak{C}_\kappa)$, associated to μ_κ, see [5] or [4]. For $V\in L(\mathfrak{C}_\kappa)$, we say that V is finitely bounded if there exists a constant $r\in\mathcal{R}_{>0}$ such that $V\subset\overline{x}_\kappa+{}^*B(\overline{0},r)$. For f measurable and finite on $\overline{x}_\kappa+\mathcal{R}^3_\kappa$, with *standard part* $^{\circ}f$*, we denote the Loeb integral by;*

$\int_V {}^{\circ}f dL(\mu_\kappa)$

Note that the Loeb measure $L(\mu_\kappa)$ *specialises to Lebesgue measure* $d\mu$ *on* $\mathcal{R}^3$*, see* [1]*. This implies that, for* $V \in L(\mathfrak{C}_\kappa)$*, finitely bounded, with* $V = st^{-1}(W)$*, for* W *Lebesgue measurable in* $\mathcal{R}^3$ *and* g S*-continuous;*

$\int_V g(\overline{x}) dL(\mu_\kappa)(\overline{x}) = \int_W g({}^{\circ}\overline{x}) d\mu({}^{\circ}\overline{x})$

Lemma 1.21. *The* $*$*-intersection* W *of an internal collection* $\{V_i : i \in I\}$*,* $V_i \subset \overline{x}_\kappa + \mathcal{R}^3_\kappa$*, indexed by internal* I*, of* $*$*-closed sets, is internal and* $*$*-closed. If the collection contains a finitely bounded set, then* W *is finitely bounded. If* f *is measurable on* $\overline{x}_\kappa + \mathcal{R}^3_\kappa$ *and finite on* V*, with* $V \subset \overline{x}_\kappa + \mathcal{R}^3_\kappa$ *finitely bounded and internal, then* $\int_V f d\mu_\kappa$ *is finite. Similarly, if* f *is finite on* $V \in L(\mathfrak{C}_\kappa)$*, with* $|f| \leq D$*,* $D \in \mathcal{R}_{>0}$*, then* $\int_V {}^{\circ}f dL(\mu_\kappa)$ *is finite.*

Proof. The claim that W is $*$-closed and internal, follows by transfer from the fact that a finite intersection of closed sets $\{V_1, \ldots, V_n\}$, $V_i \subset \mathcal{R}^3$, $1 \leq i \leq n$ is a closed set. If some V_1 is finitely bounded, the fact that W is finitely bounded follows from the fact that $W \subset \overline{x}_\kappa + V_1$, with $V_1 \subset {}^*B(\overline{0}, r)$, and $r \in \mathcal{R}_{>0}$. For the final claim, choose ${}^*B(\overline{0}, r)$, with $V \subset \overline{x}_\kappa + {}^*B(\overline{0}, r)$, and let $S(r) = {}^*[-r, r]^3 \supset {}^*B(\overline{0}, r)$. There exists a constant $C \in \mathcal{R}_{>0}$, with $|f|_V| \leq C$, otherwise, by compactness and the fact that V is internal, f would not be finite on V. Then;

$|\int_V f d\mu_\kappa| \leq \int_V |f| d\mu_\kappa \leq C \int_V d\mu_\kappa$

$= \frac{C}{\kappa^{\frac{3}{2}}} {}^*Card(\{\overline{x} \in (\overline{x}_\kappa + R_\kappa) \cap V\})$

$\leq \frac{C}{\kappa^{\frac{3}{2}}} {}^*Card(\{\overline{x} \in (\overline{x}_\kappa + R_\kappa) \cap \overline{x}_\kappa + S(r)\})$

$= \frac{C}{\kappa^{\frac{3}{2}}} (2[r\sqrt{\kappa}] + 1)^3$

$\leq \frac{C}{\kappa^{\frac{3}{2}}} (2r\sqrt{\kappa} + 3)^3$

$\leq \frac{C}{\kappa^{\frac{3}{2}}} ((2r + 1)\sqrt{\kappa})^3$

$= C(2r + 1)^3$

which is finite, as required. The second claim, using the definition of finitely bounded, the remark in Definition 1.20 and the fact that ${}^{*}B(0,r)\subset st^{-1}(B(0,r+1))$, for $r\in\mathcal{R}_{>0}$, follows from;

$|\int_{V}{}^{\circ}f dL(\mu_{\kappa})|$

$\leq\int_{V}|{}^{\circ}f|dL(\mu_{\kappa})|$

$\leq D\int_{V}dL(\mu_{\kappa})$

$\leq D\int_{\overline{x}_{\kappa}+st^{-1}(B(\overline{0},r+1))}dL(\mu_{\kappa})$

$=D\int_{B(\overline{0},r+1)}d\mu=\frac{4D\pi(r+1)^{3}}{3}$

where $d\mu$ is Lebesgue measure.

□

Definition 1.22. *Transport of Open Balls*

Let $V\subset\overline{x}_{\kappa}+\mathcal{R}_{\kappa}^{3}$, $V=\overline{x}_{\kappa}+W$, $W=st^{-1}(B(\overline{0},r))$, *for* $r\in\mathcal{R}_{>0}$. *Let* $\{t_1,t_2\}\subset\mathcal{T}_{\nu}$, *with* $t_1\leq t_2$. *Let* $S_V=R_t\cap V$ *and let;*

$S_{V,t_1,t_2}=\{s_{\overline{x}}(t_2):\overline{x}\in R, s_{\overline{x}}(t_1)\in S_V\}$

We define the transport V_{t_1,t_2} *of* V *to be the intersection of all* $V_i\subset\overline{x}_{\kappa}+\mathcal{R}_{\kappa}^{3}$, *where* $V_i\supset S_{V,t_1,t_2}$, $V_i=\overline{x}_{\kappa}+W_i$, $W_i=st^{-1}(Z_i)$, *for* Z_i *open in* $\mathcal{R}^3$. *Note that, by the remark in Definition 1.20,* $V_i\in L(\mathfrak{C}_{\kappa})$, *and, as* $L(\mathfrak{C}_{\kappa})$ *is a* σ*-algebra,* $V_{t_1,t_2}\in L(\mathfrak{C}_{\kappa})$ *as well.*

Lemma 1.23. *For any* μ_{κ} *measurable* $A\subset\mathcal{R}_{\kappa}^{3}$, *which is finitely bounded, we have that;*

$\int_{A}\rho_{\kappa}d\mu_{\kappa}\simeq\int_{A}{}^{\circ}\rho_{\kappa}dL(\mu_{\kappa})$

Proof. We claim, that ρ_{κ} is S-integrable on A, for the $*$-σ algebra $\mathcal{C}_{A,\kappa}$, generated by intersecting $\mathcal{C}_{\kappa}$ with A. As $\mu_{\kappa}(A)$ is finite, by Definition 3.17 of [5], we have to check that;

(i). $\int_{A}|\rho_{\kappa}|d\mu_{\kappa}$ is finite.

(ii). For $B\in\mathcal{C}_{A,\kappa}$, with $\mu_{\kappa}(B)\simeq 0$;

$\int_B |\rho_\kappa| d\mu_\kappa \simeq 0$

For (i), we have, by Lemma 1.19, that $\rho_\kappa|_A$ is finite. It follows, using overflow to $\{n \in {}^*\mathcal{N} : \exists x \in A, \rho_\kappa(x) > n\}$, that there exists a constant $D \in \mathcal{R}_{>0}$, with $|\rho_\kappa|_A| \leq D$. Then;

$\int_A |\rho_\kappa| d\mu_\kappa \leq C \int_A d\mu_\kappa$

$= C\mu_\kappa(A)$

which is finite, and;

$\int_B |\rho_\kappa| d\mu_\kappa \leq C \int_B d\mu_\kappa$

$= C\mu_\kappa(B) \simeq 0$

It then follows, by Theorem 3.20 of [5], that;

$^\circ \int_A \rho_\kappa d\mu_\kappa = \int_A {}^\circ\rho_\kappa dL(\mu_\kappa)$

as required.

□

Lemma 1.24. *Let notation be as in Definition 1.22, then the transport V_{t_1,t_2} of V, is finitely bounded. Moreover, $\int_V {}^\circ\rho_{\kappa,t_1} dL(\mu_\kappa)$ is finite and;*

$\int_V {}^\circ\rho_{\kappa,t_1} dL(\mu_\kappa) = \int_{V_{t_1,t_2}} {}^\circ\rho_{\kappa,t_2} dL(\mu_\kappa)$

Proof. For the first part, choose $\epsilon > 0$, with $V \subset {}^*B(\overline{0}, r+\epsilon)$. Using the proof of Lemma 1.7, we have that, for $s_{\overline{x}}$, with $s_{\overline{x}}(t_1) \in S_V$;

$|s_{\overline{x}}(t_2)| \leq |s_{\overline{x}}(t_1)| + G(t_2 - t_1)$

$\leq r + \epsilon + G(t_2 - t_1) = E$

We then have that, as $\overline{x}_\kappa$ is infinitesimal, that $\overline{x}_\kappa + st^{-1}(B(\overline{0}, E + 1)) \supset S_{V,t_1,t_2}$, so that the transport $V_{t_1,t_2} \subset \overline{x}_\kappa + st^{-1}(B(\overline{0}, E+1)) \subset \overline{x}_\kappa + {}^*B(\overline{0}, {}^\circ(E+2))$. As ${}^\circ(E+2) \in \mathcal{R}_{>0}$, we obtain the result by Definition 1.20. For the second part, keeping the same notation as above, let $W = (\overline{x}_\kappa + R_\kappa) \cap {}^*B(\overline{0}, r+\epsilon)$, then $W \supset V$ is internal and, by Lemma 1.19, ρ_{κ,t_1} is finite on W. By the argument in Lemma 1.21, we can find a constant $C \in \mathcal{R}_{>0}$, with $|\rho_{\kappa,t_1}|_W| < C$. In particular,

$|\rho_{\kappa,t_1}|_V| < C$, so that $|{}^\circ\rho_{\kappa,t_1}|_V| < C$. It follows that;

$|\int_V {}^\circ\rho_{\kappa,t_1} dL(\mu_\kappa)| \leq \int_V |{}^\circ\rho_{\kappa,t_1}| dL(\mu_\kappa)|$

$\leq C \int_V dL(\mu_\kappa)$

$= C \int_{\overline{x}_\kappa + st^{-1}(B(\overline{0},r))} dL(\mu_\kappa)$

$= C \int_{B(\overline{0},r)} d\mu = \frac{4C\pi r^3}{3}$

as required. We claim that if $\overline{y} \in R_{t_2} \cap V_{t_1,t_2}$, with $s_{\overline{x}}(t_2) = \overline{y}$, then, for all $\epsilon \in \mathcal{R}_{>0}$, there exists $\overline{z} \in R_{t_1} \cap (\overline{x}_\kappa + {}^*B(\overline{0}, r+\epsilon))$ with $s_{\overline{x}'}(t_1) = \overline{z}$ and $s_{\overline{x}'}(t_2) \simeq \overline{y}$, $(\dagger)$. Suppose not, then for every $\overline{w} \in R_{t_1} \cap {}^*B(\overline{0}, r+\epsilon)$, we have that $|s_{\overline{p}_{\overline{w}}}(t_2) - \overline{y}| \geq \delta$, for all infinitesimals δ. Let;

$Z = \{\overline{p}_{\overline{w}} : \overline{w} \in R_{t_1} \cap {}^*B(\overline{0}, r+\epsilon)\}$

Then, as $R_{t_1} \cap {}^*B(\overline{0}, r+\epsilon)$ is $*$-finite, so is Z. Define g on Z, by $g(\overline{p}_{\overline{w}}) = |\overline{y} - s_{\overline{p}_{\overline{w}}}(t_2)|$. Then, using the fact that $|g| \geq \delta$, for all infinitesimals δ, by the usual underflow argument, we can find $\gamma \in \mathcal{R}_{>0}$, with $|g| \geq \gamma$. Let $K = (\overline{x}_\kappa + st^{-1}(A^c)$ where A is the closed ball;

$A = \{\overline{z} \in \mathcal{R}^3 : |\overline{z} - {}^\circ\overline{y}| \leq \frac{\gamma}{2}\}$

Then, A^c is an open set such that $\overline{y} \notin \overline{x}_\kappa + st^{-1}(A^c)$. As $\overline{x}_\kappa + st^{-1}(A^c) \supset S_{V,t_1,t_2}$, $V_{t_1,t_2} \subset \overline{x}_\kappa + st^{-1}(A^c) \cap V_{t_1,t_2}$, so that $\overline{y} \in \overline{x}_\kappa + st^{-1}(A^c)$, a contradiction. For the next claim, let $r_n = r(1-\frac{1}{n})$, then, using the notation in Definition 1.25, we claim that $W_{t_1,r_n} \subset V$ and $W_{t_1,t_2,r_n} \subset V_{t_1,t_2}$, $(\sharp)$. The first part of $(\sharp)$ is clear. If $W_i = st^{-1}(Z_i)$, with $\overline{x}_\kappa + W_i \supset S_{V,t_1,t_2}$, we have that $\overline{x}_\kappa + W_i \supset U_{t_1,t_2,r_n}$, and, if $\overline{x} \in W_{t_1,t_2,r_n}$, we can find $\overline{w} \in U_{t_1,t_2,r_n}$, with $\overline{x} \simeq \overline{w}$. Using the fact that Z_i is open, choose $\epsilon \in \mathcal{R}_{>0}$, such that $\overline{x}_\kappa + {}^*B(\overline{w}, \epsilon)$, with $\overline{x}_\kappa + {}^*B(\overline{w}, \epsilon) \subset \overline{x}_\kappa + W_i$, so that $\overline{x} \in \overline{x}_\kappa + W_i$. It follows that $\overline{x} \in V_{t_1,t_2}$, proving $(\sharp)$. Using the result of Lemma 1.26, we have that;

$\int_{W_{t_1,r_n}} \rho_\kappa d\mu_\kappa = \int_{W_{t_1,t_2,r_n}} \rho_\kappa d\mu_\kappa$ $(\sharp\sharp)$

so that, using $(\sharp\sharp)$, Lemma 1.23, for all r_n, with $n \in \mathcal{N}$;

$\int_V {}^\circ\rho_\kappa dL(\mu_\kappa) = \int_{W_{t_1,r_n}} {}^\circ\rho_\kappa dL(\mu_\kappa) + \int_{(V \setminus W_{t_1,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$

$\simeq \int_{W_{t_1,r_n}} \rho_\kappa d\mu_\kappa + \int_{(V\setminus W_{t_1,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$

$\int_{V_{t_1,t_2}} {}^\circ\rho_\kappa dL(\mu_\kappa) = \int_{W_{t_1,t_2,r_n}} {}^\circ\rho_\kappa dL(\mu_\kappa) + \int_{(V_{t_1,t_2}\setminus W_{t_1,t_2,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$

$\simeq \int_{W_{t_1,t_2,r_n}} \rho_\kappa d\mu_\kappa + \int_{(V_{t_1,t_2}\setminus W_{t_1,t_2,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$

$= \int_{W_{t_1,r_n}} \rho_\kappa d\mu_\kappa + \int_{(V_{t_1,t_2}\setminus W_{t_1,t_2,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$

$\simeq \int_V {}^\circ\rho_\kappa dL(\mu_\kappa) - \int_{(V\setminus W_{t_1,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa) + \int_{(V_{t_1,t_2}\setminus W_{t_1,t_2,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)$ $(\sharp\sharp\sharp)$

As $max(|\rho_\kappa|_V|, |\rho_\kappa|_{V_{t_1,t_2}}|) \leq H$ for some $H \in \mathcal{R}_{>0}$, we have that;

$|\int_{(V\setminus W_{t_1,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)| \leq (H+1)L(\mu_\kappa)((V \setminus W_{t_1,r_n}))$

$|\int_{(V_{t_1,t_2}\setminus W_{t_1,t_2,r_n})} {}^\circ\rho_\kappa dL(\mu_\kappa)| \leq (H+1)L(\mu_\kappa)((V_{t_1,t_2} \setminus W_{t_1,t_2,r_n}))$ $(***)$

By the proof above, we have, for any $\delta \in \mathcal{R}_{>0}$, that if $\overline{x} \in V_{t_1,t_2}$, we can find $\overline{y} \in \overline{x}_\kappa + {}^*B(\overline{0}, r+\delta)$ with $s_{\overline{p}_{\overline{y},t_1}}(t_2) \simeq \overline{x}$. Suppose that $\overline{x} \notin W_{t_1,t_2,r_n}$, then if $\overline{y} \in \overline{x}_\kappa + {}^*B(\overline{0}, r_{n-1})$, we claim that we can find $\overline{y}' \in \overline{x}_\kappa + {}^*B(\overline{0}, r_n)$, $\overline{y}' \simeq \overline{y}$, such that $\overline{x}' \in J_{s_{\overline{p}_{\overline{y}',t_1}}(t_2),t_2}$, where $x_i'\sqrt{\kappa} = [x_i\sqrt{\kappa}]$, for $1 \leq i \leq 3$, $(\sharp\sharp\sharp\sharp)$, contradicting the fact that $\overline{x} \notin W_{t_1,t_2,r_n}$, so that $\overline{y} \notin \overline{x}_\kappa + {}^*B(\overline{0}, r_{n-1})$, $\overline{y}' \notin \overline{x}_\kappa + {}^*B(\overline{0}, r_{n-2})$, $\overline{x} \in W_{t_1,t_2,r_{n-2},1+\delta}$. To prove $(\sharp\sharp\sharp\sharp)$, let $\overline{w}_{\overline{x}'}$ be the nearest neighbour to $\overline{x}'$, and let $\overline{y}' \in R_{t_1}$, with $s_{\overline{p}_{\overline{y}'}}(t_2) = \overline{w}_{\overline{x}'}$. Suppose that $|\overline{y}' - \overline{y}| > \epsilon$, for some $\epsilon \in \mathcal{R}_{>0}$, then, using the proof of Lemma 1.12, we have that $|\overline{w}_{\overline{x}'} - s_{\overline{p}_{\overline{y},t_1}}(t_2)| \geq D\epsilon$, for some $D \in \mathcal{R}_{>0}$. As $\overline{w}_{\overline{x}'} \simeq \overline{x}' \simeq \overline{x} \simeq s_{\overline{p}_{\overline{y},t_1}}(t_2)$, we obtain a contradiction, so that $\overline{y}' \simeq \overline{y}$. As $\overline{y} \in \overline{x}_\kappa + {}^*B(\overline{0}, r_{n-1})$, and $\overline{y}' \simeq \overline{y}$, clearly $\overline{y}' \in \overline{x}_\kappa + {}^*B(\overline{0}, r_n)$, as required. It follows that $(V_{t_1,t_2} \setminus W_{t_1,t_2,r_n}) \subset W_{t_1,t_2,r_{n-2},1+\delta}$. It follows, using the final result of Lemma 1.26, that;

$L(\mu_\kappa)((V_{t_1,t_2} \setminus W_{t_1,t_2,r_n})) \leq L(\mu_\kappa)(W_{t_1,t_2,r_{n-2},1+\delta})$

$\simeq \mu_\kappa(W_{t_1,t_2,r_{n-2},1+\delta})$

$\leq D(1+\delta-r_{n-2})$

An easier calculation shows that, for arbitrary $\delta \in \mathcal{R}_{>0}$, we can find a constant $E \in \mathcal{R}_{>0}$, with;

$L(\mu_\kappa)((V \setminus W_{t_1,r_n}))$

$\leq E(1+\delta-r_n)$ $(****)$

Letting $n \rightarrow \infty$, and using the fact that $\delta \in \mathcal{R}_{>0}$, was arbitrary, the results of $(\sharp\sharp\sharp)$, $(***)$ and $(****)$, we conclude;

$\int_{V_{t_1,t_2}} {}^\circ\rho_\kappa dL(\mu_\kappa) \simeq \int_V {}^\circ\rho_\kappa dL(\mu_\kappa)$

and, therefore;

$\int_{V_{t_1,t_2}} {}^\circ\rho_\kappa dL(\mu_\kappa) = \int_V {}^\circ\rho_\kappa dL(\mu_\kappa)$

as required.

