SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 2

TRISTRAM DE PIRO

Abstract. We prove that if the frame S is decaying surface non-radiating, in the sense of Definition 1.1, then if $(\rho,\overline{J})$ is analytic, either $\rho=0$ and $\overline{J}=\overline{0}$, or S is non-radiating, in the sense of [6]. In particularly, by the result there, the charge and current satisfy certain wave equations in all the frames $S_{\overline{v}}$ connected to S by a real velocity vector $\overline{v}$, with $|\overline{v}|<c$.

This paper is divided into two parts. In the first part, we begin in Definition 1.1 with the concept of surface non-radiating, that is an all inertial frames S', relative to the base frame S, moving with velocity $\overline{v}$, $|\overline{v}|<c$, there exist real electromagnetic solutions $(\overline{E}_{\overline{v}},\overline{B}_{\overline{v}})$ to Maxwell's equations, for the transformed charge and current $(\rho_{\overline{v}},\overline{J}_{\overline{v}})$, with $div_{\overline{v}}(\overline{E}_{\overline{v}}\times\overline{B}_{\overline{v}})=0$. The general strategy of the paper is to prove that this condition implies the existence of a *complex* solution to Maxwell's equations, with the Poynting vector $\overline{E}\times\overline{B}=\overline{0}$. In order to achieve this, we need to consider rotations g of frames, and Lemma 1.4 shows that Maxwell's equations are both preserved for the rotated quantities $(\rho^g,\overline{J}^g,\overline{E}^g,\overline{B}^g)$, and so is the flux, $div(\overline{E}^g\times\overline{B}^g)=div^g(\overline{E}^g\times\overline{B}^g)$. A similar result is required for the composition of boosts and rotations in Lemma 1.8, and we give the representation of the Lorentz group as a product of boosts and rotations in Lemma 1.5. We have to move in a triangle ABC from the base frame S. Lemma 1.7, using potentials, shows that the transformation of quantities $(\rho,\overline{J},\overline{E},\overline{B})$, along the sides of the triangle is well defined, that is the transformation along AB followed by BC, gives the same result as AC. An essential component of creating triangles with boosts and rotations is the velocity composition formula of Lemma 1.9, explained in [10], and, in Lemma 1.11, we prove that any 2 sides of a triangle AB and AC can be completed with a third side BC, using boosts and rotations. In Definition 1.12, we extend the idea of real boosts and reflections, for velocities $\overline{v}$ with $|\overline{v}|<c$, to include complex and unbounded velocities, and prove generalisations of real results in Lemma 1.13. This allows us to achieve a boost with an infinite velocity to a

frame S_{∞},, which we define in Lemma 1.14. We define the stress energy tensor in Definition 1.15, and use it, in Lemma 1.16, to show how the surface non radiating condition can be transformed into a series of equations in a base frame S, $(*)$. The idea of the proof is to extend the base of the triangle AB to S_{∞}, and with the existence of a complex electromagnetic pair $(\overline{E}_{\infty}, \overline{B}_{\infty})$, with $div_{\infty}(\overline{E}_{\infty} \times \overline{B}_{\infty}) = 0$, $(**)$, use Lemma 1.19 to obtain limit equations at C, along the infinite parallel side BC, by taking a limit of $(*)$. We then, in the same Lemma, transform these limit equations back to the base frame S at A, to obtain a set of equations for $(\overline{E}, \overline{B})$ corresponding in S to $(\overline{E}_{\infty}, \overline{B}_{\infty})$. Lemmas 1.21 and 1.22 are concerned with solving these 40 linear equations in 40 unknowns, corresponding to the first derivatives of the components of the stress energy tensor. We do this we do by splitting them into 2 groups and using a symmetry lemma, which we prove in Lemma 1.26, noting that the coefficient matrix A is *not* invertible, but still obtaining that the derivatives of the Poynting vector $\overline{E} \times \overline{B}$ are zero. We exclude the non-zero constant solution, by imposing a limit condition in Remark 1.32, which seems physically reasonable, that the fields vanish at infinity, and summarise the result that $\overline{E} \times \overline{B} = \overline{0}$, in Lemma 1.23. In Lemma 1.24, we show that this implies $\overline{E} = \lambda \overline{B}$, for some $\lambda \in \mathcal{C}$, and $\rho = 0$, where the field $\overline{B} \neq \overline{0}$. In Lemma 1.25, we note that the surface non-radiating condition is invariant under real boosts with velocity vector $\overline{v}$, $|\overline{v}| < c$, reflections and rotations. The rest of the proof is concerned with deriving the property $(**)$, we do this in Lemma 1.27, using a polynomial approximation and 2 parallel boosts with velocities above and below c, and give a more rigorous explanation, involving the Stone-Weierstrass Theorem, in Lemmas 1.29 and 1.30, noting in Lemma 1.31, that the errors in this polynomial approximation are vanishing. We conclude, in Lemma 1.33, that, assuming $(\rho, \overline{J})$ is analytic, the surface non-radiating condition implies either that $\rho = 0$, $\overline{J} = \overline{0}$, or the non-radiating condition, developed in [6].

In the second part, we are concerned with some thermodynamic arguments. We define a reversed process in Definition 2.1, noting, in Lemma 2.2 that the reversed process satisfies Maxwell's equation and reverses the flux. We give the classical definition of non radiation in Definition 2.3, explored extensively in [3], and use Lemmas 2.4, 2.5 to conclude in Lemma 2.6, that, in thermal equilibrium, classically non-radiating systems satisfy the surface non-radiating condition, explored in the first part. We make the point, in Remark 2.7, that atomic systems can reasonably be thought of as classically non-radiating and in thermal equilibrium.

1. The Surface Radiating Condition

Definition 1.1. *Using notation as in the paper [6], we say that S is decaying surface non-radiating, for a given smooth real pair $(\rho, \overline{J})$ satisfying the continuity equation if;*

(i). S is surface non-radiating with respect to $(\rho, \overline{J})$

that is there exist solutions $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ to Maxwell's equations for the transformed current and charge $(\rho_{\overline{v}}, \overline{J}_{\overline{v}})$ in the frames $S_{\overline{v}}$, connected to S by a real velocity vector $\overline{v}$, with $|\overline{v}| < c$, such that $div_{S_{\overline{v}}}(\overline{E}_{\overline{v}} \times \overline{B}_{\overline{v}}) = 0$.

(ii). The solutions $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ decay at infinity in the frames $S_{\overline{v}}$, that is for coordinates (x, y, z, t) in $S_{\overline{v}}$, for given $t \in \mathcal{R}_{>0}$, we have that $lim_{|\overline{x}| \rightarrow \infty} \overline{E}(\overline{x}, t) = lim_{|\overline{x}| \rightarrow \infty} \overline{B}(\overline{x}, t) = 0$

Definition 1.2. *We let $C^1(\mathcal{R}^3, \mathcal{R}_{>0})$ denote the continuously differentiable functions f in the variables (x, y, z, t). We let $x_0 = t$, $x_1 = x$, $x_2 = y$ and $x_3 = z$. Given $g \in O(3)$, which defines a coordinate transformation $(x', y', z') = g(x, y, z)$, and $f \in C^1(\mathcal{R}^3, \mathcal{R}_{>0})$, we define f^g by;*

$f^g(x'_0, x'_1, x'_2, x'_3) = f(x_0, x_1, x_2, x_3)$

where $x'_0 = x_0$ and $g(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$. Define;

$\frac{\partial f^g}{\partial x'_0}|_{\overline{x}'} = \frac{\partial f}{\partial x_0}|_{\overline{x}}$

$\frac{\partial f^g}{\partial x'_i}|_{\overline{x}'} = (Df)|_{\overline{x}} \cdot (g^{-1})_* \overline{e}_i$, *for* $1 \leq i \leq 3$.

We observe, using the matrix representations g_{ij} and $(g^{-1})_{ij}$, with $1 \leq i, j \leq 3$, for g and g^{-1} respectively, and the fact that $g^* = g^{-1}$, that $(g^{-1})_{ji} = g_{ij}$. It follows;

$\frac{\partial f^g}{\partial x'_i}|_{\overline{x}'} = \sum_{j=1}^{3} (g^{-1})_{ji} \frac{\partial f}{\partial x_j}|_{\overline{x}} = \sum_{j=1}^{3} g_{ij} \frac{\partial f}{\partial x_j}|_{\overline{x}}$ $(*)$

Given a vector field $\overline{F}$, with components (f_1, f_2, f_3), we define $\overline{F}^g$ with components (f'_1, f'_2, f'_3) by;

$f'_i = \sum_{j=1}^{3} g_{ij} f^g_j$, *for* $1 \leq i \leq 3$

We adopt the convention that if $\{g_1, g_2\} \subset O(3)$, *then;*

$f^{g_1 g_2} = (f^{g_2})^{g_1}$, $\overline{F}^{g_1 g_2} = (\overline{F}^{g_2})^{g_1}$

Lemma 1.3. *Given* $g \in O(3)$, $f \in C^1(\mathcal{R}^3, \mathcal{R}_{>0})$ *and vector fields* $\{\overline{F}, \overline{H}\}$, *we have that;*

(i). $\frac{\partial \overline{F}^g}{\partial t'} = (\frac{\partial \overline{F}}{\partial t})^g$

(ii). $\bigtriangledown'(f^g) = (\bigtriangledown(f))^g$

(iii). $\bigtriangledown' \cdot \overline{F}^g = (\bigtriangledown \cdot \overline{F})^g$

(iv). $\bigtriangledown' \times \overline{F}^g = sign(g)(\bigtriangledown \times \overline{F})^g$

(v). $(\overline{F}^g \times \overline{H}^g) = sign(g)(\overline{F} \times \overline{H})^g$

where $sign(g) = det(g)$ *and can take values* 1 *or* -1, *as* $g \in O(3)$.

Proof. For the first part, using components (f_1, f_2, f_3) for $\overline{F}$, we have, for $1 \leq i \leq 3$, that;

$(\frac{\partial \overline{F}^g}{\partial t'})_i|_{\overline{x}'} = \frac{\partial}{\partial t'}(\sum_{j=1}^3 g_{ij} f_j^g)|_{\overline{x}'}$

$= \sum_{j=1}^3 g_{ij} \frac{\partial f_j^g}{\partial t'}|_{\overline{x}'}$

$= \sum_{j=1}^3 g_{ij} \frac{\partial f_j}{\partial t}|_{\overline{x}}$

$= \sum_{j=1}^3 g_{ij} (\frac{\partial f_j}{\partial t})^g|_{\overline{x}'}$

$= (\frac{\partial \overline{F}}{\partial t})_i^g|_{\overline{x}'}$

For the second part, we have, using the observation $(*)$ in Definition 1.2 that;

$\bigtriangledown'(f^g)_i|_{\overline{x}'} = \frac{\partial f^g}{\partial x_i'}|_{\overline{x}'}$

$= \sum_{j=1}^3 g_{ij} \frac{\partial f}{\partial x_j}|_{\overline{x}}$

$= \sum_{j=1}^3 g_{ij} \bigtriangledown (f)_j|_{\overline{x}}$

$= (\bigtriangledown(f))^g_i|_{\overline{x}'}$

For the third part, we have that;

$\bigtriangledown' \centerdot \overline{F}^g|_{\overline{x}'} = \sum_{i=1}^3 \frac{\partial(\overline{F}^g)_i}{\partial x'_i}|_{\overline{x}'}$

$= \sum_{i=1}^3 \frac{\partial}{\partial x'_i}(\sum_{j=1}^3 g_{ij} f^g_j)|_{\overline{x}'}$

$= \sum_{j=1}^3 \sum_{i=1}^3 g_{ij} \frac{\partial f^g_j}{\partial x'_i}|_{\overline{x}'}$

$= \sum_{j=1}^3 \sum_{i=1}^3 g_{ij} \sum_{k=1}^3 g_{ik} \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= \sum_{j=1}^3 \sum_{i,k=1}^3 (g^{-1})_{ji} g_{ik} \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= \sum_{j=1}^3 \sum_{k=1}^3 \delta_{jk} \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= \sum_{j=1}^3 \frac{\partial f_j}{\partial x_j}|_{\overline{x}}$

$= (\bigtriangledown \centerdot \overline{F})^g|_{\overline{x}'}$

For the fourth part, we let σ be the permutation of $(1,2,3)$, with $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$. Then, for $1 \leq i \leq 3$, we have;

$(\bigtriangledown' \times \overline{F}^g)_i|_{\overline{x}'} = (\frac{\partial(\overline{F}^g)_{\sigma^2(i)}}{\partial x'_{\sigma(i)}} - \frac{\partial(\overline{F}^g)_{\sigma(i)}}{\partial x'_{\sigma^2(i)}})|_{\overline{x}'}$

$= (\frac{\partial}{\partial x'_{\sigma(i)}}(\sum_{j=1}^3 g_{\sigma^2(i),j} f^g_j) - \frac{\partial}{\partial x'_{\sigma^2(i)}}(\sum_{j=1}^3 g_{\sigma(i),j} f^g_j))|_{\overline{x}'}$

$= (\sum_{j=1}^3 g_{\sigma^2(i),j}(\sum_{k=1}^3 g_{\sigma(i),k} \frac{\partial f_j}{\partial x_k}) - \sum_{j=1}^3 g_{\sigma(i),j}(\sum_{k=1}^3 g_{\sigma^2(i),k} \frac{\partial f_j}{\partial x_k}))|_{\overline{x}}$

$= \sum_{j,k=1}^3 (g_{\sigma^2(i),j} g_{\sigma(i),k} - g_{\sigma(i),j} g_{\sigma^2(i),k}) \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= \sum_{j,k=1,j\neq k}^3 (g_{\sigma^2(i),j} g_{\sigma(i),k} - g_{\sigma(i),j} g_{\sigma^2(i),k}) \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= \sum_{j,k=1,j\neq k}^3 \tau(j,k) cof(g)_{i,jk} \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= sign(g) \sum_{j,k=1,j\neq k}^3 \tau(j,k) g_{i,jk} \frac{\partial f_j}{\partial x_k}|_{\overline{x}}$

$= sign(g) \sum_{l=1}^3 g_{il}(\frac{\partial f_{\sigma^2(l)}}{\partial x_{\sigma(l)}} - \frac{\partial f_{\sigma(l)}}{\partial x_{\sigma^2(l)}})|_{\overline{x}}$

$= sign(g) \sum_{l=1}^3 g_{il}(\bigtriangledown \times \overline{F})_l|_{\overline{x}}$

$= sign(g)(\bigtriangledown \times \overline{F})^g_i|_{\overline{x}'}$

where $\tau(1,3) = \tau(2,1) = \tau(3,2) = 1$ and $\tau(1,2) = \tau(2,3) = \tau(3,1) = -1$, jk denotes the remaining element in the tuple $(1,2,3)$, $cof(g)_{ij}$ is the representation of the cofactor matrix of g, and we have used the fact that $g_{ij} = (g^{-1})_{ji} = sign(g)(cof(g)^*)_{ji} = sign(g)(cof(g))_{ij}$.

The proof of the fifth part is similar to the fourth, replacing the components of $\overline{F}$ with $\overline{H}$ and those of $\bigtriangledown$ with $\overline{F}$, using the fact that $\bigtriangledown' = \bigtriangledown^g$. □

Lemma 1.4. *Given a tuple $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfying Maxwell's equations in the rest frame S, and $g \in O(3)$, then the tuple $(\rho^g, \overline{J}^g, \overline{E}^g, sign(g)\overline{B}^g)$ also satisfies Maxwell's equations, and the tuple $(\rho^g, \overline{J}^g)$ satisfies the continuity equation in the rotated or reflected frame S'. Moreover;*

$\bigtriangledown' \cdot (\overline{E}^g \times sign(g)\overline{B}^g) = (\bigtriangledown \cdot (\overline{E} \times \overline{B}))^g$

In particular;

$\bigtriangledown' \cdot (\overline{E}^g \times sign(g)\overline{B}^g) = 0$ *iff* $\bigtriangledown \cdot (\overline{E} \times \overline{B}) = 0$

Proof. The second claim follows immediately from the first. For the first claim, we check the conditions using Lemma 1.3. We have that;

$(i).\ \bigtriangledown' \cdot \overline{E}^g = (\bigtriangledown \cdot \overline{E})^g = (\frac{\rho}{\epsilon_0})^g = \frac{\rho^g}{\epsilon_0}$

$(ii).\ \bigtriangledown' \times \overline{E}^g = sign(g)(\bigtriangledown \times \overline{E})^g = sign(g)(-\frac{\partial \overline{B}}{\partial t})^g = -\frac{\partial(sign(g)\overline{B}^g)}{\partial t'}$

$(iii).\ \bigtriangledown' \cdot (sign(g)\overline{B}^g) = sign(g)(\bigtriangledown \cdot \overline{B})^g = 0^g = 0$

$(iv).\ \bigtriangledown' \times (sign(g)\overline{B}^g) = sign(g)sign(g)(\bigtriangledown \times \overline{B})^g$

$= (\mu_0\overline{J} + \mu_0\epsilon_0\frac{\partial \overline{E}}{\partial t})^g = \mu_0\overline{J}^g + \mu_0\epsilon_0\frac{\partial \overline{E}^g}{\partial t'}$

For the penultimate claim, using Lemma 1.3 again, we have that;

$\bigtriangledown' \cdot (\overline{E}^g \times sign(g)\overline{B}^g) = \bigtriangledown' \cdot (sign(g)sign(g)(\overline{E} \times \overline{B})^g)$

$= \bigtriangledown' \cdot (\overline{E} \times \overline{B})^g = (\bigtriangledown \cdot (\overline{E} \times \overline{B}))^g$

The final claim follows immediately from the penultimate claim, applied to the transformations g and g^{-1}.

□

Lemma 1.5. *Let $g \in O(3)$, and let $\overline{v}$ define a boost, with matrices R_g and $B_{\overline{v}}$ respectively in the Lorentz group, then;*

$$R_g B_{\overline{v}} = B_{g(\overline{v})} R_g$$

Moreover, the representation is unique, in the sense that if;

$$R_g B_{\overline{v}} = R_h B_{\overline{w}}$$

for $\{g, h\} \subset O(3)$, and $\{\overline{v}, \overline{w}\}$ velocities defining boosts, then $g = h$ and $\overline{v} = \overline{w}$.

Proof. We first prove this as a footnote in the case when $\overline{v} = v\bar{e}_1$, ([1]). For the general case, let $\overline{v}$ be an arbitrary velocity, and choose $g \in SO(3)$ with $\overline{v} = g(v\bar{e}_1)$. By the result proved in the footnote, we

[1] $B_{v\bar{e}_1}$ is given in coordinates by;

$(B_{v\bar{e}_1})_{00} = (B_{\overline{v}})_{11} = \gamma(v)$

$(B_{v\bar{e}_1})_{22} = (B_{\overline{v}})_{33} = 1$

$(B_{v\bar{e}_1})_{10} = (B_{\overline{v}})_{01} = -\frac{v\gamma(v)}{c}$

$(B_{v\bar{e}_1})_{ij} = 0$ otherwise, $0 \leq i \leq j \leq 3$

and R_g is given in coordinates by;

$(R_g)_{00} = 1$

$(R_g)_{ij} = g_{ij}$, $1 \leq i \leq j \leq 3$

$(R_g)_{ij} = 0$ otherwise, $0 \leq i \leq j \leq 3$

where;

$\sum_{i=1}^{3} g_{ij}^2 = 1$, for $1 \leq j \leq 3$

$\sum_{i=1}^{3} g_{ij} g_{ik} = 0$, for $1 \leq j, k \leq 3$, $j \neq k$ $(*)$

so that;

$(R_g B_{v\overline{e}_1})_{00} = \gamma$

$(R_g B_{v\overline{e}_1})_{01} = -\frac{v\gamma}{c}$

$(R_g B_{v\overline{e}_1})_{02} = (R_g B_{\overline{v}})_{03} = 0$

$(R_g B_{v\overline{e}_1})_{i0} = -\frac{v\gamma g_{i1}}{c}$, for $1 \leq i \leq 3$

$(R_g B_{v\overline{e}_1})_{i1} = \gamma g_{i1}$, for $1 \leq i \leq 3$

$(R_g B_{v\overline{e}_1})_{ij} = g_{ij}$, for $1 \leq i \leq 3$, $2 \leq j \leq 3$

Using the formula in [1] for a general boost with velocity $\overline{w}$, using $(*)$, we can compute $B^{-1}_{g(v\overline{e}_1)} = B_{-vg(\overline{e}_1)}$, to obtain;

$(B_{-vg(\overline{e}_1)})_{00} = \gamma$

$(B_{-vg(\overline{e}_1)})_{0i} = \frac{\gamma v}{c} g_{i1} \sum_{k=1}^{3} g_{k1}^2 = \frac{\gamma v}{c} g_{i1}$, $1 \leq i \leq 3$

$(B_{-vg(\overline{e}_1)})_{i0} = \frac{\gamma v}{c} g_{i1}$, $1 \leq i \leq 3$

$(B_{-vg(\overline{e}_1)})_{ii} = (\gamma - 1) g_{i1}^2 + 1$, $1 \leq i \leq 3$

$(B_{-vg(\overline{e}_1)})_{21} = (B_{-vg(\overline{e}_1)})_{12} = (\gamma - 1) g_{11} g_{21}$

$(B_{-vg(\overline{e}_1)})_{31} = (B_{-vg(\overline{e}_1)})_{13} = (\gamma - 1) g_{11} g_{31}$