□

Definition 1.25. *Let $\{r, r_1, r_2\} \subset \mathcal{R}_{>0}$, with $r_1 < r_2$ and $\{t_1, t_2\} \subset \mathcal{T}_\nu$ finite, with $t_1 < t_2$, then we define;*

$U_{t_1,r} = R_t \cap (\overline{x}_K + {}^*B(\overline{0}, r))$

$V_{t_1,r} = {}^*\bigcup_{\overline{w} \in U_{t,r}} J_{\overline{w},t_1}$

$W_{t_1,r} = {}^*\bigcup_{\overline{x} \in V_{t,r}} V_{\overline{x}}$

$U_{t_1,t_2,r} = \{s_{\overline{p}_{\overline{w},t_1}}(t_2) : \overline{w} \in U_{t_1,r}\}$

$V_{t_1,t_2,r} = {}^*\bigcup_{\overline{w} \in U_{t_1,t_2,r}} J_{\overline{w},t_2}$

$W_{t_1,t_2,r} = {}^*\bigcup_{\overline{x} \in V_{t_1,t_2,r}} V_{\overline{x}}$

$U_{t_1,r_1,r_2} = R_t \cap (\overline{x}_K + {}^*A(\overline{0}, r_1, r_2))$

$V_{t_1,r_1,r_2} = {}^*\bigcup_{\overline{w} \in U_{t,r}} J_{\overline{w},t_1}$

$W_{t_1,r_1,r_2} = {}^*\bigcup_{\overline{x} \in V_{t,r}} V_{\overline{x}}$

$U_{t_1,t_2,r_1,r_2}=\{s_{\overline{p}_{\overline{w},t_1}}(t_2):\overline{w}\in U_{t_1,r}\}$

$V_{t_1,t_2,r_1,r_2}={}^*\bigcup_{\overline{w}\in U_{t_1,t_2,r}}J_{\overline{w},t_2}$

$W_{t_1,t_2,r_1,r_2}={}^*\bigcup_{\overline{x}\in V_{t_1,t_2,r}}V_{\overline{x}}$

where $B(\overline{0},r)=\{\overline{z}\in\mathcal{R}^3:|\overline{z}|<r\}$ *and* $A(\overline{0},r_1,r_2)=\{\overline{z}\in\mathcal{R}^3:r_1<|\overline{z}|<r_2\}$

Lemma 1.26. *We have that* $\{U_{t_1,r},V_{t_1,r},U_{t_1,t_2,r},V_{t_1,t_2,r}\}$ *are* $*$*-finite,* $\{W_{t_1,r},W_{t_1,t_2,r}\}$ *are* μ_κ *measurable, and there exists a constant* $C\in\mathcal{R}_{>0}$*, with* $\mu_\kappa(W_{t_1,t_2,r})\leq Cr^3$*. Moreover;*

$\int_{W_{t_1,r}}\rho_\kappa d\mu_\kappa=\int_{W_{t_1,t_2,r}}\rho_\kappa d\mu_\kappa$

We have that $\{U_{t_1,r_1,r_2},V_{t_1,r_1,r_2},U_{t_1,t_2,r_1,r_2},V_{t_1,t_2,r_1,r_2}\}$ *are* $*$*-finite,* $\{W_{t_1,r_1,r_2},W_{t_1,t_2,r_1,r_2}\}$ *are* μ_κ *measurable, and there exists a constant* $D\in\mathcal{R}_{>0}$*, with* $\mu_\kappa(W_{t_1,t_2,r})\leq D(r_2-r_1)$*,* $r_1>\frac{r_2}{3}$*,* $r_2<1$*.*

Proof. The fact that $U_{t_1,r}$ is $*$-finite follows from the facts that $(\overline{x}_K+{}^*B(\overline{0},r))$ is internal, R_t is $*$-finite, and an internal subset of a $*$-finite set is $*$-finite. $V_{t_1,r}$ is $*$-finite as each $J_{\overline{w},t_1}$ is $*$-finite, $U_{t_1,r}$ is $*$-finite and a $*$-finite union of $*$-finite sets is $*$-finite. $U_{t_1,t_2,r}$ is $*$-finite as it is in internal bijection with $U_{t_1,r}$, and $V_{t_1,t_2,r}$ is $*$-finite, for the sam reason as $V_{t_1,r}$. $W_{t_1,r}$ and $W_{t_1,t_2,r}$ are μ_κ measurable, using Definition 1.20. For the next claim, choose $\overline{w}_0\in U_{t_1,r}$, then, for any $\overline{w}\in U_{t_1,r}$, we have that $|\overline{w}-\overline{w}_0|<2r$. It follows, from the proof of Lemma 1.9, that there exists a constant $E\in\mathcal{R}_{>0}$, with $|\overline{w}'-s_{\overline{p}_{\overline{w}_0,t_1}}(t_2)|<Er$, for any $\overline{w}'\in U_{t_1,t_2,r}$. If $\overline{x}\in V_{t_1,t_2,r}$, then there exists $\overline{w}'\in U_{t_1,t_2,r}$ with $\overline{x}\in J_{\overline{w}',t_2}$, and, by (†) in Lemma 1.19, we have there exists a constant $F\in\mathcal{R}_{>0}$, with $|\overline{x}-\overline{w}'|<\frac{F}{\sqrt{\eta}}$. It follows, by the triangle inequality, and the fact that $\frac{1}{\sqrt{\eta}}$ is infinitesimal, that $|\overline{x}-s_{\overline{p}_{\overline{w}_0,t_1}}(t_2)|<Er+\frac{F}{\sqrt{\eta}}<(E+1)r$. In particular;

$W_{t_1,t_2,r}\subset{}^*B(s_{\overline{p}_{\overline{w}_0,t_1}}(t_2),(E+2)r)$

$\subset{}^*B({}^\circ s_{\overline{p}_{\overline{w}_0,t_1}}(t_2),(E+3)r)$

$\subset st^{-1}(B({}^\circ s_{\overline{p}_{\overline{w}_0,t_1}}(t_2),(E+4)r))$

It follows that;

$\mu_\kappa(W_{t_1,t_2,r}) \simeq L(\mu_\kappa)(W_{t_1,t_2,r})$

$\leq L(\mu_\kappa)(st^{-1}(B({}^\circ s_{\overline{p}_{\overline{w}_0,t_1}}(t_2),(E+4)r)))$

$= \frac{4\pi(E+1)^3r^3}{3}$

and;

$\mu_\kappa(W_{t_1,t_2,r}) \leq Cr^3$, where $C = \frac{4\pi(E+2)^3}{3}$.

We have, using Definition 1.18, that;

$\int_{W_{t_1,r}} \rho_\kappa d\mu_\kappa = \frac{1}{\kappa^{\frac{3}{2}}} {}^*\sum_{\overline{x}\in V_{t_1,r}} \frac{\kappa^{\frac{3}{2}}\rho_\eta(\overline{p}_{\overline{w_{\overline{x}}},t_1},0)}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t_1}}$

$= {}^*\sum_{\overline{x}\in V_{t_1,r}} \frac{\rho_\eta(\overline{p}_{\overline{w_{\overline{x}}},t_1},0)}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t_1}}$

$= {}^*\sum_{\overline{w}\in U_{t_1,r}} \frac{\rho_\eta(\overline{p}_{\overline{w},t_1},0)}{\eta^{\frac{3}{2}}}$

$= {}^*\sum_{\overline{w}\in U_{t_1,t_2,r}} \frac{\rho_\eta(\overline{p}_{\overline{w},t_2},0)}{\eta^{\frac{3}{2}}}$

$= {}^*\sum_{\overline{x}\in V_{t_1,t_2,r}} \frac{\rho_\eta(\overline{p}_{\overline{w_{\overline{x}}},t_2},0)}{\eta^{\frac{3}{2}}\lambda_{\overline{w_{\overline{x}}},t_2}}$

$= \frac{1}{\kappa^{\frac{3}{2}}} {}^*\sum_{\overline{x}\in V_{t_1,t_2,r}} \frac{\kappa^{\frac{3}{2}}\rho_\eta(\overline{p}_{\overline{w_{\overline{x}}},t_2},0)}{\eta^{\frac{3}{2}}\lambda_{\overline{w},t_2}} = \int_{W_{t_1,t_2,r}} \rho_\kappa d\mu_\kappa$

Similarly, $\{U_{t_1,r_1,r_2}, V_{t_1,r_1,r_2}, U_{t_1,t_2,r_1,r_2}, V_{t_1,t_2,r_1,r_2}\}$ are $*$-finite, replacing ${}^*B(\overline{0},r)$ with ${}^*A(\overline{0},r_1,r_2))$, and the same for $\{W_{t_1,r_1,r_2}, W_{t_1,t_2,r_1,r_2}\}$ being μ_κ measurable. For the last claim, we can cover $\overline{A(\overline{0},r_1,r_2)}$ with N_{r_1,r_2} open balls $B(\overline{x}_i, \frac{r_2-r_1}{2})$, where each $x_i \in \overline{A(\overline{0},r_1,r_2)}$, such that, for $r_1 > \frac{r_2}{3}$, each $B(\overline{x}_i, \frac{r_2-r_1}{2}) \subset A(\overline{0}, \frac{3r_1-r_2}{2}, \frac{3r_2-r_1}{2})$, We have that;

$N_{r_1,r_2} vol(B(\overline{x}_i, \frac{r_2-r_1}{2})) - vol(T) \leq vol(A(\overline{0}, \frac{3r_1-r_2}{2}, \frac{3r_2-r_1}{2}))$

where T is the set of overlaps, and clearly $vol(T) \leq vol(A(\overline{0}, \frac{3r_1-r_2}{2}, \frac{3r_2-r_1}{2}))$, so that;

$vol(B(\overline{x}_i, \frac{r_2-r_1}{2})) \leq \frac{2vol(A(\overline{0},\frac{3r_1-r_2}{2},\frac{3r_2-r_1}{2}))}{N_{r_1,r_2}}$ $(*)$

and a corresponding result holds for the transfers. We have that;

$V_{t_1,r_1,r_2} \subset \bigcup_{1\leq i\leq N_{r_1,r_2}} V_{\overline{x}_i,t_1,r'}$

where $V_{\overline{x}_i,t_1,r'}$ is defined by $\overline{x}_\kappa +^* B(\overline{x}_i, r')$ and, by $(*)$, $r' \leq (\frac{3vol(A(\overline{0},\frac{3r_1-r_2}{2},\frac{3r_2-r_1}{2}))}{2\pi N_{r_1,r_2}})^{\frac{1}{3}}$, $(**)$. By the previous result, using $(**)$, we have that;

$V_{t_1,t_2,r_1,r_2} \subset \bigcup_{1\leq i\leq N_{r_1,r_2}} V_{\overline{x}_i,t_1,t_2,r'}$

$\mu_\kappa(W_{t_1,t_2,r_1,r_2}) \leq \mu_\kappa(\bigcup_{1\leq i\leq N_{r_1,r_2}} V_{\overline{x}_i,t_1,t_2,r'})$

$\leq N_{r_1,r_2} max_i(\mu_\kappa(V_{\overline{x}_i,t_1,t_2,r'}))$

$\leq CN_{r_1,r_2}\frac{4\pi r'^3}{3}$

$\leq \frac{4\pi CN_{r_1,r_2}}{3}\frac{3vol(A(\overline{0},\frac{3r_1-r_2}{2},\frac{3r_2-r_1}{2}))}{2\pi N_{r_1,r_2}}$

$= 6Cvol(A(\overline{0},\frac{3r_1-r_2}{2},\frac{3r_2-r_1}{2}))$

$\leq 6\pi C(\frac{r_2-r_1}{2})(4(r_2+r_1)^2+16(r_2-r_1)^2)$

$= 3\pi C(r_2-r_1)(4(r_2+r_1)^2+16(r_2-r_1)^2)$

$\leq 3\pi C(r_2-r_1)(16+16)$

$= D(r_2-r_1)$

where $D = 96C$.

□

Lemma 1.27. *Let $\{\overline{x},\overline{y}\} \subset R$ be finite, and $t \in \mathcal{T}_\nu$ finite, with $\overline{x} \neq \overline{y}$, then, for $\nu \geq \eta^2$, there exists $C_{t,\overline{x},\overline{y}} \in \mathcal{R}_{>0}$, with;*

$((1+\frac{C_t}{\nu})^{-[\nu t]} - \frac{1}{\nu^{\frac{1}{2}}})|\overline{x}-\overline{y}| \leq |s_{\overline{x}}(t) - s_{\overline{y}}(t)| \leq (1+\frac{C_t}{\nu})^{[\nu t]}|\overline{x}-\overline{y}|$

Proof. The second inequality follows by a simple adaptation of the proof of the last part of Lemma 1.9, noting that $((1+\frac{C_t}{\nu})^{[\nu t]-1} \leq ((1+\frac{C_t}{\nu})^{[\nu t]}$. For the second part, suppose that;

$|s_{\overline{x}}(t)-s_{\overline{y}}(t)|<((1+\frac{C_t}{\nu})^{-[\nu t]}-\frac{1}{\nu^{\frac{1}{2}}})|\overline{x}-\overline{y}|$ $(*)$

As in Lemma 1.12, using $(*)$, letting $\overline{x}'=s_{\overline{x}}(t)$ and $\overline{y}'=s_{\overline{y}}(t)$, we have that;

$|s_{\overline{x}',t}(t)-s_{\overline{y}',t}(t)|\leq(1+\frac{C_t}{\nu})^{[\nu t]}|s_{\overline{x}}(t)-s_{\overline{y}}(t)|$

$<(1+\frac{C_t}{\nu})^{[\nu t]}((1+\frac{C_t}{\nu})^{-[\nu t]}-\frac{1}{\nu^{\frac{1}{2}}})|\overline{x}-\overline{y}|$

$=(1-\frac{(1+\frac{C_t}{\nu})^{[\nu t]}}{\nu^{\frac{1}{2}}})|\overline{x}-\overline{y}|$ $(**)$

By $(\dagger\dagger)$,$(\dagger\dagger\dagger)$, of Lemma 1.12, there exists $E_{t,\overline{x},\overline{y}}\in\mathcal{R}_{>0}$, with;

$max(|\overline{x}-s_{\overline{x}',t}(t)|,|\overline{y}-s_{\overline{y}',t}(t)|)|\leq\frac{E_{t,\overline{x},\overline{y}}}{\nu}$ $(***)$

so that, by the triangle inequality, using $(**),(***)$;

$|\overline{x}-\overline{y}|<\frac{2E_{t,\overline{x},\overline{y}}}{\nu}+(1-\frac{(1+\frac{C_t}{\nu})^{[\nu t]}}{\nu^{\frac{1}{2}}})|\overline{x}-\overline{y}|$ $(\dagger)$

If $\nu\geq\eta^2$, then, using the fact that $|\overline{x}-\overline{y}|\geq\frac{1}{\sqrt{\eta}}$, so that $\frac{1}{|\overline{x}-\overline{y}|}\leq\sqrt{\eta}$;

$\nu>\frac{2E_{t,\overline{x},\overline{y}}\nu^{\frac{1}{2}}\eta^{\frac{1}{2}}}{(1+\frac{C_t}{\nu})^{[\nu t]}}\geq\frac{2E_{t,\overline{x},\overline{y}}\nu^{\frac{1}{2}}}{(1+\frac{C_t}{\nu})^{[\nu t]}|\overline{x}-\overline{y}|}$

so that;

$\frac{2E_{t,\overline{x},\overline{y}}}{\nu}-\frac{(1+\frac{C_t}{\nu})^{[\nu t]}}{\nu^{\frac{1}{2}}}|\overline{x}-\overline{y}|<0$

and, from $(\dagger)$, we obtain the contradiction, that $|\overline{x}-\overline{y}|<|\overline{x}-\overline{y}|$ so $(*)$ fails, and we obtain the result.