$(B_{-vg(\overline{e}_1)})_{32} = (B_{-vg(\overline{e}_1)})_{23} = (\gamma - 1) g_{21} g_{31}$

Finally, using $(*)$ and the identity $\gamma^2(1 - \frac{v^2}{c^2}) = 1$, we compute $B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1}$ in coordinates, to obtain;

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{00} = \gamma^2 - \frac{v^2\gamma^2}{c^2}(\sum_{k=1}^{3} g_{k1}^2) = 1$

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{01} = -\frac{v\gamma^2}{c} + \frac{v\gamma^2}{c}(\sum_{k=1}^{3} g_{k1}^2) = 0$

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{0j} = \frac{\gamma v}{c}(\sum_{k=1}^{3} g_{k1} g_{kj}) = 0$ $(2 \leq j \leq 3)$

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{i0} = g_{i1}(\frac{\gamma^2 v}{c} - \frac{\gamma v}{c} - \frac{(\gamma-1)\gamma v}{c}) = 0$ $(1 \leq i \leq 3)$

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{i1} = g_{i1}(\gamma(\gamma - 1) + \gamma - \frac{\gamma^2 v^2}{c^2}) = g_{i1}$, $(1 \leq i \leq 3)$

$(B_{-vg(\overline{e}_1)} R_g B_{v\overline{e}_1})_{ij} = (\gamma - 1) g_{i1}(\sum_{k=1}^{3} g_{k1} g_{kj}) + g_{ij} = g_{ij}$

$(1 \leq i \leq 3, 2 \leq j \leq 3)$

as required

have that $B_{\overline{v}}R_g = R_g B_{v\overline{e}_1}$, $(**)$. Now let $h \in O(3)$, then, using $(**)$;

$R_h B_{\overline{v}} = R_h R_g B_{v\overline{e}_1} R_g^{-1}$ $(***)$

and the claim that $R_h B_{\overline{v}} = B_{h(\overline{v})} R_h$ is equivalent, by $(***)$, to;

$R_h R_g B_{v\overline{e}_1} R_g^{-1} = B_{h(\overline{v})} R_h$ or $R_g B_{v\overline{e}_1} R_g^{-1} = R_h^{-1} B_{h(\overline{v})} R_h$ $(****)$

We have that $h(\overline{v}) = hg(v\overline{e}_1)$ and $hg \in O(3)$, so it is sufficient, from $(****)$ to prove that;

$R_g B_{v\overline{e}_1} R_g^{-1} = R_h^{-1} B_{hg(v\overline{e}_1)} R_h$ or $B_{v\overline{e}_1} = R_g^{-1} R_h^{-1} B_{hg(v\overline{e}_1)} R_h R_g = R_{hg}^{-1} B_{hg(v\overline{e}_1)} R_{hg}$

$(*****)$

Clearly, the claim $(*****)$ or equivalently, $R_{hg} B_{v\overline{e}_1} = B_{hg(v\overline{e}_1)} R_{hg}$ follows from the proof in the footnote, as required. The second claim is noted in [10], and is straightforward to prove. We have that;

$R_h^{-1} R_g = R_{h^{-1}g} = B_{\overline{w}} B_{\overline{v}}^{-1} = B_{\overline{w}} B_{-\overline{v}}$

The following formula is given in [10] for the boost matrix $B_{\overline{w}}$;

$B_{\overline{w}} = I + \frac{\gamma_w b_{\overline{w}}}{c} + \frac{\gamma_w^2 b_{\overline{w}}^2}{c^2(\gamma_w+1)}$

where, ([2]);

$(b_{\overline{w}})_{0j} = (b_{\overline{w}})_{j0} = -w_j$ for $1 \leq j \leq 3$

$(b_{\overline{w}})_{ij} = 0$ otherwise $(0 \leq i, j \leq 3)$

It follows that;

$$B_{\overline{w}} B_{-\overline{v}} = \left(I + \frac{\gamma_w b_{\overline{w}}}{c} + \frac{\gamma_w^2 b_{\overline{w}}^2}{c^2(\gamma_w+1)}\right)\left(I - \frac{\gamma_v b_{\overline{v}}}{c} + \frac{\gamma_v^2 b_{\overline{v}}^2}{c^2(\gamma_v+1)}\right)$$

$$= I - \frac{\gamma_v b_{\overline{v}}}{c} + \frac{\gamma_w b_{\overline{w}}}{c} + \frac{\gamma_v^2 b_{\overline{v}}^2}{c^2(\gamma_v+1)} + \frac{\gamma_w^2 b_{\overline{w}}^2}{c^2(\gamma_w+1)} - \frac{\gamma_v \gamma_w b_{\overline{v}} b_{\overline{w}}}{c^2} + \frac{\gamma_v^2 \gamma_w b_{\overline{v}}^2 b_{\overline{w}}}{c^3(\gamma_v+1)} - \frac{\gamma_w^2 \gamma_v b_{\overline{w}}^2 b_{\overline{v}}}{c^3(\gamma_w+1)}$$

[2]Ungar's definition of $b_{\overline{w}}$ differs by a minus sign, as he relates unprimed to primed coordinates in the definition of the boost matrix, which seems to go slightly against the usual convention. We have also changed his formula slightly for the case when $x_0 = tc$, rather than $x_0 = t$.

$+\frac{\gamma_w^2\gamma_v^2 b_{\overline{w}}^2 b_{\overline{v}}^2}{c^4(\gamma_w+1)(\gamma_v+1)}$

Using the fact that we must have;

$(R_{h^{-1}g})_{00}=(B_{\overline{w}}B_{-\overline{v}})_{00}=1$

$(R_{h^{-1}g})_{0j}=(B_{\overline{w}}B_{-\overline{v}})_{0j}=0$, $(1\leq j\leq 3)$

we obtain the equations;

$\frac{\gamma_v^2 v^2}{c^2(\gamma_v+1)}+\frac{\gamma_w^2 v^2}{c^2(\gamma_w+1)}-\frac{\gamma_v\gamma_w\overline{v}.\overline{w}}{c^2}+\frac{\gamma_v^2\gamma_w^2 v^2 w^2}{c^4(\gamma_v+1)(\gamma_w+1)}=0$ (†)

$\frac{\gamma_w w_j}{c}-\frac{\gamma_v v_j}{c}=0$ $(1\leq j\leq 3)$ (††)

It follows from (††) that;

$v^2=\frac{\gamma_w^2 w^2}{\gamma_v^2}$ and $\overline{v}\centerdot\overline{w}=\frac{\gamma_w w^2}{\gamma_v}$

and, substituting into (†), using the relation $\frac{\gamma_w^2 w^2}{c^2}=\gamma_w^2-1$, we obtain that $\gamma_w=\gamma_v$, so that, from (††), $\overline{w}=\overline{v}$ as required.

□

Definition 1.6. *Let $g\in O(3)$ be a rotation or reflection, then, as in Definition 1.2, if $f\in C^1(\mathcal{R}^3,\mathcal{R}_{>0})$, we let f^g be defined by;*

$f^g(t',x',y',z')=f(t,x,y,z)$

where $t'=t$ and $(x',y',z')=g(x,y,z)$. For a vector field $\overline{V}$ with components (v_1,v_2,v_3), we let $\overline{V}^g$ be defined by (v_1',v_2',v_3') where;

$v_i'=\sum_{1\leq j\leq 3}g_{ij}v_j^g$, $(1\leq i\leq 3)$

and $(g_{ij})_{1\leq i,j\leq 3}$ is the matrix representation of g. For a four vector $\overline{W}$ with components (w_0,w_1,w_2,w_3), we, similarly, let $\overline{W}^g$ be defined by (w_0',w_1',w_2',w_3'), where;

$w_0'=w_0^g$

$w_i'=\sum_{1\leq j\leq 3}g_{ij}w_j^g$, $(1\leq i\leq 3)$

We introduce rotated frames S' relative to a fixed frame S, with coordinates (t',x',y',z') linked to the coordinates (t,x,y,z) in S, by the *relations;*

$t' = t$

$(x', y', z') = g(x, y, z)$

where $g \in O(3)$*. We define the Lorentz group as generated by boosts in a given direction* $\overline{v}$ *and by rotations defined by* $g \in SO(3)$*. By the augmented Lorentz group, we mean the group generated by boosts and elements* $g \in O(3)$*. It is shown in* [10] *that the composition of boosts is equivalent to a boost followed or preceded by a rotation, the so called Thomas rotation. It follows that any element* L *of the (augmented) Lorentz group can be written (uniquely) as* $L = R_g B_{\overline{v}} = B_{g(\overline{v})} R_g$*, where* $g \in SO(3)$ *(or* $g \in O(3)$*).*

Lemma 1.7. *Given potentials* $(V, \overline{A})$ *in the rest frame* S*, satisfying the relations;*

$\overline{E} = -(\bigtriangledown(V) - \frac{\partial \overline{A}}{\partial t})$

$\overline{B} = \bigtriangledown \times \overline{A}$ $(*)$

for electric and magnetic fields $\{\overline{E}, \overline{B}\}$*, the four vector* $(\frac{V}{c}, \overline{A})$ *transforms covariantly with respect to elements of the Lorentz group, augmented by* $O(3)$*, when the transformation rules are given by;*

$(\frac{V'}{c}, \overline{A}') = (\frac{V}{c}, \overline{A})^g$

when $g \in O(3)$ *is a rotation or a reflection, and by;*

$(\frac{V'}{c}, \overline{A}') = (\gamma(\frac{V}{c} - \frac{<\overline{v}, \overline{A}_{||}>}{c}), \gamma(\overline{A}_{||} - \frac{V}{c^2}\overline{v}) + \overline{A}_{\perp})$

when $\overline{v}$ *defines a boost, see* [1]*. Let the frame* S' *be defined relative to the base frame* S *by an element* τ *of the Lorentz group, which is either a reflection, rotation or a boost. Let* $(\overline{E}', \overline{B}')$ *be defined by;*

$(\overline{E}', \overline{B}') = (\overline{E}^{\tau}, sign(\tau)\overline{B}^{\tau})$

when $\tau \in O(3)$ *is a rotation or a reflection, and by;*

$(\overline{E}', \overline{B}') = (\overline{E}_{||} + \gamma(\overline{E}_{\perp} + \overline{v} \times \overline{B}), \overline{B}_{||} + \gamma(\overline{B}_{\perp} - \frac{\overline{v}}{c^2} \times \overline{E}))$

when τ defines a boost with velocity $\overline{v}$, see [1]. Then with the transformed vector $(\frac{V'}{c},\overline{A}')$, we still have that;

$\overline{E}' = -(\bigtriangledown'(V') - \frac{\partial \overline{A'}}{\partial t'})$

$\overline{B}' = \bigtriangledown' \times \overline{A'}$ $(**)$

There exists a well defined transformation of $(\overline{E},\overline{B})$ for an arbitrary element τ of the Lorentz group, augmented by orthogonal transformations. Moreover, if τ is represented by two distinct products of boosts, rotations and reflections, then the transformation of $(\overline{E},\overline{B})$ coincides with the two transformations obtained by iteration in these representations.

Proof. The fact that the $(\frac{V}{c},\overline{A})$ transforms covariantly with respect to Lorentz boosts is noted in [1], alternatively it can be shown by writing the transformation rule in matrix form and comparing it with the corresponding Lorentz matrix, the details are left to the reader. For rotations or reflections, the result is immediate from the definition. To check the second claim in the case of a rotation or reflection $\tau \in O(3)$, we have, using the definitions in the statement of the Lemma, Definition 1.2 and Lemma 1.3, that;

$-(\bigtriangledown'(V') + \frac{\partial \overline{A}'}{\partial t'})$

$= -((\bigtriangledown(V))^{\tau} + (\frac{\partial \overline{A}}{\partial t})^{\tau})$

$= (-(\bigtriangledown(V) + \frac{\partial \overline{A}}{\partial t}))^{\tau}$

$= \overline{E}^{\tau} = \overline{E}'$

and, similarly;

$\bigtriangledown' \times \overline{A}'$

$= sign(\tau)(\bigtriangledown \times \overline{A})^{\tau}$

$= sign(\tau)\overline{B}^{\tau} = \overline{B}'$

We check the second claim in the case of a boost when $\overline{v} = v\overline{e}_1$. We have, using the transformation rules above, the transformation rules for derivatives, see [1], with components $\{(a_1,a_2,a_3),(a_1',a_2',a_3')\}$ for

$\{\overline{A}, \overline{A}'\}$ respectively, that;

$$(a'_1, a'_2, a'_3) = (\gamma a_1 - \tfrac{\gamma v V}{c^2}, a_2, a_3)$$

$$V' = \gamma V - \gamma v a_1$$

$$\bigtriangledown' = (\gamma(\tfrac{\partial}{\partial x} + \tfrac{v}{c^2}\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}))$$

$$\tfrac{\partial}{\partial t'} = \gamma(\tfrac{\partial}{\partial t} + v\tfrac{\partial}{\partial x})$$

Using $(*)$ from the statement of the Lemma, with components $\{(e_1, e_2, e_3), (b_1, b_2, b_3)\}$ for $\{\overline{E}, \overline{B}\}$ respectively, we have;

$$(e_1, e_2, e_3) = (-\tfrac{\partial a_1}{\partial t} - \tfrac{\partial V}{\partial x}, -\tfrac{\partial a_2}{\partial t} - \tfrac{\partial V}{\partial y}, -\tfrac{\partial a_3}{\partial t} - \tfrac{\partial V}{\partial z})$$

$$(b_1, b_2, b_3) = (\tfrac{\partial a_3}{\partial y} - \tfrac{\partial a_2}{\partial z}, \tfrac{\partial a_1}{\partial z} - \tfrac{\partial a_3}{\partial x}, \tfrac{\partial a_2}{\partial x} - \tfrac{\partial a_1}{\partial y})$$

We then compute:

$$(-\tfrac{\partial \overline{A}'}{\partial t'} - \bigtriangledown'(V'))_1 = -\gamma(\tfrac{\partial}{\partial t} + v\tfrac{\partial}{\partial x})(\gamma a_1 - \tfrac{\gamma v V}{c^2}) - \gamma(\tfrac{\partial}{\partial x} + \tfrac{v}{c^2}\tfrac{\partial}{\partial t})(\gamma V - \gamma v a_1)$$

$$= -\gamma^2\tfrac{\partial a_1}{\partial t} + \tfrac{\gamma^2 v}{c^2}\tfrac{\partial V}{\partial t} - \gamma^2 v\tfrac{\partial a_1}{\partial x} + \tfrac{\gamma^2 v^2}{c^2}\tfrac{\partial V}{\partial x} - \gamma^2\tfrac{\partial V}{\partial x} - \tfrac{\gamma^2 v}{c^2}\tfrac{\partial V}{\partial t} + \gamma^2 v\tfrac{\partial a_1}{\partial x} + \tfrac{\gamma^2 v^2}{c^2}\tfrac{\partial a_1}{\partial t}$$

$$= -\tfrac{\partial V}{\partial x} - \gamma^2\tfrac{\partial a_1}{\partial t} + \tfrac{\gamma^2 v^2}{c^2}\tfrac{\partial a_1}{\partial t}$$

$$= -\tfrac{\partial V}{\partial x} - \tfrac{\partial a_1}{\partial t} = e_1$$

$$(-\tfrac{\partial \overline{A}'}{\partial t'} - \bigtriangledown'(V'))_2 = -\gamma(\tfrac{\partial}{\partial t} + v\tfrac{\partial}{\partial x})a_2 - \tfrac{\partial}{\partial y}(\gamma V - \gamma v a_1)$$

$$= -\gamma\tfrac{\partial a_2}{\partial t} - \gamma v\tfrac{\partial a_2}{\partial x} - \gamma\tfrac{\partial V}{\partial y} + \gamma v\tfrac{\partial a_1}{\partial y}$$

$$= \gamma e_2 + \gamma v(\tfrac{\partial a_1}{\partial y} - \tfrac{\partial a_2}{\partial x})$$

$$= \gamma e_2 - \gamma v b_3$$

$$(-\tfrac{\partial \overline{A}'}{\partial t'} - \bigtriangledown'(V'))_3 = -\gamma(\tfrac{\partial}{\partial t} + v\tfrac{\partial}{\partial x})a_3 - \tfrac{\partial}{\partial z}(\gamma V - \gamma v a_1)$$

$$= -\gamma\tfrac{\partial a_3}{\partial t} - \gamma v\tfrac{\partial a_3}{\partial x} - \gamma\tfrac{\partial V}{\partial z} + \gamma v\tfrac{\partial a_1}{\partial z}$$

$$= \gamma e_3 + \gamma v(\tfrac{\partial a_1}{\partial z} - \tfrac{\partial a_3}{\partial x})$$

$= \gamma e_3 + \gamma v b_2$

so that;

$-\frac{\partial \overline{A}'}{\partial t'} - \bigtriangledown'(V') = (e_1, \gamma e_2 - \gamma v b_3, \gamma e_3 + \gamma v b_2) = \overline{E}'$

as required. Similarly;

$(\bigtriangledown' \times \overline{A}')_1 = \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} = b_1$

$(\bigtriangledown' \times \overline{A}')_2 = \frac{\partial}{\partial z}(\gamma a_1 - \frac{\gamma v V}{c^2}) - \gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})a_3 = \gamma b_2 - \frac{\gamma v}{c^2}(\frac{\partial V}{\partial z} + \frac{\partial a_3}{\partial t})$

$= \gamma b_2 + \frac{\gamma v e_3}{c^2}$

$(\bigtriangledown' \times \overline{A}')_3 = \gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})a_2 - \frac{\partial}{\partial y}(\gamma a_1 - \frac{\gamma v V}{c^2}) = \gamma b_3 + \frac{\gamma v}{c^2}(\frac{\partial a_2}{\partial t} + \frac{\partial V}{\partial y})$

$= \gamma b_3 - \frac{\gamma v e_2}{c^2}$

so that;

$\bigtriangledown' \times \overline{A}' = (b_1, \gamma b_2 + \frac{\gamma v e_3}{c^2}, \gamma b_3 - \frac{\gamma v e_2}{c^2}) = \overline{B}'$

as required. In the general case of a boost defined by a velocity vector $\overline{v}$, we can find $g \in SO(3)$, with $\overline{v} = g(v\overline{e}_1)$. Then, by Lemma 1.5, we have that;

$B_{\overline{v}} = R_g B_{v\overline{e}_1} R_g^{-1} = R_g B_{v\overline{e}_1} R_{g^{-1}}$

By the two cases checked so far, we then have that the property $(**)$, in the statement of the Lemma, holds by iteration. If τ is an arbitrary element of the Lorentz group, it can be written uniquely as the product of a boost and a rotation or a reflection. By iteration, we can then define a transformation $(\overline{E}^{\tau}, \overline{B}^{\tau})$, of $(\overline{E}, \overline{B})$. If τ is represented by two distict products, then by the covariant property of the potential $(\frac{V}{c}, \overline{A})$, the two transformations, $(\frac{V_1}{c}, \overline{A}_1)$ and $(\frac{V_2}{c}, \overline{A}_2)$, obtained by iteration in the representations, coincide. In particularly, we have that $V_1 = V_2$ and $\overline{A}_1 = \overline{A}_2$. Again, iterating the relation $(**)$, we must have that;

$\overline{E}_1 = -(\bigtriangledown'(V_1) + \frac{\partial \overline{A}_1}{\partial t'}) = -(\bigtriangledown'(V_2) + \frac{\partial \overline{A}_2}{\partial t'}) = \overline{E}_2$

$$\overline{B}_1 = \triangledown' \times \overline{A}_1 = \triangledown' \times \overline{A}_2 = \overline{B}_2$$

where $\triangledown'$ and $\frac{\partial}{\partial t'}$ are differential operators in the new frame S' and $\{\overline{E}_1, \overline{E}_2, \overline{B}_1, \overline{B}_2\}$ are again obtained by iteration in the representations. □

Lemma 1.8. *Let $(\rho, \overline{J}, \overline{E}, \overline{B})$ be a solution to Maxwell's equations in the rest frame S, let $g \in O(3)$, and let $(\rho^g, \overline{J}^g, \overline{E}^g, sign(g)\overline{B}^g)$ in S' be as given in Lemma 1.4. Let $\overline{v}$ be a velocity, with corresponding image $\overline{w} = g(\overline{v})$. Let S'' and S''' be the frames defined by $\overline{v}$ and $\overline{w}$, relative to S and S' respectively. Then, if $(\rho'', \overline{J}'', \overline{E}'', \overline{B}'')$ is the solution to Maxwell's equations, corresponding to $(\rho, \overline{J}, \overline{E}, \overline{B})$ in S'' and $(\rho''', \overline{J}''', \overline{E}''', \overline{B}''')$ is the solution to Maxwell's equations, corresponding to $(\rho^g, \overline{J}^g, \overline{E}^g, sign(g)\overline{B}^g)$, in S''', then;*

$$[\triangledown'' \cdot (\overline{E}'' \times \overline{B}'')]^g = \triangledown''' \cdot (\overline{E}''' \times \overline{B}''')$$

In particular, we have that;

$$\triangledown'' \cdot (\overline{E}'' \times \overline{B}'') = 0 \text{ iff } \triangledown''' \cdot (\overline{E}''' \times \overline{B}''') = 0$$

Proof. The final claim follows immediately from the first. By Lemma 1.5, we have that $B_w R_g = R_g B_v$, and by Lemma 1.7, we have that $\overline{E}''' = \overline{E}''^g$, $\overline{B}''' = sign(g)\overline{B}''^g$. Then a straightforward calculation, using Lemma 1.3 shows that;

$$\triangledown''' \cdot (\overline{E}''' \times \overline{B}''') = \triangledown''' \cdot (\overline{E}''^g \times sign(g)\overline{B}''^g)$$

$$= \triangledown''' \cdot (\overline{E}'' \times \overline{B}'')^g$$

$$= [\triangledown'' \cdot (\overline{E}'' \times \overline{B}'')]^g$$

□

Lemma 1.9. *Let the frame S' move with velocity $\overline{u}$ relative to S, and let the frame S'' move with velocity $\overline{v}$ relative to S', then the velocity of S'' computed in S is given by;*

$$\overline{u} * \overline{v} = \frac{\overline{u}+\overline{v}}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}} + \frac{\gamma_u}{c^2(\gamma_u+1)} \frac{\overline{u}\times(\overline{u}\times\overline{v})}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}$$

Moreover, there exists $g \in SO(3)$ defining a rotation R_g, such that;

$B_{\overline{v}}B_{\overline{u}}=R_gB_{\overline{u}*\overline{v}}=B_{\overline{v}*\overline{u}}R_g$

where $R_g(\overline{u}*\overline{v})=\overline{v}*\overline{u}$ *and the velocity of* S *computed in* S'' *is* $-(\overline{v}*\overline{u})$.