□

Lemma 1.28. *With $\nu\geq\eta^2$, let $\overline{x}\in\overline{x}_\kappa+R_\kappa$ be finite and let $\{t_1,t_2\}\subset\mathcal{T}_\nu$ be finite, with $t_1<t_2$, and $t_1\simeq t_2$. Let $\overline{w}_1\in R_{t_1}$ and $\overline{w}_2\in R_{t_2}$ be the nearest neighbours to $\overline{x}$ at t_1 and t_2, then $\overline{p}_{\overline{w}_1,t_1}\simeq\overline{p}_{\overline{w}_2,t_2}$.*

Proof. Suppose, for contradiction, that $|\overline{p}_{\overline{w}_1,t_1}-\overline{p}_{\overline{w}_2,t_2}|\geq\epsilon$, for some $\epsilon\in\mathcal{R}_{>0}$. We have, using $(\dagger)$ of Lemma 1.19 that;

$|s_{\overline{p}_{\overline{w}_1,t_1}}(t_1)-s_{\overline{p}_{\overline{w}_2,t_2}}(t_2)|$

$= |\overline{w}_1 - \overline{w}_2|$

$\leq |\overline{w}_1 - \overline{x}| + |\overline{w}_2 - \overline{x}|$

$\leq \frac{2D}{\sqrt{\eta}} \simeq 0$ $(*)$ (can be improved for small $t_2 - t_1$)

where $D = (3 + 2F)e^{\circ t_1} + 6$. We have that, using Lemma 1.7, that;

$|s_{\overline{p}_{\overline{w}_1,t_1}}(t_2) - s_{\overline{p}_{\overline{w}_1,t_1}}(t_1)| \leq \frac{G([t_2\nu]-[t_1\nu])}{\nu} \simeq 0$ $(**)$

as $G \in \mathcal{R}_{>0}$ and $t_1 \simeq t_2$, so that, using $(*), (**)$;

$|s_{\overline{p}_{\overline{w}_1,t_1}}(t_2) - s_{\overline{p}_{\overline{w}_2,t_2}}(t_2)|$

$\leq |s_{\overline{p}_{\overline{w}_1,t_1}}(t_2) - s_{\overline{p}_{\overline{w}_1,t_1}}(t_1)| + |s_{\overline{p}_{\overline{w}_1,t_1}}(t_1) - s_{\overline{p}_{\overline{w}_2,t_2}}(t_2)| \simeq 0$ $(***)$

By Lemma 1.27, we have that;

$|s_{\overline{p}_{\overline{w}_1,t_1}}(t_2) - s_{\overline{p}_{\overline{w}_2,t_2}}(t_2)| \geq ((1 + \frac{C_{t_2}}{\nu})^{-[\nu t]} - \frac{1}{\nu^{\frac{1}{2}}})|\overline{p}_{\overline{w}_1,t_1} - \overline{p}_{\overline{w}_2,t_2}|$

$= ((1 + \frac{C_{t_2}}{\nu})^{-[\nu t]} - \frac{1}{\nu^{\frac{1}{2}}})\epsilon$

$> \frac{(1+\frac{C_{t_2}}{\nu})^{-[\nu t]}\epsilon}{2} > \frac{e^{-C_{t_2}}\epsilon}{4}$

which contradicts $(***)$, as required.

□

Definition 1.29. *For $\{n, m\} \subset \mathcal{N}$, with the rounding down convention on $[,]$, we define $R_n = \{\overline{x} : \frac{[x_i\sqrt{n}]}{\sqrt{n}} = x_i,\ for\ 1 \leq i \leq 3\}$. For $\overline{x} \in R_n$ finite, we define a path $\overline{s}_{n,m,\overline{x}}$ inductively on $\mathcal{T}_m$ by;*

$$\overline{s}_{n,m,\overline{x}}(\tfrac{j+1}{m}) - \overline{s}_{n,m,\overline{x}}(\tfrac{j}{m}) = \frac{1}{m}\frac{\overline{J}(\overline{s}_{n,m,\overline{x}}(\frac{j}{m}), \frac{j}{m})}{\rho(\overline{s}_{n,m,\overline{x}}(\frac{j}{m}), \frac{j}{m})},\ for\ 0 \leq j \leq m^2 - 1$$

For infinite $\{\eta, \nu\} \subset {}^*\mathcal{N}$, let $\overline{s}_{\eta,\nu} : R_\eta \times \mathcal{T}_\nu \rightarrow \mathcal{R}^3_\eta$, be defined by $\overline{s}_{\eta,\nu}(\overline{x}, \frac{j}{\nu}) = \overline{s}_{\eta,\nu,\overline{x}}(\frac{j}{\nu})$, $0 \leq j \leq \nu^2$, where $\overline{s}_{\eta,\nu,\overline{x}}$ is obtained by transfer from $\overline{s}_{n,m,\overline{x}}$, using $\frac{{}^*\overline{J}}{{}^*\rho}$ and the usual wrap around convention. Denote also by $\overline{s}_{\eta,\nu}$, the $\mu_\eta \times \mu_\nu$ measurable version.

Lemma 1.30. *For all infinite $\{\eta, \nu\} \subset {}^*\mathcal{N}$, finite $(\overline{x}, t) \in \mathcal{R}^3_\eta \times \mathcal{T}_\nu$;*

$(\frac{\partial\overline{s}_{\eta,\nu}}{\partial x_i})_\eta|_{\overline{x},t}$, $(\frac{\partial\overline{s}_{\eta,\nu}}{\partial t})_\nu|_{\overline{x},t}$ *and* $(\frac{\partial^2\overline{s}_{\eta,\nu}}{\partial x_i\partial t})_{\eta,\nu}|_{\overline{x},t}$

are finite, for $1 \leq i \leq 3$*, where, for* $\overline{f} : \mathcal{R}_\eta \times \mathcal{T}_\nu \rightarrow \mathcal{R}_\eta^3$*,* $\mu_\eta \times \mu_\nu$ *measurable,* $(\frac{\partial\overline{f}}{\partial x_i})_\eta$ *and* $(\frac{\partial\overline{f}}{\partial t})_\nu$ *are the nonstandard derivatives. In particularly,* $\overline{s}_{\eta,\nu}$ *is S-continuous, in the sense that, for* $\{(\overline{x},t),(\overline{x}',t')\} \subset \mathcal{R}_\eta^3 \times \mathcal{T}_\nu$*, with* $(\overline{x},t) \simeq (\overline{x}',t')$*;*

$\overline{s}_{\eta,\nu}(\overline{x},t) \simeq \overline{s}_{\eta,\nu}(\overline{x}',t')$

Proof. For $\{n,m\} \subset \mathcal{N}$, letting the initial condition $\overline{x}$ vary over $\mathcal{R}_n^3$, we have, using the chain rule and the inductive definition, that, for $0 \leq k \leq m^2-1$, the functions $\overline{s}_{k,n,m}(\overline{x}) = \overline{s}_{n,m,\overline{x}}(\overline{x},\frac{k}{m})$ are smooth. We have, for $1 \leq i \leq 3$ that;

$\frac{\partial\overline{s}_{k+1,n,m}}{\partial x_i} = \frac{\partial\overline{s}_{k,n,m}}{\partial x_i}$

$+\frac{1}{m}(grad(\frac{j_1}{\rho}) \cdot \frac{\partial\overline{s}_{k,n,m}}{\partial x_i}, grad(\frac{j_2}{\rho}) \cdot \frac{\partial\overline{s}_{k,n,m}}{\partial x_i}, grad(\frac{j_3}{\rho}) \cdot \frac{\partial\overline{s}_{k,n,m}}{\partial x_i})$

$|\frac{\partial\overline{s}_{k+1,n,m}}{\partial x_i}| \leq |\frac{\partial\overline{s}_{k,n,m}}{\partial x_i}|$

$+\frac{1}{m}(|grad(\frac{j_1}{\rho}) \cdot \frac{\partial s_{k,n,m}}{\partial x_i}| + |grad(\frac{j_2}{\rho}) \cdot \frac{\partial s_{k,n,m}}{\partial x_i}| + |grad(\frac{j_3}{\rho}) \cdot \frac{\partial s_{k,n,m}}{\partial x_i}|)$

$\leq |\frac{\partial\overline{s}_{k,n,m}}{\partial x_i}| + \frac{1}{m}max(|grad(\frac{j_1}{\rho})|, |grad(\frac{j_2}{\rho})|, |grad(\frac{j_3}{\rho})|)|\frac{\partial\overline{s}_{k,n,m}}{\partial x_i}|$

$= (1+\frac{G_i}{m})|\frac{\partial\overline{s}_{k,n,m}}{\partial x_i}|$

It follows, by a simple induction, that, for $1 \leq i \leq 3$;

$|\frac{\partial\overline{s}_{k,n,m}}{\partial x_i}| \leq (1+\frac{G_i}{m})^k|\frac{\partial\overline{s}_{0,n,m}}{\partial x_i}|$

$= (1+\frac{G_i}{m})^k$

$= ((1+\frac{G_i}{m})^m)^{\frac{k}{m}}$

$\leq (e^{G_i}+1)^t$, for $\frac{k}{m} \leq t$, $t \in \mathcal{R}_{>0}$.

In particular, $|\frac{\partial(\overline{s}_{k,n,m})_j}{\partial x_i}| \leq (e^{G_i}+1)^t$, for $1 \leq j \leq 3$.

Using the proof of Lemma 1.7, we have that, for $\{\overline{x},\overline{x}'\} \subset \mathcal{R}_n^3$, $1 \leq j \leq 3$, that;

$|(\overline{s}_{k,n,m})_j(\overline{x}') - (\overline{s}_{k,n,m})_j(\overline{x})| \leq |grad((\overline{s}_{k,n,m})_j)||\overline{x}' - \overline{x}|$

$\leq (|\frac{\partial(\overline{s}_{k,n,m})_j}{\partial x_1}| + |\frac{\partial(\overline{s}_{k,n,m})_j}{\partial x_2}| + |\frac{\partial(\overline{s}_{k,n,m})_j}{\partial x_3}|)|\overline{x}' - \overline{x}|$

$\leq ((e^{G_1}+1)^t + (e^{G_2}+1)^t + (e^{G_3}+1)^t)|\overline{x}' - \overline{x}|$

and;

$|\overline{s}_{k,n,m}(\overline{x}') - \overline{s}_{k,n,m}(\overline{x})| \leq \sum_{j=1}^{3} |(\overline{s}_{k,n,m})_j(\overline{x}') - (\overline{s}_{k,n,m})_j(\overline{x})|$

$\leq 3((e^{G_1}+1)^t + (e^{G_2}+1)^t + (e^{G_3}+1)^t)|\overline{x}' - \overline{x}|$

In particular, it follows that, for $1 \leq i \leq 3$;

$|\sqrt{m}(\overline{s}_{k,n,m})(\overline{x} + \frac{\overline{e}_i}{\sqrt{m}}) - \sqrt{m}(\overline{s}_{k,n,m})(\overline{x})|$

$\leq 3((e^{G_1}+1)^t + (e^{G_2}+1)^t + (e^{G_3}+1)^t)\sqrt{m}|\frac{\overline{e}_i}{\sqrt{m}}|$

$= 3((e^{G_1}+1)^t + (e^{G_2}+1)^t + (e^{G_3}+1)^t)$

It follows that for fixed $\{r, s\} \subset \mathcal{R}_{>0}$

$\mathcal{R} \models (\forall n \in \mathcal{N})(\forall m \in \mathcal{N})(\forall t, 0 \leq t \leq s, t \in \mathcal{T}_m)(\forall \overline{x}, |\overline{x}| \leq r$

$, \overline{x} \in \mathcal{R}_n^3)|(\frac{\partial \overline{s}_{n,m}}{\partial x_i})_m|_{\overline{x},t}| \leq C_s$

for $1 \leq i \leq 3$, where $C_s = 3((e^{G_1}+1)^s + (e^{G_2}+1)^s + (e^{G_3}+1)^s)$

By transfer, the above statement holds in ${}^*\mathcal{R}$, in particular, For all infinite $\{\eta, \nu\} \subset {}^*\mathcal{N}$, finite $(\overline{x}, t) \in \mathcal{R}_\eta^3 \times \mathcal{T}_\nu$, with $|t| \leq s$;

$|(\frac{\partial \overline{s}_{\eta,\nu}}{\partial x_i})_\eta|_{\overline{x},t}| \leq C_s$, $(*)$

and is finite, for $1 \leq i \leq 3$. We have, by the definition of a trajectory, the nonstandard partial derivative, using the bound G_t from Lemma 1.7 that;

$|(\frac{\partial \overline{s}_{\eta,\nu}}{\partial t})_\nu|_{\overline{x},t}|$

$= |\nu(\overline{s}_{\eta,\nu}(\overline{x}, t + \frac{1}{\nu}) - \overline{s}_{\eta,\nu}(\overline{x}, t)|$

$= |\frac{{}^{*}\overline{J}(\overline{s}_{\eta,\nu}(\overline{x},t),\frac{[t\nu]}{\nu})}{{}^{*}\rho(\overline{s}_{\eta,\nu}(\overline{x},t),\frac{[t\nu]}{\nu})}| \leq G_t$

For the next claim, we have, using the definition of a trajectory, that;

$(\frac{\partial^2 \overline{s}_{\eta,\nu}}{\partial x_i \partial t})_{\eta,\nu}|_{\overline{x},t} = (\frac{\partial(\overline{f}_t(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta$

$= ((\frac{\partial(f_{1t}(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta, (\frac{\partial(f_{2t}(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta, (\frac{\partial(f_{3t}(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta)$

where $\overline{f}_t(\overline{y}) = \frac{{}^{*}\overline{J}(\overline{y},\frac{[t\nu]}{\nu})}{{}^{*}\rho(\overline{y},\frac{[t\nu]}{\nu})}$, and $\overline{f}_t = (f_{1t}, f_{2t}, f_{3t})$

We have, using Taylor's theorem in the final step, see [7], and $(*)$, that, for $1 \leq j \leq 3$;

$|(\frac{\partial(f_{jt}(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta| = |\sqrt{\eta}(f_{jt}(\overline{s}_{\eta,\nu}(\overline{x}+\frac{\overline{e}_i}{\sqrt{\eta}},t)) - f_{jt}(\overline{s}_{\eta,\nu}(\overline{x},t)))|$

$= |\sqrt{\eta}((\frac{f_{jt}(y_1+\delta_1,y_2+\delta_2,y_3+\delta_3)-f_{jt}(y_1,y_2+\delta_2,y_3+\delta_3)}{\delta_1}, \frac{f_{jt}(y_1,y_2+\delta_2,y_3+\delta_3)-f_{jt}(y_1,y_2,y_3+\delta_3)}{\delta_2},$

$\frac{f_{jt}(y_1,y_2,y_3+\delta_3)-f_{jt}(y_1,y_2,y_3)}{\delta_3}) \cdot \overline{\delta})|$

$\leq |(\frac{f_{jt}(y_1+\delta_1,y_2+\delta_2,y_3+\delta_3)-f_{jt}(y_1,y_2+\delta_2,y_3+\delta_3)}{\delta_1}, \frac{f_{jt}(y_1,y_2+\delta_2,y_3+\delta_3)-f_{jt}(y_1,y_2,y_3+\delta_3)}{\delta_2},$

$\frac{f_{jt}(y_1,y_2,y_3+\delta_3)-f_{jt}(y_1,y_2,y_3)}{\delta_3})||\sqrt{\eta}\overline{\delta}|$

$\simeq |({}^{*}\frac{\partial g_j}{\partial x_1}|_{(y_1,y_2+\delta_2,y_3+\delta_3,t)}, {}^{*}\frac{\partial g_j}{\partial x_2}|_{(y_1,y_2,y_3+\delta_3,t)}, {}^{*}\frac{\partial g_j}{\partial x_3}|_{(y_1,y_2,y_3,t)})||(\frac{\partial \overline{s}_{\eta,\nu}}{\partial x_i})_\eta|_{\overline{x},t}|$

$\leq C_s(|{}^{*}\frac{\partial g_j}{\partial x_1}|_{(y_1,y_2+\delta_2,y_3+\delta_3,t)}| + |{}^{*}\frac{\partial g_j}{\partial x_2}|_{(y_1,y_2,y_3+\delta_3,t)}| + |{}^{*}\frac{\partial g_j}{\partial x_3}|_{(y_1,y_2,y_3,t)}|)$

$\leq C_s(\sum_{k=1}^{3} H_{j,k,t''})$

where;

$\overline{\delta} = (\delta_1,\delta_2,\delta_3) = \overline{s}_{\eta,\nu}(\overline{x}+\frac{\overline{e}_i}{\sqrt{\eta}},t) - \overline{s}_{\eta,\nu}(\overline{x},t), \overline{y} = (y_1,y_2,y_3) = \overline{s}_{\eta,\nu}(\overline{x},t)$

$g_j(\overline{x},t) = (\frac{\overline{J}(\overline{x},t)}{\rho(\overline{x},t)})_j$

and we have transferred the fact that, for $1 \leq k \leq 3$ and $\{t', t''\} \subset \mathcal{R}_{>0}$, $t' \leq t''$, $t < t''$;

$|\frac{\partial g_{j,t'}}{\partial x_k}| = |\frac{\partial(\frac{j_{j,t'}}{\rho_{t'}})}{\partial x_k}|$

$= |\frac{\frac{\partial j_{j,t'}}{\partial x_k}\rho_{t'} - j_{j,t'}\frac{\partial \rho}{\partial x_k}}{\rho_{t'}^2}|$

$\leq |\frac{\partial j_{j,t}}{\partial x_k}||\rho_{t'}| + |j_{j,t'}||\frac{\partial \rho_{t'}}{\partial x_k}|$

$\leq D_{j,k,t''}E_{j,t''} + F_{j,t''}G_{j,k,t''} = H_{j,k,t''}$

for some $\{D_{j,k,t''}, E_{j,t''}, F_{j,t''}, G_{j,k,t''}, H_{j,k,t''}\} \subset \mathcal{R}_{>0}$

It follows that;

$|(\frac{\partial^2 \overline{s}_{\eta,\nu}}{\partial x_i \partial t})_{\eta,\nu}|_{\overline{x},t}| \leq \sum_{j=1}^3 |\frac{\partial (f_{jt}(\overline{s}_{\eta,\nu}(\overline{x},t)))}{\partial x_i})_\eta|$

$\leq C_s \sum_{j,k=1}^3 H_{j,k,t''}$

as required.