Proof. The first formula is given in [10]. We use the original formula there for the boost matrix;

$$B_{\overline{u}}=I+\gamma_u b+\frac{\gamma_u^2}{\gamma_u+1}b^2$$

as it is unnecessary to introduce the variable $x_0=ct$, but see the footnote in Lemma 1.5. The frame S moves with velocity $-\overline{u}$ relative to S' and, using primed coordinates for S, and unprimed for S', we have, using the matrix $B_{-\overline{u}}$ to relate the two frames, that;

$$dt'=\gamma_u dt+c^{-2}\gamma_u u_1dx_1+c^{-2}\gamma_u u_2dx_2+c^{-2}\gamma_u u_3dx_3$$

$$dx_1'=\gamma_u u_1dt+(1+\frac{\gamma_u^2u_1^2}{c^2(\gamma_u+1)})dx_1+\frac{\gamma_u^2u_1u_2}{c^2(\gamma_u+1)}dx_2+\frac{\gamma_u^2u_1u_3}{c^2(\gamma_u+1)}dx_3$$

$$dx_2'=\gamma_u u_2dt+\frac{\gamma_u^2u_1u_2}{c^2(\gamma_u+1)}dx_1+(1+\frac{\gamma_u^2u_2^2}{c^2(\gamma_u+1)})dx_2+\frac{\gamma_u^2u_2u_3}{c^2(\gamma_u+1)}dx_3$$

$$dx_3'=\gamma_u u_3dt+\frac{\gamma_u^2u_1u_3}{c^2(\gamma_u+1)}dx_1+\frac{\gamma_u^2u_2u_3}{c^2(\gamma_u+1)}dx_2+(1+\frac{\gamma_u^2u_3^2}{c^2(\gamma_u+1)})dx_3$$

Using the facts that $v_i=\frac{dx_i}{dt}$, for $1\leq i\leq 3$, $\frac{1}{\gamma_u}=1-\frac{u^2\gamma_u}{c^2(\gamma_u+1)}$ and the formula $\overline{u}\times(\overline{u}\times\overline{v})=\overline{u}(\overline{u}\cdot\overline{v})-\overline{v}(\overline{u}\cdot\overline{u})$

we compute;

$$v_1'=\frac{dx_1'}{dt'}=\frac{\gamma_u u_1dt+(1+\frac{\gamma_u^2u_1^2}{c^2(\gamma_u+1)})dx_1+\frac{\gamma_u^2u_1u_2}{c^2(\gamma_u+1)}dx_2+\frac{\gamma_u^2u_1u_3}{c^2(\gamma_u+1)}dx_3}{\gamma_u dt+c^{-2}\gamma_u u_1dx_1+c^{-2}\gamma_u u_2dx_2+c^{-2}\gamma_u u_3dx_3}$$

$$=\frac{\gamma_u u_1+(1+\frac{\gamma_u^2u_1^2}{c^2(\gamma_u+1)})v_1+\frac{\gamma_u^2u_1u_2}{c^2(\gamma_u+1)}v_2+\frac{\gamma_u^2u_1u_3}{c^2(\gamma_u+1)}v_3}{\gamma_u+c^{-2}\gamma_u u_1v_1+c^{-2}\gamma_u u_2v_2+c^{-2}\gamma_u u_3v_3}$$

$$=\frac{v_1+u_1(\gamma_u+\frac{\gamma_u^2\overline{u}\cdot\overline{v}}{c^2(\gamma_u+1)})}{\gamma_u(1+\frac{\overline{u}\cdot\overline{v}}{c^2})}$$

$$=\frac{\frac{v_1}{\gamma_u}+u_1(1+\frac{\gamma_u\overline{u}\cdot\overline{v}}{c^2(\gamma_u+1)})}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}$$

$$=\frac{(1-\frac{u^2\gamma_u}{c^2(\gamma_u+1)})v_1}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}+\frac{(1+\frac{\gamma_u\overline{u}\cdot\overline{v}}{c^2(\gamma_u+1)})u_1}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}$$

$$=\frac{u_1+v_1}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}+\frac{\gamma_u((\overline{u}\cdot\overline{v})u_1-u^2v_1)}{1+\frac{\overline{u}\cdot\overline{v}}{c^2}}$$

$= \frac{u_1+v_1}{1+\frac{\overline{u}\bullet\overline{v}}{c^2}} + \frac{\gamma_u(\overline{u}\times(\overline{u}\times\overline{v}))_1}{1+\frac{\overline{u}\bullet\overline{v}}{c^2}}$

and similarly;

$v_i' = \frac{u_i+v_i}{1+\frac{\overline{u}\bullet\overline{v}}{c^2}} + \frac{\gamma_u(\overline{u}\times(\overline{u}\times\overline{v}))_i}{1+\frac{\overline{u}\bullet\overline{v}}{c^2}}$, for $2 \leq i \leq 3$

so the result follows. The first of the second set of formulae is also given in [10], ([3]). We can deduce the second formula from the first. We

---

[3]In fact Ungar claims that $B_{\overline{u}}B_{\overline{v}} = B(\overline{u} * \overline{v})R_h$, for some $h \in SO(3)$. Remembering that the boost matrix we use in this paper, for t coordinates, reverses the signs of $\overline{u}$, $\overline{v}$ and $\overline{u} * \overline{v}$, we obtain;

$B_{\overline{u}}^{-1}B_{\overline{v}}^{-1} = B_{-\overline{u}}B_{-\overline{v}} = B_{-(\overline{u}*\overline{v})}R_h = B_{\overline{u}*\overline{v}}^{-1}R_h$

so that;

$B_{\overline{v}}B_{\overline{u}} = R_{h^{-1}}B_{\overline{u}*\overline{v}}$ $(*)$

and we can take $g = h^{-1}$. This formula also holds for the boost matrices with $x_0 = ct$ coordinates, as letting A_c be defined by;

$(A_c)_{00} = c$

$(A_c)_{ii} = 1$, for $1 \leq i \leq 3$

$(A_c)_{ij} = 0$, otherwise, for $0 \leq i, j \leq 3$

we obtain from $(*)$ that;

$(A_cB_{\overline{v}}A_c^{-1})(A_cB_{\overline{u}}A_c^{-1}) = (A_cR_gA_c^{-1})(A_cB_{\overline{u}*\overline{v}}A_c^{-1}) = R_g(A_cB_{\overline{u}*\overline{v}}A_c^{-1})$

and the boost matrices in $x_0 = ct$ coordinates are given by $\{A_cB_{\overline{u}}A_c^{-1}, A_cB_{\overline{v}}A_c^{-1}, A_cB_{\overline{u}*\overline{v}}A_c^{-1}\}$

The explicit representation of h and, therefore, g is known, and included by Ungar. The formula is given by;

$R_g = R_{h^{-1}} = I - c_1\Omega + c_2\Omega^2$

where;

$(\Omega)_{ii} = 0$, $1 \leq i \leq 3$

$(\Omega)_{ij} = (-1)^{i+j}\omega_{ij}$, $1 \leq i < j \leq 3$

have that;

$$B_{\overline{u}*\overline{v}}B_{-\overline{u}}B_{-\overline{v}} = R_{g^{-1}}$$

and, as this holds for any velocities $\{\overline{u}, \overline{v}\}$, making the substitutions $-\overline{v}$ for $\overline{u}$ and $-\overline{u}$ for $\overline{v}$, and, using the fact that $-(\overline{v} * \overline{u}) = -\overline{v} * -\overline{u}$, we can find $h \in SO(3)$ with;

$$B_{-(\overline{v}*\overline{u})}B_{\overline{v}}B_{\overline{u}} = B_{-\overline{v}*-\overline{u}}B_{\overline{v}}B_{\overline{u}} = R_{h^{-1}}$$

so that;

$$B_{\overline{v}}B_{\overline{u}} = B_{\overline{v}*\overline{u}}R_{h^{-1}} \ (\dagger)$$

By the first part of Lemma 1.5, and using the first result, we have that;

$$B_{\overline{v}}B_{\overline{u}} = R_g B_{\overline{u}*\overline{v}} = B_{R_g(\overline{u}*\overline{v})}R_g$$

and by the uniqueness part of Lemma 1.5 and (†), we have that;

$$R_{h^{-1}} = R_g \text{ and } \overline{v} * \overline{u} = R_g(\overline{u} * \overline{v})$$

It then follows from (†) that;

$$B_{\overline{v}}B_{\overline{u}} = B_{\overline{v}*\overline{u}}R_g$$

as well. For the final claim, observe that S' moves with velocity $-\overline{v}$ relative to S'' and S moves with velocity $-\overline{u}$ relative to S'. Using the first claim, we have that the velocity of S computed in the frame S'' is $-\overline{v} * -\overline{u} = -(\overline{v} * \overline{u})$ as required.