To verify the last claim, it is enough, by transitivity of $\simeq$, to show that, for $\overline{x}$ finite, and $\epsilon \simeq 0$, $\delta \simeq 0$;

$\overline{s}_{\eta,\nu}(\overline{x} + \epsilon \overline{e}_1, t) \simeq \overline{s}_{\eta,\nu}(\overline{x}, t)$

$\overline{s}_{\eta,\nu}(\overline{x}, t + \delta) \simeq \overline{s}_{\eta,\nu}(\overline{x}, t)$

We can write $\epsilon = \frac{r}{\sqrt{\eta}} + r'$, where $0 \leq r' < \frac{1}{\sqrt{\eta}}$, then, using measurabilty, $(*)$, and without loss of generality, assuming that $r \geq 0$;

$|\overline{s}_{\eta,\nu}(\overline{x} + \epsilon \overline{e}_1, t) - \overline{s}_{\eta,\nu}(\overline{x}, t)|$

$= |\overline{s}_{\eta,\nu}(\overline{x} + \frac{r}{\sqrt{\eta}}\overline{e}_1, t) - \overline{s}_{\eta,\nu}(\overline{x}, t)|$

$\leq \sum_{k=0}^{r-1} |\overline{s}_{\eta,\nu}(\overline{x} + \frac{k+1}{\sqrt{\eta}}\overline{e}_1, t) - \overline{s}_{\eta,\nu}(\overline{x} + \frac{k}{\sqrt{\eta}}\overline{e}_1, t)|$

$\leq \sum_{k=0}^{r-1} \frac{C_t}{\sqrt{\eta}}$

$= \frac{C_t r}{\sqrt{\eta}} \simeq 0$

Similarly, writing $\delta = \frac{s}{\nu} + s'$, where $0 \leq s' < \frac{1}{\nu}$, and without loss of generality, assuming $s \geq 0$, we can show that;

$|\overline{s}_{\eta,\nu}(\overline{x}, t + \delta) - \overline{s}_{\eta,\nu}(\overline{x}, t)|$

$\leq \frac{G_{t+\delta+1}s}{\nu} \simeq 0$

as required. □

Definition 1.31. *Let* $\{\overline{w_i}, \overline{w_j}\} \subset R_t$*, with* $\overline{w_i} \neq \overline{w_j}$*, then we let* $H_{ij,t} \subset {}^*\mathcal{R}^3$ *be the internal hyperplane, defined by;*

$H_{ij,t} = \{\overline{z} \in {}^*\mathcal{R}^3 : |\overline{z} - \overline{w_i}| = |\overline{z} - \overline{w_j}|\}$

Fixing $\overline{w_i}$*, we let* $H_{i,t} \subset {}^*\mathcal{R}^3$ *be the internal hyperplane arrangement, defined by;*

$H_{i,t} = {}^*\bigcup_{\overline{x_j} \in R_t, \overline{x_j} \neq \overline{x_i}} H_{ij,t}$

We define an alcove $Z \subset {}^*\mathcal{R}^3 \setminus H_{i,t}$ *to be a* $*$*-connected component of* ${}^*\mathcal{R}^3 \setminus H_{i,t}$*, and, noting that* $\overline{w}_i \notin H_{i,t}$*, let* Z_i *be the unique alcove containing* $\overline{w_i}$*.*

Lemma 1.32. *Let notation be as in Definition 1.31 and 1.18, then, for* $\overline{x}_i$ *finite;*

$J_{\overline{x}_i,t} = Z_i \cap (\overline{x}_\kappa + R_\kappa)$

In particularly, Z_i *is* $*$*-bounded.*

Proof. By Lemma 1.17, we have that $\overline{x}_\kappa + R_\kappa \subset {}^*\mathcal{R}^3 \setminus H_{i,t}$. We claim, firstly, that $Z_i \cap (\overline{x}_\kappa + R_\kappa) \subset J_{\overline{x}_i,t}$. Suppose, for contradiction, that we can find $\overline{w} \in Z_i \cap (\overline{x}_\kappa + R_\kappa)$, with $\overline{w} \notin J_{\overline{x}_i,t}$, then we can find $\overline{w}_j \in R_t$, with $|\overline{w} - \overline{w}_j| < |\overline{w} - \overline{w}_i|$. It follows that any $*$-continuous path from $\overline{w}$ to $\overline{w}_i$, would pass through $H_{ij,t}$, in particular Z_i cannot be $*$-connected. We claim, second, that $J_{\overline{x}_i,t} \subset Z_i \cap (\overline{x}_\kappa + R_\kappa)$. Suppose, for contradiction, that we can find $\overline{w} \in J_{\overline{x}_i,t}$, with $\overline{w} \notin Z_i \cap (\overline{x}_\kappa + R_\kappa)$, then, letting $l_{\overline{w},\overline{x}_i}$, be the line connecting $\overline{w}$ and $\overline{x}_i$, we must have that $l_{\overline{w},\overline{x}_i}$ intersects one of the hyperplanes $H_{ij,t}$, otherwise we could construct a $*$-connected space $Z_i \cup l_{\overline{w},\overline{x}_i} \supset Z_i$, contradicting the definition of a component Z_i as maximal. It follows that $|\overline{w} - \overline{x}_j| < |\overline{w} - \overline{x}_i|$ and $\overline{x}_i$ is not the nearest neighbour. For the final claim, if Z_i is not $*$-bounded, we can, using the transfer principle, for a given $\epsilon \in \mathcal{R}_{>0}$, find $\overline{w}_{\eta,\epsilon}$ such that ${}^*B(\overline{w}_{\eta,\epsilon}, \epsilon) \subset Z_i$, and $\overline{w}_{\eta,\epsilon} \in \overline{x}_\kappa + \mathcal{R}^3_\kappa$ is infinite. Clearly, we can find $\overline{s} \in (\overline{x}_\kappa + R_\kappa) \cap {}^*B(\overline{w}_{\eta,\epsilon}, \epsilon) \subset Z_i$, so that $\overline{s}$ is infinite as well. By the previous result, we have that $\overline{s} \in J_{\overline{x}_i,t}$, contradicting the fact in (†) of Lemma 1.19, that $\overline{s} \in {}^*B(\overline{x}_i, \frac{C}{\sqrt{\eta}})$, so $\overline{s}$ is finite.

□

Definition 1.33. *Let notation be as in Definition 1.31, let Z_i be an alcove corresponding to a finite $\overline{w}_i \in R_t$, let $S_{i,t}$ be the $*$-finite set enumerating the $*$-convex components $V_{ik,t}$ of δZ_i, with $dim(V_{ik,t}) = 2$, corresponding to hyperplanes $H_{ik,t}$, so that $\delta Z_i = {}^*\bigcup_{k\in S_{i,t}} V_{ik,t}$. We define $T_{i,t}$, the relevant points to $\overline{w}_i$, by;*

$$T_{i,t} = \{\overline{w}_k \in R_t : k \in S_{i,t}, V_{ik,t} \subset H_{ik,t}\}$$

We have that $\overline{Z_i}$ is convex, so that, by the Krein-Milman Theorem, $\overline{Z_i}$ is the $*$-convex hull of the $*$-finite set $V_{i,t}$ of extreme points, which we refer to as vertices.

Lemma 1.34. *If $\overline{v} \in V_{i,t}$, then there exist $\{\overline{w}_{k_1}, \overline{w}_{k_2}, \overline{w}_{k_3}\} \subset T_{i,t}$, such that;*

$$\overline{v} = H_{ik_1,t} \cap H_{ik_2,t} \cap H_{ik_3,t}$$

Proof. As $\overline{v} \in V_{i,t} \subset \delta Z_i$, we can find $\overline{w}_{k_1} \in T_{i,t}$, with $\overline{v} \in H_{ik_1,t}$. Choose a point $\overline{p} \in V_{ik_1,t}$ with $\overline{p} \neq \overline{v}$, Suppose that $\overline{v} \notin {}^*\bigcup_{k\in S_{i,t}, k\neq k_1} H_{ik,t}$, so that, as ${}^*\bigcup_{k\in S_{i,t}, k\neq k_1} H_{ik,t}$ is $*$-closed, there exists $\epsilon \in {}^*\mathcal{R}_{>0}$, with ${}^*B(\overline{v}, \epsilon) \cap {}^*\bigcup_{k\in S_{i,t}, k\neq k_1} H_{ik,t} = \emptyset$, $(*)$. As $\overline{v} \in \delta Z_i$, we can find a $*$-sequence $(\overline{v}_n) \subset {}^*B(\overline{v}, \epsilon) \cap Z_i$, with $n \in {}^*\mathcal{N}$, such that $|\overline{v}_n - \overline{v}| \to 0$ and $l_{\overline{v}_n,\overline{p}} \cap H_{ik_1,t} = \overline{p}$, $(**)$. As $\overline{Z_i}$ is bounded, and using $(**)$, there exist points $\overline{p}_n = l_{\overline{v}_n,\overline{p}} \cap {}^*\bigcup_{k\in S_{i,t}, k\neq k_1} H_{ik,t}$, and, without loss of generality, we can assume that $\overline{p}_n = l_{\overline{v}_n,\overline{p}} \cap V_{ik,t}$ for some $k \neq k_1$. By $(*)$, we have that $|\overline{p}_n - \overline{v}| \geq \epsilon$, so the same is true for the $*$-limit $\overline{p}' \in V_{ik,t}$. Clearly, $\overline{v} \in l_{\overline{p}'\overline{p}}$, which contradicts the fact that $\overline{v}$ is a vertex. Suppose that $\overline{v} \in H_{ik_1,t} \cap H_{ik_2,t}$, for $k_1 \neq k_2$, but $\overline{v} \notin {}^*\bigcup_{k\in S_{i,t}, k\notin\{k_1,k_2\}} H_{ik,t}$, then, as $dim(V_{ik_1,t} \cap V_{ik_2,t}) = 1$, we can choose $\overline{p} \in V_{ik_1,t} \cap V_{ik_2,t}$, with $\overline{p} \neq \overline{v}$ again. Repeating the same argument, we can choose $(\overline{v}_n)$ again such that $l_{\overline{v}_n,\overline{p}} \cap H_{ik_1,t} = l_{\overline{v}_n,\overline{p}} \cap H_{ik_2,t} = \overline{p}$, and obtain similarly a contradiction.

□

Definition 1.35. *Suppose that $\{t_1, t_2\} \subset \mathcal{T}_\nu$, with $t_1 < t_2$, and let Z_{i,t_1} be an alcove, corresponding to $\overline{w}_i \in R_{t_1}$, with $\overline{w}_i$ finite. For $t_1 \leq t \leq t_2$, we define the shifted alcove $Z_{i,t}$ to be the alcove corresponding to $s_{\overline{p}_{\overline{w}_i}}(t)$, which we later abbreviate to $\overline{w}_{it}$. We define the shift to be simple, if;*

(i). $T_{i,t} = \{s_{\overline{p}_{\overline{w}}}(t) : \overline{w} \in T_{i,t_1}\}$

(ii). For every pair of vertices $(\overline{v},\overline{v}')\in V_{i,t_1}\times V_{i,t}$ *there exist unique triples* (k_1,k_2,k_3) *and* (k_1',k_2',k_3') *with;*

$\overline{v}=H_{ik_1,t_1}\cap H_{ik_2,t_1}\cap H_{ik_3,t_1}$

$\overline{v}'=H_{ik_1',t_2}\cap H_{ik_2',t}\cap H_{ik_3',t}$

(iii). $\overline{v}\in V_{i,t_1}$ *iff the intersection* $H_{ik_1,t}\cap H_{ik_2,t}\cap H_{ik_3,t}$ *of the shifted planes defines a vertex* $\overline{v}'\in V_{i,t}$.

If $\overline{w}_{jt}\in T_{i,t}$*, we define the centre* $\overline{c}_{ij,t}=\frac{\overline{w}_{it}+\overline{w}_{jt}}{2}$.

Lemma 1.36. *We have that there exists* $S_{it_1}\in\mathcal{R}_{>0}$*, with;*

$|(\overline{v}_t)'|_{t_1}|\leq S_{it_1}$

Proof. We have that $\overline{c}_{ij,t}\in H_{ij,t}$ and for $\overline{z}\in H_{ij,t}$, we have;

$<\overline{z}-\overline{c}_{ij,t},\overline{w}_{jt}-\overline{w}_{it}>=0$

so that with coordinates $\overline{z}=(z_1,z_2,z_3)$, $\overline{w}_{jt}=(w_{j1t},w_{j2t},w_{j3t})$, $\overline{w}_{it}=(w_{i1t},w_{i2t},w_{i3t})$, $\overline{c}_{ij,t}=(c_{ijt1},c_{ijt2},c_{ijt3})=(\frac{w_{i1t}+w_{j1t}}{2},\frac{w_{i2t}+w_{j2t}}{2},\frac{w_{i3t}+w_{j3t}}{2})$, the equation of the plane $H_{ij,t}$ is given by;

$\sum_{k=1}^{3}(w_{jkt}-w_{ikt})z_k=\sum_{k=1}^{3}c_{ijtk}(w_{jkt}-w_{ikt})$

By (ii), a vertex $\overline{v}_t\in H_{i,t}$, with coordinates $\overline{v}_t=(v_{1t},v_{2t},v_{3t})$, given by the intersection $H_{ik_1,t}\cap H_{ik_2,t}\cap H_{ik_3,t}$, is the intersection of the planes;

$\sum_{k=1}^{3}(w_{k_1kt}-w_{ikt})v_{kt}=\sum_{k=1}^{3}c_{ik_1tk}(w_{k_1kt}-w_{ikt})$

$=\sum_{k=1}^{3}\frac{(w_{ikt}+w_{k_1kt})(w_{k_1kt}-w_{ikt})}{2}=\sum_{k=1}^{3}\frac{w_{k_1kt}^2-w_{ikt}^2}{2}$

$\sum_{k=1}^{3}(w_{k_2kt}-w_{ikt})v_{kt}=\sum_{k=1}^{3}c_{ik_2tk}(w_{k_2kt}-w_{ikt})$

$=\sum_{k=1}^{3}\frac{(w_{ikt}+w_{k_2kt})(w_{k_2kt}-w_{ikt})}{2}=\sum_{k=1}^{3}\frac{w_{k_2kt}^2-w_{ikt}^2}{2}$

$\sum_{k=1}^{3}(w_{k_3kt}-w_{ikt})v_{kt}=\sum_{k=1}^{3}c_{ik_3tk}(w_{k_3kt}-w_{ikt})$

$=\sum_{k=1}^{3}\frac{(w_{ikt}+w_{k_3kt})(w_{k_3kt}-w_{ikt})}{2}=\sum_{k=1}^{3}\frac{w_{k_3kt}^2-w_{ikt}^2}{2}$

which we can write in matrix form, as;

$A_{ik_1k_2k_3t}\overline{v}_t = \overline{d}_{ik_1k_2k_3t}$

where, for $1 \leq l \leq 3$, $1 \leq k \leq 3$;

$(A_{ik_1k_2k_3t})_{lk} = w_{k_lkt} - w_{ikt}$, $(\overline{d}_{ik_1k_2k_3t})_l = \sum_{k=1}^{3} \frac{w_{k_lkt}^2 - w_{ikt}^2}{2}$

By the intersection property, we have that;

$\overline{v}_t = (A_{ik_1k_2k_3t})^{-1}\overline{d}_{ik_1k_2k_3t}$

and, letting $()'_\nu$ denote the nonstandard derivative;

$(\overline{v}_t)'_\nu|_{t_1} = ((A_{ik_1k_2k_3t})^{-1}\overline{d}_{ik_1k_2k_3t})'_\nu|_{t_1}$

$= \nu((A_{ik_1k_2k_3(t+\frac{1}{\nu})})^{-1}\overline{d}_{ik_1k_2k_3(t+\frac{1}{\nu})} - (A_{ik_1k_2k_3t})^{-1}\overline{d}_{ik_1k_2k_3t})|_{t_1}$

$= \nu((A_{ik_1k_2k_3(t+\frac{1}{\nu})})^{-1} - (A_{ik_1k_2k_3t})^{-1})\overline{d}_{ik_1k_2k_3(t+\frac{1}{\nu})}$

$+(A_{ik_1k_2k_3t})^{-1}(\overline{d}_{ik_1k_2k_3(t+\frac{1}{\nu})} - \overline{d}_{ik_1k_2k_3(t+\frac{1}{\nu})}))|_{t_1}$

$= (((A_{ik_1k_2k_3t})^{-1})'_\nu\overline{d}_{ik_1k_2k_3(t+\frac{1}{\nu})} + (A_{ik_1k_2k_3t})^{-1}(\overline{d}_{ik_1k_2k_3t})'_\nu)|_{t_1}$

$= (\frac{cof(A_{ik_1k_2k_3t})^T}{det(A_{ik_1k_2k_3t})})'_\nu|_{t_1}\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})} + (\frac{cof(A_{ik_1k_2k_3t_1})^T}{det(A_{ik_1k_2k_3t_1})})(\overline{d}_{ik_1k_2k_3t})'_\nu)|_{t_1}$ (E)

We have, using Lemmas 1.7 and 1.9, that for $1 \leq l \leq 3$, there exists $D_{t_1+\frac{1}{\nu},l} \in \mathcal{R}_{>0}$, with;

$|(\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})})_l| = |\sum_{k=1}^{3} \frac{w_{k_lk(t_1+\frac{1}{\nu})}^2 - w_{ik(t_1+\frac{1}{\nu})}^2}{2}|$

$= \frac{1}{2}||\sigma_{\overline{p}_{\overline{w}_i}}(t_1+\frac{1}{\nu})|^2 - |\sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1+\frac{1}{\nu})|^2||$

$= \frac{1}{2}|(|\sigma_{\overline{p}_{\overline{w}_i}}(t_1+\frac{1}{\nu})| + |\sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1+\frac{1}{\nu})|)(|\sigma_{\overline{p}_{\overline{w}_i}}(t_1+\frac{1}{\nu})| - |\sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1+\frac{1}{\nu})|)|$

$\leq D_{t_1+\frac{1}{\nu},l}||\sigma_{\overline{p}_{\overline{w}_i}}(t_1+\frac{1}{\nu})| - |\sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1+\frac{1}{\nu})||$