□

$(\Omega)_{ij} = -(\Omega)_{ji}$, $1 \leq j < i \leq 3$

$\omega = \overline{u} \times \overline{v} = (\omega_1, \omega_2, \omega_3)$

$c_1 = \frac{-\gamma_u\gamma_v(\gamma_u+\gamma_v+\gamma_{u*v}+1)}{c^2(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)}$

$c_2 = \frac{\gamma_u^2\gamma_v^2}{c^4(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)}$

Remarks 1.10. *The presence of the Thomas rotation resolves the so called Mocanu paradox, explained in [9], that the relative velocities of two frames S and S'', connected by two successive boosts $\{\overline{u},\overline{v}\}$, do not differ by a minus sign, when computed in S and S'' respectively. An interesting perspective on the relativistic effects of rotations is given in [2].*

Lemma 1.11. *Given a scalar $v \in \mathcal{R}$, with $|v| < c$, and a velocity $\overline{u}$, with $u < c$, there exists a unique velocity $\overline{w}$, such that;*

$$B_{\overline{w}}B_{\overline{u}} = R_g B_{v\overline{e}_1}$$

where $g \in SO(3)$. Moreover, if $\overline{u}$ is fixed and $v \rightarrow \infty$, when $u_1 \neq 0$ and $\overline{u} \neq u_1\overline{e}_1$, we have that;

$$\overline{w} \rightarrow iv(0, u_2, u_3)$$

When, $u_1 \neq 0$ and $\overline{u} = u_1\overline{e}_1$, we have that;

$$\overline{w} \rightarrow -\frac{c^2\overline{e}_1}{u_1}$$

When $u_1 = 0$

$$\overline{w} \rightarrow v(1 - \frac{iu_1}{c}, -\frac{iu_2}{c}, -\frac{iu_3}{c})$$

Proof. By Lemma 1.9, we have that;

$$B_{\overline{u}}B_{-v\overline{e}_1} = R_g B_{-v\overline{e}_1 * \overline{u}}$$

where $g \in SO(3)$, and, rearranging;

$$B_{v\overline{e}_1 * -\overline{u}}B_{\overline{u}} = B_{-(-v\overline{e}_1 * \overline{u})}B_{\overline{u}} = B^{-1}_{-v\overline{e}_1 * \overline{u}}B_{\overline{u}} = R_g B^{-1}_{-v\overline{e}_1} = R_g B_{v\overline{e}_1}$$

so it is sufficient to take $\overline{w} = v\overline{e}_1 * -\overline{u}$. The uniqueness claim is clear by the second part of Lemma 1.5. By the formula in Lemma 1.9, we have that;

$$\overline{w} = \frac{v\overline{e}_1 - \overline{u}}{1 - \frac{v\overline{e}_1 \cdot \overline{u}}{c^2}} - \frac{\gamma_v(v\overline{e}_1 \times (v\overline{e}_1 \times \overline{u}))}{c^2(\gamma_v + 1)(\frac{1 - v\overline{e}_1 \cdot \overline{u}}{c^2})}$$

$$= \frac{v\overline{e}_1 - \overline{u}}{1 - \frac{u_1 v}{c^2}} - \frac{u_1 v^2 \overline{e}_1 - v^2 \overline{u}}{c^2(1 - \frac{u_1 v}{c^2})(1 + \sqrt{1 - \frac{v^2}{c^2}})}$$

For the first case, taking the limit as $v\rightarrow\infty$, keeping $\overline{u}$ fixed, we obtain;

$$\overline{w}_{\infty}=\frac{\overline{e}_1}{-\frac{u_1}{c^2}}-\frac{vu_1}{\frac{-iu_1}{c}}\overline{e}_1+\frac{v}{\frac{-iu_1}{c}}\overline{u}$$

$$=-icv\overline{e}_1+icv\overline{e}_1+icv(0,\tfrac{u_2}{u_1},\tfrac{u_3}{u_1})$$

$$=iv(0,u_2,u_3)$$

In the second case, we obtain the finite limit;

$$\overline{w}_{\infty}=\frac{\overline{e}_1}{-\frac{u_1}{c^2}}-icv\overline{e}_1+icv\overline{e}_1=-\frac{c^2\overline{e}_1}{u_1}$$

and, in the final case, we obtain;

$$\overline{w}_{\infty}=lim_{v\rightarrow\infty}(v\overline{e}_1-\overline{u}+\frac{v^2\overline{u}}{c^2(1+\sqrt{1-\frac{v^2}{c^2}})})$$

$$=v\overline{e}_1+\frac{v\overline{u}}{c^2(\frac{i}{c})}$$

$$=v(1-\tfrac{iu_1}{c},-\tfrac{iu_2}{c},-\tfrac{iu_3}{c})$$

as required.

□

Definition 1.12. *We extend the definition of boost matrices in Lemma 1.5 to include complex vectors $\overline{v}$, with $\overline{v}^2\neq c^2$, where $\overline{v}^2=v_1^2+v_2^2+v_3^2$, and $\gamma_{\overline{v}}=\frac{1}{\sqrt{1-\frac{\overline{v}^2}{c^2}}}$, where we can take either square root, provided we do so consistently in the definition. We denote the two complex boost matrices obtained from a complex vector $\overline{v}$ by $B^1_{\overline{v}}$ and $B^2_{\overline{v}}$. The fact that $(B^i_{\overline{v}})^{-1}=(B^i_{-\overline{v}})$, for $1\leq i\leq 2$, follows easily from the real case and the fact that it holds generically, noting that $(-\overline{v})^2=\overline{v}^2$, so we can take compatible square roots. Similarly, we extend the definition of $\overline{u}*\overline{v}$, to include complex vectors $\overline{u}$ and $\overline{v}$, with $\overline{u}^2\notin\{0,c^2\}$, $\overline{v}^2\neq c^2$, and $1+\frac{\overline{u}\centerdot\overline{v}}{c^2}\neq 0$, taking $\overline{u}\centerdot\overline{v}$ to be $u_1v_1+u_2v_2+u_3v_3$, and noting there are two possibilities, $(\overline{u}*\overline{v})^1$ and $(\overline{u}*\overline{v})^2$, depending on the choice of square root in $\gamma_{\overline{u}}$. We extend the orthogonal group $O(3)$, to consist of complex transformations R_g in the spatial coordinates with $R_gR_g^t=R_g^tR_g=I$, where t denotes transpose. We denote this group by $G(3)$, noting that it is a group, as $R_gR_h(R_gR_h)^t=R_gR_hR_h^tR_g^t=R_gR_g^t=I$, and, similarly, $(R_gR_h)^tR_gR_h=I$. We denote by $SG(3)$ the subgroup of $G(3)$ consisting of complex transformations with $det(R_g)=1$. We let*

$C^{an}(\mathcal{C}^4)$ denote the set of analytic functions in the variables (x,y,z,t), and, given $f \in C^{an}(\mathcal{C}^4), g \in G(3)$, define f^g by extension from the real case. Using analytic derivatives, we similarly define the transformation $\overline{F}^g$ of a complex vector field. We extend the definition of the operator $\bigtriangledown$ using analytic derivatives. Let the frame $S_{\overline{v}}$ be connected to the base frame S by one of the boost matrices $B^i_{\overline{v}}$, $1 \leq i \leq 2$, for the complex vector $\overline{v}$, with $\overline{v}^2 \neq c^2$. Given a real valued tuple $(\rho, \overline{J})$, and a complex valued tuple $(\overline{E}, \overline{B})$, such that $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfy Maxwell's equations in S, we extend the transformation rules, given in [1], for $(\rho, \overline{J})$ and $(\overline{E}, \overline{B})$. We use the above definitions of $\overline{u} \centerdot \overline{v}$ and $\gamma_{\overline{v}}$, for complex vectors $\{\overline{u}, \overline{v}\}$, with $\overline{v}^2 \neq c^2$, and use the choice of square root, determined by i. We can link the coordinates with the boost matrix $B^i_{\overline{v}}$, and we are only interested in the coordinates of $\mathcal{C}^4$ determined by the image of $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$, under the boost matrix. Similarly, we define the transformation of derivatives from $S_{\overline{v}}$ to S, using the boost matrix $(B^i_{\overline{v}})^{-1}$, in coordinates (x'_0, x'_1, x'_2, x'_3) for the frame $S_{\overline{v}}$, by the rule $\frac{\partial}{\partial x'_i} = \sum_{j=0}^4 ((B^i_{\overline{v}})^{-1})_{ji} \frac{\partial}{\partial x_j}$.

Lemma 1.13. *With the notation of Definition 1.12, we still have that the results of Lemmas 1.3, 1.5 and 1.9 hold. That is, the results $(i)-(v)$ of Lemma 1.3 hold, with analytic derivatives, the extension of the $\bigtriangledown$ operator, and taking $g \in G(3)$. Moreover, noting $g(\overline{v})^2 = \overline{v}^2$, so we can take compatible square roots;*

$$R_g B^i_{\overline{v}} = B^i_{g(\overline{v})} R_g, \text{ for } 1 \leq i \leq 2$$

when $g \in G(3)$, $\overline{v}^2 \neq c^2$, and we still have uniqueness of representation, that is if;

$$R_g B^{i_1}_{\overline{v}} = R_h B^{i_2}_{\overline{w}}$$

for $\{g,h\} \subset G(3)$, and $\{\overline{v}, \overline{w}\}$, with $\overline{v}^2 \neq c^2$ and $\overline{w}^2 \neq c^2$, then $g = h$, $\overline{v} = \overline{w}$ and $i_1 = i_2$ in the sense of taking compatible square roots in $\gamma_{\overline{v}}$ or $\gamma_{\overline{w}}$. Finally, for $\{\overline{u}, \overline{v}\}$, with $\{\overline{u}^2, \overline{v}^2\} \cap \{0, c^2\} = \emptyset$, $1 + \frac{\overline{u} \centerdot \overline{v}}{c^2} \neq 0$ and $(1 + \frac{\overline{u} \centerdot \overline{v}}{c^2})^2 - (1 - \frac{\overline{u}^2}{c^2})(1 - \frac{\overline{v}^2}{c^2}) \neq 0$, $(\dagger)$, taking;

$$\gamma_{\overline{u} * \overline{v}} = \gamma_{\overline{v} * \overline{u}} = \gamma_{\overline{u}} \gamma_{\overline{v}} (1 + \tfrac{\overline{u} \centerdot \overline{v}}{c^2}) \ (*)$$

we have that there exists $g_{i_1 i_2} \in SG(3)$, with;

$$B_{\overline{v}}^{i_2}B_{\overline{u}}^{i_1}=R_{g_{i_1i_2}}B_{(\overline{u}*\overline{v})^{i_3}}^{i_4}=B_{(\overline{v}*\overline{u})^{i_5}}^{i_4}R_{g_{i_1i_2}}$$

and;

$$R_{g_{i_1i_2}}((\overline{u}*\overline{v})^{i_3})=(\overline{v}*\overline{u})^{i_5}$$

where i_3 is determined by the choice of i_1, i_5 is determined by the choice of i_2, i_4 is determined by the formula $(*)$, and $g_{i_1i_2}$ is determined by the choices of i_1 and i_2.

Let the frames $S_{\overline{v}}$ and S as in the above definition, be connected by the boost matrices $B_{\overline{v}}^i$ and $B_{-\overline{v}}^i$. Then, if $(\rho_{\overline{v}}^i, \overline{J}_{\overline{v}}^i)$ and $(\overline{E}_{\overline{v}}^i, \overline{B}_{\overline{v}}^i)$ are the transformed quantities, we have that;

$$\rho=(\rho_{\overline{v}}^i)_{-\overline{v}}^i,\ \overline{J}=(\overline{J}_{\overline{v}}^i)_{-\overline{v}}^i,\ \overline{E}=(\overline{E}_{\overline{v}}^i)_{-\overline{v}}^i,\ \overline{B}=(\overline{B}_{\overline{v}}^i)_{-\overline{v}}^i$$

Let $\{\overline{u},\overline{v}\}$ satisfy the conditions $(\dagger)$ above, with frames $\{S,S',S'',S''',S''''\}$, such that S' is connected to S by $B_{\overline{u}}^{i_1}$, S'' is connected to S' by $B_{\overline{v}}^{i_2}$, S''' is connected to S by $B_{(\overline{u}*\overline{v})_{i_3}}^{i_4}$, S'''' is connected to S by $R_{g_{i_1i_2}}$, S'' is connected to S''' by $R_{g_{i_1i_2}}$, S'' is connected to S'''' by $B_{(\overline{v}*\overline{u})_{i_5}}^{i_4}$, for the appropriate choice of $\{i_1,i_2,i_3,i_4,i_5\}$, then we have, for the transformations, corresponding to the pairs $(\rho,\overline{J})$ and $(\overline{E},\overline{B})$, that;

$$((\rho_{\overline{u}}^{i_1})_{\overline{v}}^{i_2},(\overline{J}_{\overline{u}}^{i_1})_{\overline{v}}^{i_2})=((\rho_{(\overline{u}*\overline{v})_{i_3}}^{i_4})_{g_{i_1i_2}},(\overline{J}_{(\overline{u}*\overline{v})_{i_3}}^{i_4})_{g_{i_1i_2}})=((\rho_{g_{i_1i_2}})_{(\overline{v}*\overline{u})_{i_5}}^{i_4},(\overline{J}_{g_{i_1i_2}})_{(\overline{v}*\overline{u})_{i_5}}^{i_4})$$

$$((\overline{E}_{\overline{u}}^{i_1})_{\overline{v}}^{i_2},(\overline{B}_{\overline{u}}^{i_1})_{\overline{v}}^{i_2})=((\overline{E}_{(\overline{u}*\overline{v})_{i_3}}^{i_4})_{g_{i_1i_2}},(\overline{B}_{(\overline{u}*\overline{v})_{i_3}}^{i_4})_{g_{i_1i_2}})=((\overline{E}_{g_{i_1i_2}})_{(\overline{v}*\overline{u})_{i_5}}^{i_4},(\overline{B}_{g_{i_1i_2}})_{(\overline{v}*\overline{u})_{i_5}}^{i_4})$$

Moreover, the transformation of derivatives from S'' to S is independent of the path taken, in the sense that for coordinates $\{x'',y'',z'',t''\}$ in S'';

$$(B_{(\overline{u}*\overline{v})_{i_3}}^{i_4})^{-1}(R_{g_{i_1i_2}})^{-1}(\tfrac{\partial}{\partial x_i''})=(B_{\overline{u}}^{i_1})^{-1}(B_{\overline{v}}^{i_2})^{-1}(\tfrac{\partial}{\partial x_i''})=(R_{g_{i_1i_2}})^{-1}(B_{(\overline{v}*\overline{u})_{i_5}}^{i_4})^{-1}(\tfrac{\partial}{\partial x_i''})$$

Proof. For the first part, in (i), we use the complex linearity of the analytic derivatives. For (ii), we still have the property $(g^{-1})_{ji}=g_{ij}$ for the complex entries of $g\in G(3)$, the rest follows using the chain rule for analytic functions. For (iii), we again use the properties mentioned to prove $(i),(ii)$. (iv) is similar, using the definition of the inverse of a complex matrix $A^{-1}=\frac{1}{det(A)}(cof(A))^t$. (v) is similar to (iv). For the second part, the fact that $g(\overline{v})^2=\overline{v}^2$ for $g\in G(3)$ is a simple calculation using the definitions. We first prove the footnote relations;

$$R_g B^i_{v\overline{e}_1}=B^i_{g(v\overline{e}_1)}R_g \ (1\leq i\leq 2)$$

for any $v\in\mathcal{C}$ with $v^2\neq c^2$. These follow, using the properties of $g\in G(3)$ and the fact that the identity $\gamma^2_{v\overline{e}_1}(1-\frac{v^2}{c^2})=1$ holds for either choice of square root in $\gamma_{v\overline{e}_1}$. The rest of the proof follows similarly to Lemma 1.5. We note that if $\overline{v}^2\neq c^2$, then for a choice $v\in\mathcal{C}$ of the square root of $\overline{v}^2$, we have that $(\frac{\overline{v}}{v})^2=1$. We can then find $g\in SG(3)$ with $g(\overline{e}_1)=\frac{\overline{v}}{v}$, by taking a matrix with first column $\frac{\overline{v}}{v}$, then choosing $\overline{h}$, with $\frac{\overline{v}}{v}\centerdot\overline{h}=0$, and $h_1^2+h_2^2+h_3^2=1$ as the second column, and finally taking the complex cross product $\frac{\overline{v}}{v}\times\overline{h}$ as the third column. Interchanging the second and third columns if necessary, we can ensure $det(g)=1$. Then $g(v\overline{e}_1)=\overline{v}$ by linearity and $v^2\neq c^2$. For the uniqueness claim, we can compute $B^{i_2}_{\overline{w}}B^{i_2}_{-\overline{v}}$ and using the definition of $G(3)$, obtain the same relations, including;

$$\gamma_{\overline{w}}w_j=\gamma_{\overline{v}}v_j,\text{ for } (1\leq j\leq 3)\ (*)$$

for the appropriate choices of a square root in $\gamma_{\overline{v}}$ and $\gamma_{\overline{w}}$. As in the proof, we conclude that $\gamma_{\overline{v}}=\gamma_{\overline{w}}$, which from $(*)$ implies that $\overline{v}=\overline{w}$ and that we have taken compatible square roots i_1 and i_2. For the final part, we note that, using the real formula for $\overline{u}*\overline{v}$, the matrices $\{B_{\overline{u}},B_{\overline{v}},B_{\overline{u}*\overline{v}},R_g\}$ depend algebraically and rationally on the parameters $\{\overline{u},\overline{v},\gamma_{\overline{u}},\gamma_{\overline{v}},\gamma_{\overline{u}*\overline{v}}\}$, where $\{\overline{u},\overline{v}\}$ are real vectors with $max(\overline{u}^2,\overline{v}^2)<c^2$. It follows that the identity;

$$B_{\overline{v}}B_{\overline{u}}=R_gB_{\overline{u}*\overline{v}}\ (**)$$

amounts to a set of rational algebraic identities $R_i=0$, $1\leq i\leq 16$, involving $\{\overline{u},\overline{v},\gamma_{\overline{u}},\gamma_{\overline{v}},\gamma_{\overline{u}*\overline{v}}\}$ as well. As noted in [10], these identities are still true when we make the substitution $\gamma_{\overline{u}}\gamma_{\overline{v}}(1+\frac{\overline{u}\centerdot\overline{v}}{c^2})$, $(***)$, for $\gamma_{\overline{u}*\overline{v}}$. We let $V\subset\mathcal{C}^6$, be the open subvariety defined by;

$$V=\{(\overline{u},\overline{v})\in\mathcal{C}^6:\{\overline{u}^2,\overline{v}^2\}\cap\{0,c^2\}=\emptyset,1+\frac{\overline{u}\centerdot\overline{v}}{c^2}\neq 0,(1+\frac{\overline{u}\centerdot\overline{v}}{c^2})^2$$

$$-(1-\frac{\overline{u}^2}{c^2})(1-\frac{\overline{v}^2}{c^2})\neq 0\}$$

and $C\subset V\times\mathcal{C}^2$ be the double cover of V defined by;

$C = \{(\overline{u}, \overline{v}, w_1, w_2) \in V \times \mathcal{C}^2 : w_1^2 = \frac{1}{1-\frac{\overline{u}^2}{c^2}}, w_2^2 = \frac{1}{1-\frac{\overline{v}^2}{c^2}}\}$

Making the substitutions w_1 for $\gamma_{\overline{u}}$ and w_2 for $\gamma_{\overline{v}}$, the closed rational algebraic relations $R_i(\overline{u}, \overline{v}, w_1, w_2) = 0$ hold generically on C. The conditions on $\{\overline{u}, \overline{v}\}$ are necessary to ensure the denominators in the definitions of $\{B_{\overline{u}}, B_{\overline{v}}, B_{\overline{u}*\overline{v}}, R_g\}$ are non-zero on C, so, by construction the rational functions R_i have no poles. It follows that the R_i are identically zero on C. In particular, the identity $(**)$ holds for all complex vectors $\{\overline{u}, \overline{v}\}$ with $\{\overline{u}^2, \overline{v}^2\} \cap \{0, c^2\} = \emptyset$, $1 + \frac{\overline{u}\cdot\overline{v}}{c^2} \neq 0$, $(1 + \frac{\overline{u}\cdot\overline{v}}{c^2})^2 - (1 - \frac{\overline{u}^2}{c^2})(1 - \frac{\overline{v}^2}{c^2}) \neq 0$, and choices of square root in $\{\gamma_{\overline{u}}, \gamma_{\overline{v}}\}$, $(\dagger)$. We need to take $\gamma_{\overline{u}*\overline{v}} = \gamma_{\overline{u}}\gamma_{\overline{v}}(1 + \frac{\overline{u}\cdot\overline{v}}{c^2})$, and the coefficients $\{c_1, c_2\}$ in $g_{i_1 i_2}$, see the footnote in Lemma 1.9, must be determined by the choices in $(\dagger)$. Again, formulating the property $g \in SG(3)$ as a set of closed conditions, and using the fact that they holds generically, we obtain that $g_{i_1 i_2} \in SG(3)$. Finally, we need to check that the identity $(***)$ holds up to a minus sign, for any choice of root in $\{\gamma_{\overline{u}}, \gamma_{\overline{v}}, \gamma_{\overline{u}*\overline{v}}\}$, and for $\{\overline{u}, \overline{v}\}$ satisfying the usual conditions. This follows by verifying the identity given in [10];

$(\overline{u} * \overline{v})^2 = \frac{(\overline{u}+\overline{v})^2}{(1+\frac{\overline{u}\cdot\overline{v}}{c^2})^2} - \frac{1}{c^2}\frac{(\overline{u}\times\overline{v})^2}{(1+\frac{\overline{u}\cdot\overline{v}}{c^2})^2}$

This is a straightforward calculation involving verifying the identities;

$\overline{u} \times (\overline{u} \times \overline{v}) = (\overline{u} \cdot \overline{v})^2 - \overline{u}^2\overline{v}^2$, $(\overline{u} \times \overline{v})^2 = \overline{u}^2\overline{v}^2 - (\overline{u} \cdot \overline{v})^2$

for complex vectors. Then, computing $\gamma_{\overline{u}*\overline{v}}$ gives the result. We thus obtain the relation;