$\leq D_{t_1+\frac{1}{\nu},l}|\sigma_{\overline{p}_{\overline{w}_i}}(t_1+\frac{1}{\nu}) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1+\frac{1}{\nu})|$

$\leq D_{t_1+\frac{1}{\nu},l}e^{A_{D_{t_1+\frac{1}{\nu}}}{}^{\circ}(t_1+\frac{1}{\nu})}|\overline{p}_{\overline{w}_i} - \overline{p}_{\overline{w}_{k_l}}|$

Choosing κ sufficiently large, using Lemma 1.32, (†) of Lemma 1.19 and the fact that $\overline{w}_{k_l,t} \in T_{i,t}$, we can find $\overline{z} \in H_{ik_l,t}$, with $|\overline{z}-\overline{w_{i,t}}| < \frac{C+1}{\sqrt{\eta}}$. By the definition of $H_{ik_l,t}$ and the fact that $|c_{ik_l,t} - \overline{w_{i,t}}| \leq |\overline{z} - \overline{w_{i,t}}| < \frac{C+1}{\sqrt{\eta}}$, we must have that $|\overline{w}_{k_l,t} - \overline{w_{i,t}}| < \frac{2(C+1)}{\sqrt{\eta}}$, and, by Lemma 1.27;

$$|\overline{p}_{\overline{w}_i} - \overline{p}_{\overline{w}_{k_l}}| < \frac{2(C+1)}{\sqrt{\eta}} 2(1 + \frac{C_{t_l}}{\nu})^{[\nu t_l]} < \frac{4(C+1)}{\sqrt{\eta}}(e^{C_{t_l} \circ t_l} + 1) = \frac{L_{ilt_l}}{\sqrt{\eta}}$$

It follows that there exists $K_{ilt_1} \in \mathcal{R}_{>0}$, with;

$$|(\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})})_l| < D_{t_1+\frac{1}{\nu},l} e^{A_{D_{t+\frac{1}{\nu}}} \circ (t+\frac{1}{\nu})} \frac{4(C+1)}{\sqrt{\eta}}(e^{C_t \circ t} + 1) = \frac{K_{ilt_1}}{\sqrt{\eta}} \quad (\sharp)$$

We have, using Lemma 1.30, that;

$$|((\overline{d}_{ik_1k_2k_3t})'_\nu)_l|_{t_1}| = |(\sum_{k=1}^{3} \frac{w^2_{k_lkt} - w^2_{ikt}}{2})'_\nu|_{t_1}|$$

$$= |\sum_{k=1}^{3} \frac{(w^2_{ikt})'_\nu - (w^2_{k_lkt})'_\nu}{2}|_{t_1}|$$

$$= |\nu \sum_{k=1}^{3} \frac{(w^2_{ik(t+\frac{1}{\nu})} - w^2_{ikt}) - (w^2_{k_lk(t+\frac{1}{\nu})} - w^2_{k_lkt})}{2}|_{t_1}|$$

$$= |\nu \sum_{k=1}^{3} \frac{(w_{ik(t+\frac{1}{\nu})} + w_{ikt})(w_{ik(t+\frac{1}{\nu})} - w_{ikt}) - (w_{k_lk(t+\frac{1}{\nu})} + w_{k_lkt})(w_{k_lk(t+\frac{1}{\nu})} - w_{k_lkt})}{2}|_{t_1}|$$

$$= |\sum_{k=1}^{3} \frac{(w_{ik(t_1+\frac{1}{\nu})} + w_{ikt_1})(w_{ikt})'_\nu|_{t_1} - (w_{k_lk(t_1+\frac{1}{\nu})} + w_{k_lkt_1})(w_{k_lkt})'_\nu|_{t_1}}{2}|$$

$$\simeq |\sum_{k=1}^{3} w_{ikt_1}(w_{ikt})'_\nu|_{t_1} - w_{k_lkt_1}(w_{k_lkt})'_\nu|_{t_1}|$$

$$= |\sum_{k=1}^{3} (w_{ikt_1}(w_{ikt})'_\nu|_{t_1} - w_{k_lkt_1}(w_{ikt})'_\nu|_{t_1} + w_{k_lkt_1}(w_{ikt})'_\nu|_{t_1} - w_{k_lkt_1}(w_{k_lkt})'_\nu|_{t_1})|$$

$$= |(\sigma_{\overline{p}_{\overline{w}_i}}(t_1) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1))(\sigma_{\overline{p}_{\overline{w}_i}}(t))'_\nu|_{t_1} + (\sigma_{\overline{p}_{\overline{w}_i}}(t) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t))'_\nu|_{t_l} \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1)|$$

$$\leq |\sigma_{\overline{p}_{\overline{w}_i}}(t_1) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1)||(\sigma_{\overline{p}_{\overline{w}_i}}(t))'_\nu|_{t_1}| + |(\sigma_{\overline{p}_{\overline{w}_i}}(t) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t))'_\nu|_{t_l}||\sigma_{\overline{p}_{\overline{w}_{k_l}}}(t_1)|$$

$$\leq e^{A_{D_{t_1}} \circ (t_1)} |\overline{p}_{\overline{w}_i} - \overline{p}_{\overline{w}_{k_l}}||(\sigma_{\overline{p}_{\overline{w}_i}}(t))'_\nu|_{t_1}| + G_{l,t_1}|(\sigma_{\overline{p}_{\overline{w}_i}}(t) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t))'_\nu|_{t_l}|$$

$$\leq e^{A_{D_{t_1}} \circ (t_1)} \frac{L_{ilt_l}}{\sqrt{\eta}} |(\sigma_{\overline{p}_{\overline{w}_i}}(t))'_\nu|_{t_1}| + G_{l,t_1}|(\sigma_{\overline{p}_{\overline{w}_i}}(t) - \sigma_{\overline{p}_{\overline{w}_{k_l}}}(t))'_\nu|_{t_l}|$$

$$\leq \frac{M_{ilt_l}}{\sqrt{\eta}} N_{it_l} + \sqrt{3} G_{l,t_1} max_{\overline{z} \in {}^*B(\overline{p}_{\overline{w}_{k_l}}, r_{il}+1)}(|(\frac{\partial^2 \overline{s}_{\eta,\nu}}{\partial x_i \partial t})_{\eta,\nu}|_{\overline{z},t_1}|)|\overline{p}_{\overline{w}_i} - \overline{p}_{\overline{w}_{k_l}}|$$

$$\leq \frac{M_{ilt_l}}{\sqrt{\eta}} N_{it_l} + \sqrt{3} G_{l,t_1} H_{lr_{il}} \frac{L_{ilt_l}}{\sqrt{\eta}} = \frac{K_{ilt_l}}{\sqrt{\eta}} \quad (\dagger)$$

so that;

$|((\overline{d}_{ik_1k_2k_3t})'_\nu)|_{t_1}| \leq \sum_{l=1}^3 \frac{K_{ilt_l}}{\sqrt{\eta}} = \frac{P_{it_l}}{\sqrt{\eta}}$ (††)

where $\{H_{lr_{il}}, M_{ilt_l}, K_{ilt_l}, N_{it_l}, G_{l,t_1}\} \subset \mathcal{R}_{>0}$, $r_{il} = |\overline{p}_{\overline{w}_i} - \overline{p}_{\overline{w}_{k_l}}|$.
We have that;

$det(A_{ik_1k_2k_3t_1}) = \overline{\Delta}_{ik_1t_1} \centerdot (\overline{\Delta}_{ik_2t_1} \times \overline{\Delta}_{ik_3t_1})$

where $\overline{\Delta}_{ik_jt} = \overline{w}_{k_j,t} - \overline{w}_{i,t}$, for $1 \leq j \leq 3$

We have that;

$\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_2t} \times \overline{\Delta}_{ik_3t}$

$= (\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_2t}) \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})} + (\overline{\Delta}_{ik_3(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_3t}) \times \overline{\Delta}_{ik_2t}$ $(*)$

so that, using $(*)$;

$det(A_{ik_1k_2k_3(t+\frac{1}{\nu})}) - det(A_{ik_1k_2k_3t})$

$= \overline{\Delta}_{ik_1(t+\frac{1}{\nu})} \centerdot (\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})}) - \overline{\Delta}_{ik_1t} \centerdot (\overline{\Delta}_{ik_2t} \times \overline{\Delta}_{ik_3t})$

$= (\overline{\Delta}_{ik_1(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_1t}) \centerdot (\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})}) - \overline{\Delta}_{ik_1t} \centerdot (\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} \times$

$\overline{\Delta}_{ik_3(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_2t} \times \overline{\Delta}_{ik_3t})$

$= (\overline{\Delta}_{ik_1(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_1t}) \centerdot (\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})}) - \overline{\Delta}_{ik_1t} \centerdot ((\overline{\Delta}_{ik_2(t+\frac{1}{\nu})} -$

$\overline{\Delta}_{ik_2t}) \times \overline{\Delta}_{ik_3(t+\frac{1}{\nu})} + (\overline{\Delta}_{ik_3(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_3t}) \times \overline{\Delta}_{ik_2t})$ (††)

By Lemma 1.30, and using the argument (†) above, we have, for $1 \leq j \leq 3$, that;

$|\overline{\Delta}_{ik_jt}| \leq \frac{M_{ik_jt}}{\sqrt{\eta}}$

$|\overline{\Delta}_{ik_j(t+\frac{1}{\nu})} - \overline{\Delta}_{ik_jt}| = |\frac{1}{\nu}(\overline{s}_{\overline{p}_{\overline{w}_{k_j}}} - \overline{s}_{\overline{p}_{\overline{w}_i}})'_\nu|_t| \leq \frac{L_{ik_jt}}{\nu\sqrt{\eta}}$

where $\{M_{ik_jt}, L_{ik_jt}\} \subset \mathcal{R}_{>0}$. It follows, from (††), that;

$|det(A_{ik_1k_2k_3(t+\frac{1}{\nu})})-det(A_{ik_1k_2k_3t})|$

$\leq|(\overline{\Delta}_{ik_1(t+\frac{1}{\nu})}-\overline{\Delta}_{ik_1t})||\overline{\Delta}_{ik_2(t+\frac{1}{\nu})}||\overline{\Delta}_{ik_3(t+\frac{1}{\nu})}|+|\overline{\Delta}_{ik_1t}||(\overline{\Delta}_{ik_2(t+\frac{1}{\nu})}-\overline{\Delta}_{ik_2t})||\overline{\Delta}_{ik_3(t+\frac{1}{\nu})}|$

$+|\overline{\Delta}_{ik_1t}||(\overline{\Delta}_{ik_3(t+\frac{1}{\nu})}-\overline{\Delta}_{ik_3t})||\overline{\Delta}_{ik_2t})|$

$\leq\frac{1}{\nu\eta^{\frac{3}{2}}}(M_{ik_2t}M_{ik_3t}Lik_1t+M_{ik_1t}M_{ik_3t}Lik_2t+M_{ik_1t}M_{ik_2t}Lik_3t)$

$=\frac{O_{ik_1k_2k_3t}}{\nu\eta^{\frac{3}{2}}}$ (!)

where $O_{ik_1k_2k_3t}\in\mathcal{R}_{>0}$. In particularly;

$|(det(A_{ik_1k_2k_3t}))'_\nu|\leq\frac{O_{ik_1k_2k_3t}}{\eta^{\frac{3}{2}}}$, (!!)

It follows that if $t=\frac{s}{\nu}\leq t_1$;

$|det(A_{ik_1k_2k_3(t_0+t)})-det(A_{ik_1k_2k_3t_0})|$

$=|^*\sum_{i=0}^{s-1}(det(A_{ik_1k_2k_3(t_0+\frac{i+1}{\nu})})-det(A_{ik_1k_2k_3(t_0+\frac{i}{\nu})}))|$

$\leq{}^*\sum_{i=0}^{s-1}|det(A_{ik_1k_2k_3(t_0+\frac{i+1}{\nu})})-det(A_{ik_1k_2k_3(t_0+\frac{i}{\nu})})|$

$\leq{}^*\sum_{i=0}^{s-1}\frac{O_{ik_1k_2k_3(t_0+\frac{i}{\nu})}}{\nu\eta^{\frac{3}{2}}}$

$\leq\frac{sO_{ik_1k_2k_3t_0t_1}}{\nu\eta^{\frac{3}{2}}}=\frac{t\nu O_{ik_1k_2k_3t_0t_1}}{\nu\eta^{\frac{3}{2}}}=\frac{tO_{ik_1k_2k_3t_0t_1}}{\eta^{\frac{3}{2}}}$

for $max_{0\leq i\leq s-1}(O_{ik_1k_2k_3}(t_0+\frac{i}{\nu}))\leq O_{ik_1k_2k_3t_0t_1}$ and $O_{ik_1k_2k_3t_0t_1}\in\mathcal{R}_{>0}$.

so that;

$|det(A_{ik_1k_2k_3t})-det(A_{ik_1k_2k_30})|\leq\frac{tO_{ik_1k_2k_30t_1}}{\eta^{\frac{3}{2}}}$

....Correction. $det(A_{ik_1k_2k_30})\neq 0$, with coordinates $\frac{i}{\sqrt{\eta}}$, $i\in\mathcal{Z}$, implies that $|det(A_{ik_1k_2k_30})|\geq\frac{M}{\eta^{\frac{3}{2}}}$, where $M\in\mathcal{Z}_{>0}$. Same argument as below, find $t_0\in\mathcal{R}_{>0}$, such that all triples $(k_1k_2k_3)$ with $det(A_{ik_1k_2k_30})\neq 0$ remain vertices fot $0\leq t<t_0$. Continuity argument, to exclude contribution of vertices with $det(A_{ik_1k_2k_3t})=0$, for some $0\leq t<t_0$....

We have that $|det(A_{ik_1k_2k_30})|=\frac{1}{\eta^{\frac{3}{2}}}$, so that, choosing some $t_3\in\mathcal{R}_{>0}$ for $0<t\leq min(\frac{1}{2O_{ik_1k_2k_30t_3}},t_3)$, with $t\in R_{>0}$, we have that;

$\frac{1}{2\eta^{\frac{3}{2}}} \leq |det(A_{ik_1k_2k_3t})| \leq \frac{3}{2\eta^{\frac{3}{2}}}$ $(\sharp\sharp\sharp)$

We restrict attention to $0 \leq t_1 < t_2 \leq min(\frac{1}{2O_{ik_1k_2k_30t_3}}, t_3)$.

We have , letting $||,||$ denote the maximum modulus matrix norm, and using the property, by transfer, from Lemma 1.37, that in ${}^*\mathcal{R}^3$;

$|A\overline{v}| \leq 3\sqrt{3}||A|||\overline{v}|$

$||(cof(A_{ik_1k_2k_3t_1})^T|_{t_1}|| = ||cof(A_{ik_1k_2k_3t_1}|_{t_1}||$

$= max_{1\leq l,k\leq 3}(|cof(A_{ik_1k_2k_3t_1})_{lk}|)$

$= max_{1\leq l,k\leq 3}(|(-1)^{l+k}(w_{k_{l'}k't} - w_{ik't})(w_{k_{\sigma(l')}\sigma(k')t} - w_{i\sigma(k')t}) - (w_{\sigma(k_l)kt}$

$-w_{ikt})(w_{k_l\sigma(k)t} - w_{i\sigma(k)t})|)$

where for $1 \leq k,l \leq 3$, $k' = \mu k(k \in A_k)$, $A_k = \{1,2,3\} \setminus \{k\}$, $l' = \mu l(l \in A_l)$, $A_l = \{1,2,3\} \setminus \{l\}$, $\sigma : A_k \rightarrow A_k$, $\sigma \neq Id$, $\sigma^2 = Id$, and similarly for A_l.

We have, using the calculation in $(\sharp)$, for some $C \in \mathcal{R}_{>0}$, $1 \leq k,l \leq 3$, that;

$|(-1)^{l+k}(w_{k_{l'}k't} - w_{ik't})(w_{k_{\sigma(l')}\sigma(k')t} - w_{i\sigma(k')t}) - (w_{\sigma(k_l)kt} - w_{ikt})(w_{k_l\sigma(k)t}$

$-w_{i\sigma(k)t})|$

$\leq |w_{k_{l'}k't} - w_{ik't}||w_{k_{\sigma(l')}\sigma(k')t} - w_{i\sigma(k')t}| + |w_{\sigma(k_l)kt} - w_{ikt}||w_{k_l\sigma(k)t} - w_{i\sigma(k)t}|$

$\leq 2(\frac{2\sqrt{3}(C+1)}{\sqrt{\eta}})^2 = \frac{24(C+1)^2}{\eta}$

so that;

$||(cof(A_{ik_1k_2k_3t_1})^T|_{t_1}|| \leq \frac{24(C+1)^2}{\eta}$ $(\sharp\sharp\sharp\sharp)$

It follows from $(\sharp\sharp), (\sharp\sharp\sharp), (\sharp\sharp\sharp\sharp)$, and using Lemma 1.37, that;

$|\frac{cof(A_{ik_1k_2k_3t_1})^T(\overline{d}_{ik_1k_2k_3t})'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})}| \leq \frac{2.24(C+1)^2 P_{it_1}\eta^{\frac{3}{2}}}{\eta\sqrt{\eta}}$

$= 48(C+1)^2 P_{it_1} = Q_{it_1}$ (D)

We have, using the chain rule, that;

$(\frac{cof(A_{ik_1k_2k_3t})^T}{det(A_{ik_1k_2k_3t})})'_\nu|_{t_1}\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}$

$= ((\frac{1}{det(A_{ik_1k_2k_3t})})'_\nu|_{t_1}cof(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})^T + \frac{(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})})\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}$

$= (-\frac{(det(A_{ik_1k_2k_3t}))'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})det(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})}cof(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})^T + \frac{(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})})\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}$ (C)

By (♯), (♯♯♯), (♯♯♯♯), (!!), and using Lemma 1.37 we have that;

$|(-\frac{(det(A_{ik_1k_2k_3t}))'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})det(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})}cof(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})^T)\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|$

$\leq 3\sqrt{3}||-\frac{(det(A_{ik_1k_2k_3t}))'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})det(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})}cof(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})^T|||\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|$

$= 3\sqrt{3}|\frac{(det(A_{ik_1k_2k_3t}))'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})det(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})}||\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|||cof(A_{ik_1k_2k_3(t_1+\frac{1}{\nu})})^T||$

$\leq 3\sqrt{3}\frac{O_{ik_1k_2k_3t_1}}{\eta^{\frac{3}{2}}}4\eta^3\frac{K_{ilt_1}}{\sqrt{\eta}}\frac{24(C+1)^2}{\eta}$

$= 288\sqrt{3}(C+1)^2 O_{ik_1k_2k_3t_1}K_{ilt_1}$ (A)

We have, using (♯), (♯♯♯), Lemma 1.37, that;

$|(\frac{(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})})\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|$

$\leq 3\sqrt{3}||\frac{(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})}|||\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|$

$= 3\sqrt{3}|\frac{1}{det(A_{ik_1k_2k_3t_1})}||\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|||(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}||$

$\leq 3\sqrt{3}2\eta^{\frac{3}{2}}K_{\frac{ilt_1}{\sqrt{\eta}}}||(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}||$, (B)

We have that;

$||(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}|| = ||(cof(A_{ik_1k_2k_3t})'_\nu|_{t_1}||$

$= max_{1\leq l,k\leq 3}(|(cof(A_{ik_1k_2k_3t})_{lk})'|_{t_1}|)$

$= max_{1\leq l,k\leq 3}(|[(-1)^{l+k}(w_{k_{l'}k't} - w_{ik't})(w_{k_{\sigma(l')}\sigma(k')t} - w_{i\sigma(k')t}) - (w_{\sigma(k_l)kt}$

$-w_{ikt})(w_{k_l\sigma(k)t}-w_{i\sigma(k)t})]'|_{t_1}|)$

We have, using the product rule, the calculations in $(\sharp)$ and $(\dagger)$, that for some $\{C_l, D_l\}\subset\mathcal{R}_{>0}$, $1\leq k,l\leq 3$;

$|[(-1)^{l+k}(w_{k_{l'}k't}-w_{ik't})(w_{k_{\sigma(l')}\sigma(k')t}-w_{i\sigma(k')t})-(w_{\sigma(k_l)kt}-w_{ikt})(w_{k_l\sigma(k)t}$

$-w_{i\sigma(k)t})]'_{t_1}|$

$=|(-1)^{l+k}(w_{k_{l'}k't}-w_{ik't})'|_{t_1}(w_{k_{\sigma(l')}\sigma(k')(t_1+\frac{1}{\nu})}-w_{i\sigma(k')(t_1+\frac{1}{\nu})})+(-1)^{l+k}(w_{k_{l'}k't_1}$

$-w_{ik't_1})(w_{k_{\sigma(l')}\sigma(k')t}-w_{i\sigma(k')t})'|_{t_1}-(w_{\sigma(k_l)kt}-w_{ikt})'|_{t_1}(w_{k_l\sigma(k)(t_1+\frac{1}{\nu})}-$

$w_{i\sigma(k)(t_1+\frac{1}{\nu})})-(w_{\sigma(k_l)kt_1}-w_{ikt_1})(w_{k_l\sigma(k)t}-w_{i\sigma(k)t})'|_{t_1}|$

$\leq|(w_{k_{l'}k't}-w_{ik't})'|_{t_1}||w_{k_{\sigma(l')}\sigma(k')(t_1+\frac{1}{\nu})}-w_{i\sigma(k')(t_1+\frac{1}{\nu})}|+|w_{k_{l'}k't_1}$

$-w_{ik't_1}||(w_{k_{\sigma(l')}\sigma(k')t}-w_{i\sigma(k')t})'|_{t_1}|+|(w_{\sigma(k_l)kt}-w_{ikt})'|_{t_1}|$

$|w_{k_l\sigma(k)(t_1+\frac{1}{\nu})}-w_{i\sigma(k)(t_1+\frac{1}{\nu})}|+|w_{\sigma(k_l)kt_1}-w_{ikt_1}||(w_{k_l\sigma(k)t}-w_{i\sigma(k)t})'|_{t_1}|$

$\leq 4\frac{C_1}{\sqrt{\eta}}\frac{D_1}{\sqrt{\eta}}=\frac{4C_1D_1}{\eta}$

so that;

$||(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}||\leq\frac{4C_1D_1}{\eta}\ (X)$

and, from (B);

$|(\frac{(cof(A_{ik_1k_2k_3t})^T)'_\nu|_{t_1}}{det(A_{ik_1k_2k_3t_1})})\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|\leq 3\sqrt{3}2\eta^{\frac{3}{2}}\frac{K_{ilt_1}}{\sqrt{\eta}}\frac{4C_1D_1}{\eta}$

$=24\sqrt{3}K_{ilt_1}C_1D_1$

so that, using $(A),(C)$;

$|(\frac{cof(A_{ik_1k_2k_3t})^T}{det(A_{ik_1k_2k_3t})})'_\nu|_{t_1}\overline{d}_{ik_1k_2k_3(t_1+\frac{1}{\nu})}|$

$\leq 288\sqrt{3}(C+1)^2O_{ik_1k_2k_3t_1}K_{ilt_1}+24\sqrt{3}K_{ilt_1}C_1D_1=R_{it_1}$

and from $(D),(E)$, that;

$|(\overline{v}_t)'|_{t_1}| \leq Q_{it_1} + R_{it_1} = S_{it_1}$

as required.

□

Lemma 1.37. *Let $||,||$ be the maximum modulus norm on matrices and $|,|$ the usual vector modulus. Then, for $\lambda \in \mathcal{R}$, $\overline{v} \in \mathcal{R}^3$, we have that;*

$||\lambda A|| = |\lambda|||A||$

$|A\overline{v}| \leq 3\sqrt{3}||A|||\overline{v}$

Proof. The first claim is clear. We have, for $1 \leq i \leq 3$, and $\overline{w}_i = \sum_{j=1}^3 \beta_{ji}\overline{e}_j$, $\overline{w} = \sum_{j=1}^3 \beta_j\overline{e}_j$, where $\beta_{ji} \in \{-1,1\}$, $\beta_j \in \{-1,1\}$ noting that $|\overline{w}_i| = |\overline{w}| = \sqrt{3}$, that;

$|A(\overline{e_i})| = |(|a_{1i}|,|a_{2i}|,|a_{3i}|) \centerdot \overline{w}_i|$

$\leq |(|a_{1i}|,|a_{2i}|,|a_{3i}|)||\overline{w}_i|$

$= \sqrt{3}|(|a_{1i}|,|a_{2i}|,|a_{3i}|)|$

$= \sqrt{3}(\sum_{j=1}^3 |a_{ji}|^2)^{\frac{1}{2}}$

$\leq \sqrt{3}(\sum_{j=1}^3 ||A||^2)^{\frac{1}{2}}$

$= 3||A||$

so that;

$|A\overline{v}| = |\sum_{i=1}^3 v_iA(\overline{e_i})|$

$\leq \sum_{i=1}^3 |v_i|A(\overline{e_i}|$

$\leq 3||A||\sum_{i=1}^3 |v_i|$

$= 3||A|||\overline{v} \centerdot \overline{w}|$

$\leq 3||A|||\overline{v}||\overline{w}|$

$= 3\sqrt{3}||A|||\overline{v}|$

$\square$

Definition 1.38. *Let* $\overline{s}_{\eta,\nu} : \mathcal{R}_\eta^3 \times \mathcal{T}_\nu \rightarrow {}^*\mathcal{R}^3$ *be as in Definition 2.29, and let* $\overline{s}_{\eta,\nu}(\overline{x},t)$ *be written in coordinates as* $(w_{\overline{x}1t}, w_{\overline{x}2t}, w_{\overline{x}3t})$*. We let* $\overline{d}_{\eta,\nu} : \mathcal{R}_\eta^{12} \times \mathcal{T}_\nu \rightarrow {}^*\mathcal{R}^3$ *be defined by;*

$\overline{d}_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) = (\sum_{k=1}^3 \frac{w^2_{\overline{x}_1kt} - w^2_{\overline{x}_0kt}}{2}, \sum_{k=1}^3 \frac{w^2_{\overline{x}_2kt} - w^2_{\overline{x}_0kt}}{2}, \sum_{k=1}^3 \frac{w^2_{\overline{x}_3kt} - w^2_{\overline{x}_0kt}}{2})$

and $\overline{A}_{\eta,\nu} : \mathcal{R}_\eta^{12} \times \mathcal{T}_\nu \rightarrow {}^*\mathcal{R}^9$ *be defined, for* $1 \leq l, k \leq 3$ *by;*

$(\overline{A}_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t))_{lk} = w_{\overline{x}_lkt} - w_{\overline{x}_0kt}$

Let $U \subset R_\eta^{12} \times T_\nu$ *be the* $*$*-finite set for which* $det(\overline{A}_{\eta,\nu}) \neq 0$*, and* $V = \{(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) : (\frac{[\overline{x}_0\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[\overline{x}_1\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[\overline{x}_2\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[\overline{x}_3\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[t\nu]}{\nu}) \in U\}$*. We define* $\overline{\Gamma}_{\eta,nu} : V \rightarrow {}^*\mathcal{R}^3$ *by;*

$\overline{\Gamma}_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) = (\overline{A}_{\eta,\nu})^{-1}\overline{d}_{\eta,\nu}$

Lemma 1.39. *Using similar notation to Lemma 2.30, we have, for* $1 \leq i, j \leq 3$*, that,*

$(\frac{\partial^2 \overline{\Gamma}_{\eta,\nu}}{\partial x_{ij} \partial t})_{\eta,\nu}|_{(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t)} = \overline{0}$

Proof. Let $\tau_{12} : \mathcal{R}_\eta^{12} \times \mathcal{T}_\nu \rightarrow \mathcal{R}_\eta^{12} \times \mathcal{T}_\nu$ be the elementary permutation which permutes $\{\overline{x}_1, \overline{x}_2\}$, and fixes $\{\overline{x}_3, t\}$. We have that;

$(\overline{A}_{\eta,\nu}(\tau_{12}(\overline{x}_0, \overline{x}_2, \overline{x}_1, \overline{x}_3, t)))_{lk} = (\overline{A}_{\eta,\nu}(\overline{x}_0, \overline{x}_2, \overline{x}_1, \overline{x}_3, t))_{lk}$

$= w_{\overline{x}_{\tau_{12}(l)}kt} - w_{\overline{x}_0kt}$

$= (\overline{A}_{\eta,\nu}(\overline{x}_0, \overline{x}_2, \overline{x}_1, \overline{x}_3, t))_{\tau_{12}(l)k}$

and, swapping the first two rows;

$det(\overline{A}_{\eta,\nu})(\tau_{12}(\overline{x}_0, \overline{x}_2, \overline{x}_1, \overline{x}_3, t)) = det(\overline{A}_{\eta,\nu})(\overline{x}_0, \overline{x}_2, \overline{x}_1, \overline{x}_3, t)$

$= -det(\overline{A}_{\eta,\nu})(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t))$

It follows that τ_{12} leaves U invariant. Moreover, as, for $(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)\in U$, $(\overline{A}_{\eta,\nu})^{-1}\overline{d}_{\eta,\nu}|_{\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t}$ defines the point intersection of the three hyperplanes $\{H_{\overline{x}_0\overline{x}_kt}:1\leq k\leq 3\}$, which are permuted by τ_{12}, we have that $\Gamma_{\eta,\nu}\circ\tau_{12}=\Gamma_{\eta,\nu}$. It follows that;

$$(\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=\nu(\Gamma_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3,(t+\frac{1}{\nu}))}-\Gamma_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3,t)})$$

$$=\nu(\Gamma_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}-\Gamma_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)})$$

$$=(\Gamma_{\eta,\nu}\circ\tau_{12})'_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$$

$$=(\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$$

so that $(\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12}=(\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu$ as well.

If follows that, for $1\leq j\leq 3$;

$$\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{1j}\partial t}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=\frac{\partial}{\partial x_{1j}}((\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$$

$$=\sqrt{\eta}[((\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12})|_{(\overline{x}_0,\frac{(\overline{x}_1+\overline{e}_j)}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}-((\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}]$$

$$=\sqrt{\eta}[((\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu)|_{(\overline{x}_0,\overline{x}_2,\frac{(\overline{x}_1+\overline{e}_j)}{\sqrt{\eta}},\overline{x}_3,t)}-((\frac{\partial\Gamma_{\eta,\nu}}{\partial t})_\nu)|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3,t)}]$$

$$=(\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3,t)}$$

$$=((\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}\circ\tau_{12})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$$

so that $\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{1j}\partial t}=\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{2j}\partial t}\circ\tau_{12}$, (N).

As $\overline{A}_{\eta,\nu}\overline{A}_{\eta,\nu}^{-1}=Id$, we have that;

$$\frac{\partial(\overline{A}_{\eta,\nu}\overline{A}_{\eta,\nu}^{-1})}{\partial x_{1j}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=O$$

so that;

$$(\frac{\partial}{\partial x_{1j}})_\eta(\overline{A}_{\eta,\nu})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}+\overline{A}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial}{\partial x_{1j}})_\eta(\overline{A}_{\eta,\nu}^{-1})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$$

$$=O$$

and;

$(\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=-\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{A}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$=-\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$ (K)

where;

$((\overline{B}_{\eta,\nu})|_{(\overline{x}_1,t)})_{1k}=(\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_1,t)})_k$

$((\overline{B}_{\eta,\nu})|_{(\overline{x}_1,t)})_{lk}=0$, for $2\leq l\leq 3$

Similarly;

$(\frac{\partial^2\overline{A}^{-1}_{\eta,\nu}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$=\frac{\partial}{\partial t}(-\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu$ (P)

We have, using the product rule, that;

$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=(\frac{\partial^2(\overline{A}^{-1}_{\eta,\nu}\overline{d}_{\eta,\nu})}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$=\frac{\partial}{\partial t}(((\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)})\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}+(\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)})_\nu$

$=(\frac{\partial^2\overline{A}^{-1}_{\eta,\nu}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+(\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$+(\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial^2\overline{d}_{\eta,\nu}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$
(L)

By symmetry, we have that;

$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$=(\frac{\partial^2\overline{A}^{-1}_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}+(\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial x_{2j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

$+(\frac{\partial\overline{A}^{-1}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial x_{2j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial^2\overline{d}_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$
(M)

Combining $(L),(M),(N)$, we obtain that;

$(\frac{\partial^2\overline{A}_{\eta,\nu}^{-1}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$+(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial x_{1j}})_\eta|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial^2\overline{d}_{\eta,\nu}}{\partial x_{1j}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$=(\frac{\partial^2\overline{A}_{\eta,\nu}^{-1}}{\partial x_{2j}\partial t})_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{d}_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}+(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial x_{2j}})_\eta\circ$
$\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

$+(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_\nu\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial\overline{d}_{\eta,\nu}}{\partial x_{2j}})_\eta\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}_{\eta,\nu}^{-1}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial^2\overline{d}_{\eta,\nu}}{\partial x_{2j}\partial t})_{\eta,\nu}\circ$
$\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$ (O)

We have that;

$\overline{d}_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3,t)}=\overline{d}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

where, for a vector $\overline{v}\in{}^*\mathcal{R}^3$, $\overline{v}^{\tau_{12}}$ is obtained from $\overline{v}$ by swapping the first and second entries. A similar calculation shows that;

$(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

We have that;

$\overline{A}_{\eta,\nu}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}=\overline{A}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

where, for a matrix $\overline{A}\in{}^*\mathcal{R}^9$, $\overline{A}^{\tau_{12}}$ is obtained from $\overline{A}$ by swapping the first and second rows.

It follows, as $\overline{A}_{\eta,\nu}\overline{A}_{\eta,\nu}^{-1}=Id$, that;

$I=I\circ\tau_{12}=(\overline{A}_{\eta,\nu}\overline{A}_{\eta,\nu}^{-1})\circ\tau_{12}$

$=(\overline{A}_{\eta,\nu}\circ\tau_{12})(\overline{A}_{\eta,\nu}^{-1}\circ\tau_{12})$

$=\overline{A}_{\eta,\nu}^{\tau_{12}}(\overline{A}_{\eta,\nu}^{-1}\circ\tau_{12})$

so that;

$\overline{A}_{\eta,\nu}^{-1}\circ\tau_{12}=(\overline{A}_{\eta,\nu}^{\tau_{12}})^{-1}=(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}$

where, for a matrix $\overline{B}\in{}^{*}\mathcal{R}^{9}$, $\overline{B}_{\tau_{12}}$ is obtained from $\overline{B}$ by swapping the first and second columns.

Similarly, we have that;

$(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_{\nu}\circ\tau_{12}=((\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_{\nu})_{\tau_{12}}$

We have, using Lemma 2.30, that;

$\frac{\partial}{\partial x_{1j}}(\overline{d}_{\eta,\nu}|)_{\eta}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3t))}$

$=\frac{\partial}{\partial x_{1j}}(\sum_{k=1}^{3}\frac{w^2_{\overline{x}_1kt}-w^2_{\overline{x}_0kt}}{2},\sum_{k=1}^{3}\frac{w^2_{\overline{x}_2kt}-w^2_{\overline{x}_0kt}}{2},\sum_{k=1}^{3}\frac{w^2_{\overline{x}_3kt}-w^2_{\overline{x}_0kt}}{2})$

$=\frac{\partial}{\partial x_j}(\sum_{k=1}^{3}\frac{w^2_{\overline{x}_1kt}}{2},0,0)|_{(\overline{x}_1,t)}$

$=(\frac{\partial}{\partial x_j}(\frac{\sigma_{\overline{x}_1}(t)\centerdot\sigma_{\overline{x}_1}(t)}{2}),0,0)|_{\overline{x}_1}$

$=(\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t)+\sigma_{\overline{x}_1}(t)),0,0)|_{\overline{x}_1}$

$\simeq((\frac{\partial\overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_1,t)}\centerdot\overline{s}_{\eta,\nu}|_{(\overline{x}_1,t)},0,0)$ (F)

Similarly, using (F), we have that;

$\frac{\partial^2}{\partial x_{1j}\partial t}(\overline{d}_{\eta,\nu}|)_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3t)}$

$=\frac{\partial}{\partial t}((\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t)+\sigma_{\overline{x}_1}(t+\frac{1}{\nu})),0,0)|_{\overline{x}_1})|_t$

$\simeq(((\frac{\partial^2\overline{s}_{\eta,\nu}}{\partial x_j\partial t})_{\eta,\nu}|_{(\overline{x}_1,t)}\centerdot\overline{s}_{\eta,\nu}|_{(\overline{x}_1,t)}+(\frac{\partial\overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_1,t)}\centerdot(\frac{\partial\overline{s}_{\eta,\nu}}{\partial t})_{\nu}|_{(\overline{x}_1,t)}),0,0)$ (G)

It follows, by symmetry, that;

$\frac{\partial}{\partial x_{2j}}(\overline{d}_{\eta,\nu}|)_{\eta}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3t))}$

$=(0,\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_2}(t))\centerdot(\sigma_{\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}}}(t)+\sigma_{\overline{x}_2}(t)),0)|_{\overline{x}_2}$

$\simeq(0,(\frac{\partial\overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_2,t)}\centerdot\overline{s}_{\eta,\nu}|_{(\overline{x}_2,t)},0)$ (F')

and;

$\frac{\partial}{\partial x_{2j}}(\overline{d}_{\eta,\nu}|)_{\eta}\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3t))}$

$= \frac{\partial}{\partial x_{2j}}(\overline{d}_{\eta,\nu}|)_{\eta}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3 t))}$

$= (0, \frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t)) \centerdot (\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t) + \sigma_{\overline{x}_1}(t)), 0)|_{\overline{x}_1}$

$\simeq (0, (\frac{\partial \overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_1,t)} \centerdot \overline{s}_{\eta,\nu}|_{(\overline{x}_1,t)}, 0)$ (F'')

Similarly;

$\frac{\partial^2}{\partial x_{2j}\partial t}(\overline{d}_{\eta,\nu}|)_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3 t)}$

$= \frac{\partial}{\partial t}((0, \frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_2}(t)) \centerdot (\sigma_{\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}}}(t) + \sigma_{\overline{x}_2}(t+\frac{1}{\nu})), 0)|_{\overline{x}_2})|_t$

$\simeq (0, ((\frac{\partial^2 \overline{s}_{\eta,\nu}}{\partial x_j \partial t})_{\eta,\nu}|_{(\overline{x}_2,t)} \centerdot \overline{s}_{\eta,\nu}|_{(\overline{x}_2,t)} + (\frac{\partial \overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_2,t)} \centerdot (\frac{\partial \overline{s}_{\eta,\nu}}{\partial t})_{\nu}|_{(\overline{x}_2,t)}), 0)$ (G')

and;

$\frac{\partial^2}{\partial x_{2j}\partial t}(\overline{d}_{\eta,\nu}|)_{\eta,\nu} \circ \tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3 t)}$

$= \frac{\partial^2}{\partial x_{2j}\partial t}(\overline{d}_{\eta,\nu}|)_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_2,\overline{x}_1,\overline{x}_3 t)}$

$= \frac{\partial}{\partial t}((0, \frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t)) \centerdot (\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t) + \sigma_{\overline{x}_1}(t+\frac{1}{\nu})), 0)|_{\overline{x}_1})|_t$

$\simeq (0, ((\frac{\partial^2 \overline{s}_{\eta,\nu}}{\partial x_j \partial t})_{\eta,\nu}|_{(\overline{x}_1,t)} \centerdot \overline{s}_{\eta,\nu}|_{(\overline{x}_1,t)} + (\frac{\partial \overline{s}_{\eta,\nu}}{\partial x_j})_{\eta}|_{(\overline{x}_1,t)} \centerdot (\frac{\partial \overline{s}_{\eta,\nu}}{\partial t})_{\nu}|_{(\overline{x}_1,t)}), 0)$ (G'')

By symmetry, using (K), we have that;

$(\frac{\partial \overline{A}^{-1}_{\eta,\nu}}{\partial x_{2j}})_{\eta}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)} = -\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_2,t)}\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$
(K')

where;

$((\overline{C}_{\eta,\nu})|_{(\overline{x}_2,t)})_{2k} = (\frac{\partial \overline{s}}{\partial x_j}|_{(\overline{x}_2,t)})_k$

$((\overline{C}_{\eta,\nu})|_{(\overline{x}_2,t)})_{lk} = 0$, for $l = 1$ or $l = 3$

and;

$(\frac{\partial \overline{A}^{-1}_{\eta,\nu}}{\partial x_{2j}})_{\eta} \circ \tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$= -\overline{A}^{-1}_{\eta,\nu} \circ \tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu} \circ \tau_{12}|_{(\overline{x}_2,t)}\overline{A}^{-1}_{\eta,\nu} \circ \tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

$= -(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$ (K'')

and, similarly;

$(\frac{\partial^2\overline{A}_{\eta,\nu}^{-1}}{\partial x_{2j}\partial t})_{\eta,\nu} \circ \tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$

$= \frac{\partial}{\partial t}(-(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu$ (P')

It follows from (O), that;

$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}$

$+(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$-(\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}((\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t+\frac{1}{\nu}))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t+\frac{1}{\nu})+\sigma_{\overline{x}_1}(t+\frac{1}{\nu})),0,0)|_{\overline{x}_1})$

$-\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial}{\partial t}((\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t)+\sigma_{\overline{x}_1}(t+\frac{1}{\nu})),0,0)|_{\overline{x}_1})|_t)$

$= \frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu\overline{d}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+((\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

$-((\frac{\partial\overline{A}_{\eta,\nu}^{-1}}{\partial t})_\nu)_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}((0,\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t+\frac{1}{\nu}))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t+\frac{1}{\nu})+\sigma_{\overline{x}_1}(t+\frac{1}{\nu})),0)|_{\overline{x}_1})$

$-(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}(\frac{\partial}{\partial t}((0,\frac{1}{2}\frac{\partial}{\partial x_j}(\sigma_{\overline{x}_1}(t))\centerdot(\sigma_{\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}}}(t)+\sigma_{\overline{x}_1}(t+\frac{1}{\nu})),0)|_{\overline{x}_1})|_t)$

(O')

Cancelling terms, it follows that;

$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}$

$+(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$= \frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu\overline{d}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+((\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

(O'')

so that, by the product rule;

$(\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}$

$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}$

$+(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$=(\frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}\frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}^{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+((\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

(O''')

and, noting that;

$\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}=(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{C}_{\eta,\nu}|_{(\overline{x}_1,t)}$

we obtain;

$(\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}$

$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}$

$+(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$

$=(\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}^{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$

$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})_{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial\overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$

(O'''')

As;

$$(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu \overline{d}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$= (\overline{A}_{\eta,\nu}^{-1})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu \overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$((\frac{\partial(\overline{A}_{\eta,\nu}^{-1})}{\partial t})_\nu)_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)} \overline{d}_{\eta,\nu}^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$= (\frac{\partial(\overline{A}_{\eta,\nu}^{-1})}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)} \overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$(\overline{A}_{\eta,\nu}^{-1})_{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu^{\tau_{12}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$$

$$= (\overline{A}_{\eta,\nu}^{-1})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$$

we obtain that;

$$(\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu \overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))} + \overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}$$

$$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$+(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}$$

$$= (\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu \overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}\frac{\partial}{\partial t}((\overline{A}_{\eta,\nu}^{-1})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu)\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}$$

$$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}(\overline{A}_{\eta,\nu}^{-1})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}$$

(O''''')

Rearranging, we obtain;

$$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu[\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))} -$$

$$\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}\overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}]$$

$$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}[\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})_\nu \overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))} -$$

$$\frac{\partial}{\partial t}(\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu \overline{d}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}]$$

$$+\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}[\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)} -$$

$$\overline{A}_{\eta,\nu}^{-1}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})(\frac{\partial \overline{d}_{\eta,\nu}}{\partial t})_\nu|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}] = \overline{0}\ (O'''''')$$

so that, using the product rule twice;

$$\frac{\partial}{\partial t}(\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)})_\nu[\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,(t+\frac{1}{\nu}))}-\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,(t+\frac{1}{\nu}))}]$$

$$+\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}[\frac{\partial}{\partial t}(\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}-\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)})_\nu]$$

$$=\frac{\partial}{\partial t}(\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}[\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_2,\overline{x}_3,t)}-\overline{\Gamma}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2+\frac{\overline{e}_j}{\sqrt{\eta}},\overline{x}_3,t)}])_\nu$$

$$=\overline{0}\ (O''''''')$$

It follows that;

$$=\frac{\partial}{\partial t}(\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}[(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}])_\nu=\overline{0}$$

and, by the definition of $\overline{B}_{\eta,\nu}|_{(\overline{x}_1,t)}$ and the fact that $\overline{A}^{-1}_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}$ is invertible, we have that;

$$\frac{\partial}{\partial t}(<\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_1,t)},(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}>)_\nu=0,\ (H)$$

so that;

$$<\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_1,t)},(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}>=c\ (I)$$

Applying τ_{12}, we obtain that;

$$<\frac{\partial\overline{s}}{\partial x_j}\circ\tau_{12}|_{(\overline{x}_1,t)},(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})\circ\tau_{12}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}>=c$$

and, using the proof of (N), for $1\leq j\leq 3$, we have that;

$$\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{1j}}\circ\tau_{12}=\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{2j}}$$

$$\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{2j}}\circ\tau_{12}=\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{1j}}$$

so that, for $1\leq j\leq 3$;

$$<\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_2,t)},(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}>=c\ (I')$$

Subtracting (I') from (I), we have, for $1\leq j\leq 3$, that;

$$<\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_1,t)}-\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_2,t)},(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)}>=0\ (I'')$$

Fix j, and choose $(\overline{x}_{10},\overline{x}_{20},t_0)$ with;

$\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{10},t_0)}-\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{20},t_0)}\neq\overline{0}$

and a path $\overline{x}_{20}(s)$, $0\leq s\leq s_0$, $s_0\in\mathcal{R}_{>0}$, with $\overline{x}_{20}(0)=\overline{x}_{20}$, such that;

$\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{10},t_0)}-\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{20}(s),t_0+s)}$

is fixed.

Let;

$R(\overline{z},s)=(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_{10},\overline{x}_{20}(s),\overline{z},t_0+s)}$

Then, for any $\overline{x}_3$ with $(\overline{x}_0,\overline{x}_{10},\overline{x}_{20},\overline{x}_3,t_0)\in U$, on some ${}^*B(\overline{x}_3,\epsilon)$, $\epsilon\in\mathcal{R}_{>0}$, we have that R maps ${}^*B(\overline{x}_3,\epsilon)\times[0,s_0]$ into $\mathcal{H}_{(\overline{x}_0,\overline{x}_{10},\overline{x}_{20},t_0)}\subset{}^*\mathcal{R}^3$, where $\mathcal{H}_{(\overline{x}_0,\overline{x}_{10},\overline{x}_{20},t_0)}$ is the *-hyperplane defined by;

$\mathcal{H}_{(\overline{x}_0,\overline{x}_{10},\overline{x}_{20},t_0)}=\{\overline{v}:<\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{10},t_0)}-\frac{\partial\overline{s}}{\partial x_j}|_{(\overline{x}_{20},t_0)},\overline{v}>=0\}$

As $R|_{{}^*B(\overline{x}_3,\epsilon)}\times[0,s_0]$ is $*$-analytic on ${}^*B(\overline{x}_3,\epsilon)\times[0,s_0]$, in particularly $*$-smooth on ${}^*B(\overline{x}_3,\epsilon)\times[0,s_0]$, by the $*$-inverse function theorem, there exists a smooth path $\gamma_\epsilon:(-\delta,\delta)\rightarrow{}^*B(\overline{x}_3,\epsilon)\times[0,s_0]$, such that $R|_{Im(\gamma_\epsilon)}=\overline{f}$. In particularly, transferring the property that the only real analytic functions with a limit point of zeros, are zero, we can conclude that $R|_{{}^*B(\overline{x}_3,\epsilon)\times[0,s_0]}=\overline{f}$, so that;

$(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}}+\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2(s),\overline{z},t_0+s)}=\overline{f}(\overline{x}_0,\overline{x}_1,\overline{x}_2,t_0)$, $\overline{z}\in{}^*B(\overline{x}_3,\epsilon)$ (I''')

In particular, differentiating with respect to t;

$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{1j}}+\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2(s),\overline{x}_3,t_0+s)}$

$+(\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}})_1)\centerdot\overline{x}_2'(s),\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}})_2)\centerdot\overline{x}_2'(s),\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}})_3)\centerdot\overline{x}_2'(s))|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2(s),\overline{x}_3,t_0+s)}$

$+(\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})_1)\centerdot\overline{x}_2'(s),\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})_2)\centerdot\overline{x}_2'(s),\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})_3)\centerdot\overline{x}_2'(s))|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2(s),\overline{x}_3,t_0+s)}$

$=\overline{0}$, (I'''')

From (I'''), differentiating with respect to $\overline{x}_3$, taking $s=0$, we have for $1\leq k\leq 3$, that;

$$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{3k}\partial x_{1j}}+\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{3k}\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t_0)}=\overline{0}$$

so that, taking the limit as $\overline{x}_2\rightarrow\overline{x}_1$, we have, as below, that, for $1\leq k\leq 3$;

$$\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{3k}\partial x_{1j}}=\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{3k}\partial x_{2j}}=\overline{0}$$

By symmetry we can conclude that, for $1\leq j\leq 3$,$1\leq k\leq 3$, $1\leq a<b\leq 3$;

$$\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{ak}\partial x_{bj}}=\overline{0}$$

By a similar argument to (N), we have that;

$$\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{1j}}=\frac{\partial\Gamma_{\eta,\nu}}{\partial x_{2j}}\circ\tau_{12}$$

so that, for $1\leq j\leq 3$, $1\leq k\leq 3$;

$$\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{1k}x_{1j}}=\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{1k}x_{2j}}\circ\tau_{12}=\overline{0}$$

and, by symmetry, we can conclude that, for $1\leq a\leq 3$, $1\leq j\leq 3$, $1\leq k\leq 3$;

$$\frac{\partial^2\Gamma_{\eta,\nu}}{\partial x_{ak}x_{aj}}=\overline{0}$$

as well. In particularly, for $1\leq d\leq 3$;

$$(\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{1j}})_d)=(\bigtriangledown_{\overline{x}_2}((\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial x_{2j}})_d)=\overline{0}\ (I''''')$$

From (I''''), (I'''''), we obtain that;

$$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{1j}}+\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{2j}})|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2(s),\overline{x}_3,t_0+s)}=\overline{0},\ (I'''''')$$

Using (N), taking $s=0$, we have that;

$$\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{1j}}\circ\tau_{1j}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t_0)}=-\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial t\partial x_{1j}}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t_0)}$$

Taking the limit as $\overline{x}_1 \rightarrow \overline{x}_2$, we obtain that;

$\frac{\partial^2 \overline{\Gamma}_{\eta,\nu}}{\partial t \partial x_{1j}}|_\Delta = \overline{0}$

where $\Delta = \{(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) \in {}^*\mathcal{R}^{12} \times \mathcal{T}_\nu, \overline{x}_1 = \overline{x}_2\}$

and we extend $\frac{\partial^2 \overline{\Gamma}_{\eta,\nu}}{\partial t \partial x_{1j}}$ $*$-analytically to Δ. By the limiting property of $*$-analytic functions, we obtain that;

$\frac{\partial^2 \overline{\Gamma}_{\eta,\nu}}{\partial t \partial x_{1j}} = \overline{0}$

and, by symmetry, we have, for $1 \leq k \leq 3$, $1 \leq j \leq 3$, that;

$\frac{\partial^2 \overline{\Gamma}_{\eta,\nu}}{\partial t \partial x_{kj}} = \overline{0}$

$\square$

Definition 1.40. *We say that two vertices $\{\overline{v}_1, \overline{v}_2\} \subset V_{it}$, in the alcove $Z_{i,t}$, defined by $\overline{x}_0$, are connected if they are defined by relevant points $\{(s_{\overline{x}_0}(t), s_{\overline{x}_{11}}(t), s_{\overline{x}_{21}}(t), s_{\overline{x}_{31}}(t))\}$ and $\{(s_{\overline{x}_0}(t), s_{\overline{x}_{12}}(t), s_{\overline{x}_{22}}(t), s_{\overline{x}_{32}}(t))\}$, for $t \in (t_0, t_0 + \epsilon)$, with $\epsilon \in \mathcal{R}_{>0}$, with the tuples $(\overline{x}_0, \overline{x}_{11}, \overline{x}_{21}, \overline{x}_{31}, t)$ and $(\overline{x}_0, \overline{x}_{12}, \overline{x}_{22}, \overline{x}_{32}, t)$ lying in the same $*$-connected component of U.*

Given an alcove $Z_{i,t}$, defined by $\overline{x}_0$, with vertices $V_{i,t} = \{\overline{v}_i : 1 \leq i \leq \kappa(\overline{x}_0)\}$, we have that, for any two distinct vertices, $\{\overline{v}_1, \overline{v}_2\} \subset V_{it}$, the edges $\overline{e}_{\overline{x}_0,\overline{v}_1}$ and $\overline{e}_{\overline{x}_0,\overline{v}_2}$ intersect at $\overline{x}_0$. Otherwise, without loss of generality, $\overline{v}_2 \in \overline{e}_{\overline{x}_0,\overline{v}_2}$, contradicting the definition of a vertex. For three distinct vertices $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$, we define $T_{\overline{x}_0,\overline{v}_1,\overline{v}_2,\overline{v}_3}$ to be the $*$-convex hull of $\{\overline{x}_0, \overline{v}_1, \overline{v}_2, \overline{v}_3\}$, so that $T_{\overline{x}_0,\overline{v}_1,\overline{v}_2,\overline{v}_3} \subset Z_{i,t}$. Let $p \in Z_{i,t}$, with $p \neq \overline{x}_0$, then $l_{\overline{x}_0,p}$ intersects $\delta Z_{i,t}$ at q, with $q \in V_{ik,t}$, a $*$-connected component of $\delta Z_{i,t}$, $dim(V_{ik,t}) = 2$. If $\overline{v} \in V_{ik,t}$ is an extreme point of $V_{ik,t}$, then it is a vertex, otherwise it would lie on a line $l_{\overline{y}_1,\overline{y}_2}$, with $dim(l_{\overline{y}_1,\overline{y}_2} \cap V_{ik,t}) = 1$, $\{\overline{y}_1, \overline{y}_2\} \subset Z_{i,t}$, and $\{\overline{y}_1, \overline{y}_2\}$ distinct from $\overline{v}$, contradicting the fact that $\overline{v} \in V_{ik,t}$. As $dim(V_{ik,t}) = 2$, it has at least 3 vertices, and we can pick $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$, such that $q \in \mathcal{H}_{\overline{v}_1,\overline{v}_2,\overline{v}_3}$. In particular, it follows that;

$Z_{i,t} = {}^*\bigcup_{(\overline{v}_1,\overline{v}_2,\overline{v}_3) \in V_{i,t}^3, \overline{v}_i \neq \overline{v}_j, 1 \leq i < j \leq 3} T_{\overline{x}_0,\overline{v}_1,\overline{v}_2,\overline{v}_3}$

Let $R_{\overline{x}_0}$ be the $*$-planar graph, with vertices $\{a_i : 1 \leq i \leq \kappa(\overline{x}_0)\}$. and an edge e_{ij} assigned if $\overline{e}_{\overline{v}_i,\overline{v}_j} \subset \delta Z_{i,t}$. Clearly, $R_{\overline{x}_0}$ is 4-connected, so, by [9], there exists a $*$-Hamiltonian cycle C, with $C(0) = C(\kappa(\overline{x}_0))$.

Then, clearly;

$Z_{i,t} = {}^*\bigcup_{0\leq j\leq \kappa(\overline{x}_0)-2} T_{\overline{x}_0,\overline{v}_{C(j)},\overline{v}_{C(j+1)},\overline{v}_{C(j+2)}}$

and, moreover, for $0 \leq j < k \leq \kappa(\overline{x}_0) - 2$*;*

$dim(T_{\overline{x}_0,\overline{v}_{C(j)},\overline{v}_{C(j+1)},\overline{v}_{C(j+2)}} \cap T_{\overline{x}_0,\overline{v}_{C(k)},\overline{v}_{C(k+1)},\overline{v}_{C(k+2)}}) \leq 2$

Lemma 1.41. *We have, for two connected vertices* $\{\overline{v}_1, \overline{v}_2\} \subset V_{it}$*, that, for* $t \in (t_0, t_0 + \epsilon)$*;*

$(\overline{v}_1)'_\nu(t) = (\overline{v}_2)'_\nu(t)$

and the velocities are all finite. Moreover, if;

$\overline{e}_{\overline{v}_1,\overline{v}_2}(t) = \overline{v}_1(t) - \overline{v}_2(t)$

and;

$l_{\overline{v}_1,\overline{v}_2}(t) = |\overline{e}_{\overline{v}_1,\overline{v}_2}(t)|$

is the length of the edge connecting $\overline{v}_1$ *and* $\overline{v}_2$*, then* $(l_{\overline{v}_1,\overline{v}_2})'_\nu(t) = 0$*, for* $t \in (t_0, t_0 + \epsilon)$*.*

We have, for a vertex $\overline{v}$*, that there exists* $\{L_{\overline{v},\overline{x}_0,t}, K_{\overline{v},\overline{x}_0,t}\} \subset \mathcal{R}_{>0}$*, with;*

$|(\overline{v})_\nu(t) - (s_{\overline{x}_0})_\nu(t)| \leq \frac{L_{\overline{v},\overline{x}_0,t}}{\sqrt{\eta}}$

$|(\overline{v})'_\nu(t) - (s_{\overline{x}_0})'_\nu(t)| \leq \frac{K_{\overline{v},\overline{x}_0,t}}{\sqrt{\eta}}$

Proof. By Lemma 1.39, we have that, for the alcove $Z_{i,t}$ defined by $\overline{x}_0$, that, for $1 \leq i \leq 3$, $1 \leq j \leq 3$;

$(\frac{\partial^2\overline{\Gamma}_{\eta,\nu}}{\partial x_{ij}\partial t})_{\eta,\nu}|_{(\overline{x}_0,\overline{x}_1,\overline{x}_2,\overline{x}_3,t)} = \overline{0}$

so that $(\frac{\partial\overline{\Gamma}_{\eta,\nu}}{\partial t})_\nu$ is constant on the $*$-connected components of U. and;

$\overline{v}_1(t) = \overline{\Gamma}_{\eta,\nu}|(\overline{x}_0, \overline{x}_{11}, \overline{x}_{21}, \overline{x}_{31}, t), \overline{v}_2(t) = \overline{\Gamma}_{\eta,\nu}|(\overline{x}_0, \overline{x}_{12}, \overline{x}_{22}, \overline{x}_{32}, t)$

where for $1 \leq i \leq 3$;

$s_{\overline{x}_{i1}}(t) = \overline{w}_{i1,t}$, $s_{\overline{x}_{i2}}(t) = \overline{w}_{i2,t}$

$\overline{v}_1(t) = \mathcal{H}_{k_{11}t} \cap \mathcal{H}_{k_{21}t} \cap \mathcal{H}_{k_{31}t}$, $\overline{v}_2(t) = \mathcal{H}_{k_{12}t} \cap \mathcal{H}_{k_{22}t} \cap \mathcal{H}_{k_{32}t}$

It follows that, for $t \in (t_0, t_0 + \epsilon)$;

$|(\overline{v}_1)'_\nu(t) - (\overline{v}_2)'_\nu(t)| = |\frac{\partial \overline{\Gamma}_{\eta,\nu}}{\partial t}|(\overline{x}_0, \overline{x}_{11}, \overline{x}_{21}, \overline{x}_{31}, t) - \frac{\partial \overline{\Gamma}_{\eta,\nu}}{\partial t}|(\overline{x}_0, \overline{x}_{12}, \overline{x}_{22}, \overline{x}_{32}, t)|$

$= 0$

so that $(\overline{v}_1)'_\nu(t) = (\overline{v}_2)'_\nu(t)$, $(*)$. By Lemma 1.36, all the velocities are finite. We have, for $t \in (t_0, t_0 + \epsilon)$, that;

$(\overline{e}_{\overline{v}_1,\overline{v}_2})'_\nu(t) = (\overline{v}_1)'_\nu(t) - (\overline{v}_2)'_\nu(t)$

$= \overline{0}$

so that $\overline{e}_{\overline{v}_1,\overline{v}_2}$ is constant on $(t_0, t_0 + \epsilon)$. In particularly, $l_{\overline{v}_1,\overline{v}_2}$ is constant on $(t_0, t_0 + \epsilon)$, and $(l_{\overline{v}_1,\overline{v}_2})'_\nu(t) = 0$, for $t \in (t_0, t_0 + \epsilon)$.

For the penultimate claim, we have that, for $(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) \in U$, using the calculations $(\sharp\sharp\sharp), (\sharp\sharp\sharp\sharp)$, from Lemma 1.36, that;

$|(\overline{v})_\nu(t) - (s_{\overline{x}_0})_\nu(t)| = |\Gamma_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) - s_{\overline{x}_0,t}|$

$= |\overline{A}^{-1}_{\eta,\nu,t}\overline{d}_{\eta,\nu,t} - s_{\overline{x}_0,t}|$

$= |\overline{A}^{-1}_{\eta,\nu,t}(\overline{d}_{\eta,\nu,t} - \overline{A}_{\eta,\nu,t}s_{\overline{x}_0,t})|$

$\leq 3\sqrt{3}||\overline{A}^{-1}_{\eta,\nu,t}|||\overline{d}_{\eta,\nu,t} - \overline{A}_{\eta,\nu,t}s_{\overline{x}_0,t}|, \leq 3\sqrt{3}(48(C+1)^2)\eta^{\frac{1}{2}}|\overline{d}_{\eta,\nu,t} - \overline{A}_{\eta,\nu,t}s_{\overline{x}_0,t}|$

$= 144\sqrt{3}(C+1)^2\eta^{\frac{1}{2}}|\overline{d}_{\eta,\nu,t} - \overline{A}_{\eta,\nu,t}s_{\overline{x}_0,t}|$ $(\dagger)$

We have that, for $1 \leq j \leq 3$;

$|\overline{d}_{j,\eta,\nu,t} - (\overline{A}_{\eta,\nu,t}s_{\overline{x}_0,t})_j|$

$= |s_{\overline{x}_j,t}|^2 - |s_{\overline{x}_0,t}|^2 - (s_{\overline{x}_j,t} - s_{\overline{x}_0,t}) \centerdot s_{\overline{x}_0,t}$

$= |s_{\overline{x}_j,t} \centerdot (s_{\overline{x}_j,t} - s_{\overline{x}_0,t})|$

so that, from (†);

$|\Gamma_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) - s_{\overline{x}_0,t}|$

$\leq 144\sqrt{3}(C+1)^2\eta^{\frac{1}{2}} \sum_{j=1}^{3} |s_{\overline{x}_j,t} \centerdot (s_{\overline{x}_j,t} - s_{\overline{x}_0,t})|$ (††)

For a given t_0, using a translation by $s_{\overline{x}_0,t_0}$, it is easy to see that;

$|\Gamma_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) - s_{\overline{x}_0,t}| = |\Gamma'_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t) - s'_{\overline{x}_0,t}|$

where the hashed trajectory $s^{\sharp}_{\overline{x}_0,t} = s_{\overline{x}_0,t} - s_{\overline{x}_0,t_0}$, and, in the definition of Γ', we replace the trajectories $\{s_{\overline{x}_j,t} : 0 \leq j \leq 4\}$ by $\{s^{\sharp}_{\overline{x}_j,t} : 0 \leq j \leq 4\}$, with $s^{\sharp}_{\overline{x}_j,t} = s_{\overline{x}_j,t} - s_{\overline{x}_0,t_0}$, for $1 \leq j \leq 3$. We can, therefore, assume that $s_{\overline{x}_0,t_0} = \overline{0}$, and, using (†) of Lemma 1.19 and Lemma 1.32, $|s_{\overline{x}_j,t_0}| \leq \frac{R_{\overline{v},\overline{x}_0,t_0}}{\sqrt{\eta}}$, $1 \leq j \leq 3$, where $R_{\overline{v},\overline{x}_0,t_0} \in \mathcal{R}_{>0}$, to obtain, from (††), that;

$|\Gamma_{\eta,\nu}(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t_0) - s_{\overline{x}_0,t_0}|$

$\leq 144\sqrt{3}(C+1)^2\eta^{\frac{1}{2}} \sum_{j=1}^{3} |s_{\overline{x}_j,t_0}|^2$

$\leq 3.144\sqrt{3}(C+1)^2\eta^{\frac{1}{2}} \sum_{j=1}^{3} \frac{R^2_{\overline{v},\overline{x}_0,t_0}}{\eta}$

$= \frac{L_{\overline{v},\overline{x}_0,t_0}}{\eta^{\frac{1}{2}}}$

As t_0 was arbitrary, we obtain the result. For the final claim, suppose that $s_{\overline{x}_0,t_0} = s'_{\overline{x}_0,t_0} = \overline{0}$, so that $s_{\overline{x}_0,t_0+\frac{1}{\nu}} = \overline{0}$ and $(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t_0) \in U$. Using the calculations (♯♯♯), (♯♯♯♯), (!!), (X) from Lemma 1.36 we have that;

$|(\overline{v})'_\nu(t_0) - (s_{\overline{x}_0})'_\nu(t_0)| = |(\frac{\partial \Gamma_{\eta,\nu}}{\partial t})_\nu(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, t_0) - s'_{\overline{x}_0,t_0}|$

$= |\frac{\partial}{\partial t}(\overline{A}^{-1}_{\eta,\nu,t_0}\overline{d}_{\eta,\nu,t_0} - \overline{s}_{\eta,\nu,\overline{x}_0,t_0})_\nu|$

$= |\frac{\partial}{\partial t}[\overline{A}^{-1}_{\eta,\nu,t_0}(\overline{d}_{\eta,\nu,t_0} - \overline{A}_{\eta,\nu,t_0}\overline{s}_{\eta,\nu,\overline{x}_0,t_0})]_\nu|$

$= |(\frac{\partial \overline{A}^{-1}_{\eta,\nu,t_0}}{\partial t})_\nu(\overline{d}_{\eta,\nu,t_0+\frac{1}{\nu}} - \overline{A}_{\eta,\nu,t_0+\frac{1}{\nu}}\overline{s}_{\eta,\nu,\overline{x}_0,t_0+\frac{1}{\nu}}) + \overline{A}^{-1}_{\eta,\nu,t_0}(\overline{d}_{\eta,\nu,t_0} - \overline{A}_{\eta,\nu,t_0}\overline{s}_{\eta,\nu,\overline{x}_0,t_0})'_\nu|$

$\leq 3\sqrt{3}||(\frac{\partial \overline{A}_{\eta,\nu,t_0}^{-1}}{\partial t})_\nu|||\overline{d}_{\eta,\nu,t_0+\frac{1}{\nu}} - \overline{A}_{\eta,\nu,t_0+\frac{1}{\nu}}\overline{s}_{\eta,\nu,\overline{x}_0,t_0+\frac{1}{\nu}}|$

$+3\sqrt{3}||\overline{A}_{\eta,\nu,t_0}^{-1}|||(\overline{d}_{\eta,\nu,t_0} - \overline{A}_{\eta,\nu,t_0}\overline{s}_{\eta,\nu,\overline{x}_0,t_0})'_\nu|$

$\leq 3\sqrt{3}D\sqrt{\eta}\sum_{j=1}^3 |s_{\overline{x}_j,t_0} \centerdot (s_{\overline{x}_j,t_0+\frac{1}{\nu}} - s_{\overline{x}_0,t_0+\frac{1}{\nu}})|$

$+3\sqrt{3}E\sqrt{\eta}(\sum_{j=1}^3 |s'_{\overline{x}_j,t_0} \centerdot (s_{\overline{x}_j,t_0+\frac{1}{\nu}} - s_{\overline{x}_0,t_0+\frac{1}{\nu}}) + \sum_{j=1}^3 |s_{\overline{x}_j,t_0} \centerdot (s'_{\overline{x}_j,t_0} - s'_{\overline{x}_0,t_0})|)$

$= 3\sqrt{3}D\sqrt{\eta}\sum_{j=1}^3 |s_{\overline{x}_j,t_0} \centerdot s_{\overline{x}_j,t_0+\frac{1}{\nu}}|$

$+3\sqrt{3}E\sqrt{\eta}(\sum_{j=1}^3 |s'_{\overline{x}_j,t_0} \centerdot s_{\overline{x}_j,t_0+\frac{1}{\nu}}| + \sum_{j=1}^3 |s_{\overline{x}_j,t_0} \centerdot s'_{\overline{x}_j,t_0}|)$

$\leq 3\sqrt{3}D\sqrt{\eta}\sum_{j=1}^3 |s_{\overline{x}_j,t_0}||s_{\overline{x}_j,t_0+\frac{1}{\nu}}|$

$+3\sqrt{3}E\sqrt{\eta}(\sum_{j=1}^3 |s'_{\overline{x}_j,t_0}||s_{\overline{x}_j,t_0+\frac{1}{\nu}}| + \sum_{j=1}^3 |s_{\overline{x}_j,t_0}||s'_{\overline{x}_j,t_0}|)$ (Y)

By Lemma 1.30, the fact that $s_{\overline{x}_0,t_0+\frac{1}{\nu}} = s'_{\overline{x}_0,t_0} = \overline{0}$, we have, for some $F \in \mathcal{R}_{>0}$, that;

$max(\{|s_{\overline{x}_j,t_0+\frac{1}{\nu}}| : 1 \leq j \leq 3\} \cup \{|s'_{\overline{x}_j,t_0}| : 1 \leq j \leq 3\}) \leq \frac{F}{\sqrt{\eta}}$

so that, using (Y);

$|(\overline{v})'_\nu(t_0) - (s_{\overline{x}_0})'_\nu(t_0)| \leq 3\sqrt{3}D\sqrt{\eta}(\frac{3F^2}{\eta}) + 3\sqrt{3}E\sqrt{\eta}(\frac{6F^2}{\eta})$

$\leq \frac{\sqrt{3}(9D+18E)F^2}{\sqrt{\eta}} = \frac{G}{\sqrt{\eta}}$

Suppose that $(s_{\overline{x}_0}(t_0), t_0) \in {}^*B(\overline{r}_0, \epsilon) \times (t_1, t_2)$, where $\epsilon \in \mathcal{R}_{>0}$, $t_1 < t_2$, $\{t_1, t_2\} \subset \mathcal{R}_{>0}$, $\overline{r}_0 \in \mathcal{R}^3$. Choose a smooth function $h(\overline{x}, t)$, with $h|_{B(\overline{r}_0,\epsilon)\times(t_1,t_2)} = \overline{J}|_{B(\overline{r}_0,\epsilon)\times(t_1,t_2)}$ and h supported on $B(\overline{r}_0, \epsilon') \times (t'_1, t'_2)$, with $\epsilon < \epsilon'$, $t'_1 < t_1 < t_2 < t'_2$, $\epsilon' \in \mathcal{R}_{>0}$, $\{t'_1, t'_2\} \subset \mathcal{R}_{>0}$. Then, clearly, ${}^*h(s_{\overline{x}_0}(t_0), t_0) = {}^*\overline{J}(s_{\overline{x}_0}(t_0), t_0)$. Let $\overline{J}_1(\overline{x}, t) = \overline{J}(\overline{x}, t) - h(\overline{x}, t)$, then, we have that ${}^*\overline{J}_1(s_{\overline{x}_0}, t_0) = \overline{0}$. By the result of Lemma 2.11, replacing $(\rho, \overline{J})$ with $(\rho, \overline{J}_1)$, we can find, $\overline{y}_0 \in R$, with;

$|s^\sharp_{\overline{y}_0}(t_0) - s_{\overline{x}_0}(t_0)| \leq$.

$\square$

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