$B_{\overline{v}}^{i_2} B_{\overline{u}}^{i_1} = R_{g_{i_1 i_2}} B^{i_4}_{(\overline{u}*\overline{v})^{i_3}}$ $(\dagger\dagger)$

for some $g_{i_1 i_2} \in SG(3)$. For the rest of the proof, we can follow the argument in Lemma 1.9. Note that the conditions in this Lemma on $\{\overline{u}, \overline{v}\}$ are symmetric, so that making the substitutions, $-\overline{v}$ for $\overline{u}$ and $-\overline{u}$ for $\overline{v}$, observing the choice of root for $\gamma_{\overline{v}*\overline{u}}$ is admissible for $\gamma_{-\overline{v}*-\overline{u}}$ or $\gamma_{-(\overline{v}*\overline{u})}$, we can use the first part of this lemma, to conclude that there exists $h \in SG(3)$ with;

$B_{\overline{v}}^{i_2} B_{\overline{u}}^{i_1} = B^{i_6}_{(\overline{v}*\overline{u})^{i_5}} R_h$ $(\dagger\dagger\dagger)$

where i_6 is determined by the formula $(*)$ in the statement of the Lemma. Then, we can use the conjugation result $(\dagger\dagger)$, applied to $(\dagger)$, noting that $((\overline{u}*\overline{v})^{i_3})^2\neq c^2$, by $(*)$, to obtain;

$$B_{\overline{v}}^{i_2}B_{\overline{u}}^{i_1}=B_{g_{i_1i_2}((\overline{u}*\overline{v})^{i_3})}^{i_4}R_{g_{i_1i_2}}$$

By the uniqueness of representation, noting that $((\overline{v}*\overline{v})^{i_5})^2\neq c^2$ and $(g_{i_1i_2}((\overline{u}*\overline{v})^{i_3}))^2\neq c^2$, we conclude that $h=g_{i_1i_2}$, $i_4=i_6$, and $g_{i_1i_2}((\overline{u}*\overline{v})^{i_3})=(\overline{v}*\overline{u})^{i_5}$ as required.

The next claim can be seen by following through the computation for real $\overline{v}$, with $|\overline{v}|<c$. For the penultimate claim, we can use the fact that $(\rho,\overline{J})$ transforms as a four-vector, and the components of $\{\overline{E},\overline{B}\}$ transform as part of the covariant field tensor, see [3]. Then we can use the result that the identities holds for generically independent real $\{\overline{u},\overline{v}\}$, with $|\overline{u}|<c$ and $|\overline{v}|<c$. For the last claim, we can just use the identification of matrices in $(**)$ of the lemma. □

Lemma 1.14. *Limit Frames*

Given a series of frames $S_{f(s)\overline{v}}$, where $|\overline{v}|=1$, connected to the base frame S by the boost matrices $B_{f(s)\overline{v}}$, with $lim_{s\rightarrow\infty}f(s)=\infty$, f smooth and positive real-valued on $\mathcal{R}_{>0}$, the boost matrix $lim_{s\rightarrow\infty,f(s)\neq c}B_{f(s)\overline{v}}$ exists, with a given choice of square root, and defines a limit frame S_∞. Given a series of transformations $(\rho_{f(s)\overline{v}},\overline{J}_{f(s)\overline{v}})$ and $(\overline{E}_{f(s)\overline{v}},\overline{B}_{f(s)\overline{v}})$ of $(\rho,\overline{J})$ and $(\overline{E},\overline{B})$ from the base frame S, we define the transformation to S_∞ by taking the limit as $s\rightarrow\infty$, of the transformation rules, see [1], for the pairs $(\rho_{f(s)\overline{v}},\overline{J}_{f(s)\overline{v}})$ and $(\overline{E}_{f(s)\overline{v}},\overline{B}_{f(s)\overline{v}})$. If $((\rho,\overline{J})$ is a real pair and $(\overline{E},\overline{B})$ is a complex valued pair, then the limit exists at the corresponding coordinates in S_∞. Defining the transformation of derivatives from S_∞ to S in the usual way by $(lim_{s\rightarrow\infty,f(s)\neq c}B_{f(s)\overline{v}})^{-1}$, we have that the transformation is given by $lim_{s\rightarrow\infty,f(s)\neq c}((B_{f(s)\overline{v}})^{-1})$. In particularly, the limit definitions are independent of the choice of f with $lim_{s\rightarrow\infty}f(s)=\infty$.

Proof. For the first claim, using the formula for the boost matrix $B_{f(s)\overline{v}}$, given in Lemma 1.5;

$$B_{f(s)\overline{v}}=I+\frac{\gamma_s b_{f(s)\overline{v}}}{c}+\frac{\gamma_s^2 b_{f(s)\overline{v}}^2}{c^2(\gamma_s+1)}$$

where $\gamma_s=\frac{1}{\sqrt{1-\frac{f(s)^2}{c^2}}}$. We can compute the limit, taking a positive square root;

$lim_{s\rightarrow\infty,f(s)\neq c}(\gamma_s+1)=1+lim_{s\rightarrow\infty,f(s)\neq c}\frac{1}{\sqrt{1-\frac{f(s)^2}{c^2}}}$

$=1+lim_{s\rightarrow\infty,f(s)\neq c}\frac{1}{f(s)\sqrt{\frac{1}{f(s)}^2-\frac{1}{c^2}}}=1$

$lim_{s\rightarrow\infty,f(s)\neq c}f(s)\gamma_s=lim_{s\rightarrow\infty,f(s)\neq c}\frac{f(s)}{\sqrt{1-\frac{f(s)^2}{c^2}}}$

$=lim_{s\rightarrow\infty,f(s)\neq c}\frac{1}{\sqrt{\frac{1}{f(s)^2}-\frac{1}{c^2}}}=-ic$

$lim_{s\rightarrow\infty,f(s)\neq c}f(s)^2\gamma_s^2=lim_{s\rightarrow\infty,f(s)\neq c}\frac{f(s)^2}{1-\frac{f(s)^2}{c^2}}=-c^2$

It is then straightforward to see that that $lim_{s\rightarrow\infty,f(s)\neq c}(\frac{\gamma_s b_{f(s)\overline{v}}}{c})_{ij}$ and $lim_{s\rightarrow\infty,f(s)\neq c}(\frac{\gamma_s^2 b_{f(s)\overline{v}}^2}{c^2(\gamma_s+1)})_{ij}$ exist for $0\leq i,j\leq 4$, as required. For the second claim, we have that;

$\rho_{f(s)\overline{v}}=\gamma_s(\rho-\frac{<f(s)\overline{v},\overline{J}>}{c^2})$

$\overline{J}_{f(s)\overline{v}}=\gamma_s(\overline{J}_{||,s}-f(s)\overline{v}\rho)+\overline{J}_{\perp,s}$

We have , for a vector field $\overline{F}$ and scalar ρ, that;

$lim_{s\rightarrow\infty}\overline{F}_{||,s}=lim_{s\rightarrow\infty}\frac{<\overline{F},f(s)\overline{v}>f(s)\overline{v}}{f(s)^2}=<\overline{F},\overline{v}>\overline{v}=\overline{F}_{||}$

so that;

$lim_{s\rightarrow\infty,f(s)\neq c}\overline{F}_{\perp,s}=\overline{F}-\overline{F}_{||,s}=\overline{F}_{\perp}$

and $lim_{s\rightarrow\infty,f(s)\neq c}\gamma_s\overline{F}_{||,s}=lim_{s\rightarrow\infty}\gamma_s\frac{<\overline{F},f(s)\overline{v}>f(s)\overline{v}}{f(s)^2}=0$

As above, we have that $lim_{s\rightarrow\infty,f(s)\neq c}\gamma_s\rho=0$, and

$lim_{s\rightarrow\infty,f(s)\neq c}\gamma_s<f(s)\overline{v},\overline{F}>=lim_{s\rightarrow\infty}\gamma_s f(s)<\overline{v},\overline{F}>=-ic<\overline{v},\overline{F}>$

$lim_{s\rightarrow\infty,f(s)\neq c}\gamma_s f(s)\overline{v}\rho=-ic\overline{v}\rho$

so that;

$lim_{s\rightarrow\infty,f(s)\neq c}\rho_{f(s)\overline{v}}=\frac{i}{c}<\overline{v},\overline{J}>$

$lim_{s\rightarrow\infty,f(s)\neq c}\overline{J}_{f(s)\overline{v}} = ic\overline{v}\rho + \overline{J}_{\perp}$

Similarly, we have that;

$\overline{E}_{f(s)\overline{v}} = \overline{E}_{||,s} + \gamma_s(\overline{E}_{\perp,s} + f(s)\overline{v}\times\overline{B})$

$\overline{B}_{f(s)\overline{v}} = \overline{B}_{||,s} + \gamma_s(\overline{B}_{\perp,s} - \frac{f(s)\overline{v}\times\overline{E}}{c^2})$

so that, using the above computations again, replacing $<\overline{v},\overline{F}>$ by $\overline{v}\times\overline{F}$;

$lim_{s\rightarrow\infty,f(s)\neq c}\overline{E}_{f(s)\overline{v}} = \overline{E}_{||} - ic(\overline{v}\times\overline{B})$

$lim_{s\rightarrow\infty,f(s)\neq c}\overline{B}_{f(s)\overline{v}} = \overline{B}_{||} + \frac{i}{c}(\overline{v}\times\overline{E})$

For the penultimate claim we have that $lim_{s\rightarrow\infty,f(s)\neq c}B_{f(s)\overline{v}}$ is invertible, which can be seen from the inverse boost matrix, replacing $\overline{v}$ by $-\overline{v}$, and noting the limit exists again. Then, we can use the formula for the inverse of a matrix, and elementary properties of limits. The last claim is clear from the above calculation.

□

Definition 1.15. *In the context of special relativity, we choose coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. We have the coordinate relationship for the Lorentz transformation;*

$x^{0'} = \gamma(x^0 - \frac{vx^1}{c})$

$x^{1'} = \gamma(x^1 - \frac{vx^0}{c})$

$x^{2'} = x^2$

$x^{3'} = x^3$

where v is the velocity of a boost in the x-direction, and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

We can encode the transformation with the Lorentz matrix given by $\overline{\Lambda}$, which is defined by;

$(\overline{\Lambda})_{00} = (\overline{\Lambda})_{11} = \gamma$

$(\overline{\Lambda})_{01} = (\overline{\Lambda})_{10} = -\gamma\beta$

$(\overline{\Lambda})_{22} = (\overline{\Lambda})_{33} = 1$

$(\overline{\Lambda})_{ij} = 0$, *otherwise, for* $0 \leq i, j \leq 3$

where $\beta = \frac{v}{c}$. *We let;*

$\sigma = \frac{\epsilon_0}{2}(e^2 + c^2 b^2)$

where $e = |\overline{E}|$, $b = |\overline{B}|$, *and* $\{\overline{E}, \overline{B}\}$ *are electric and magnetic fields, satisfying Maxwell's equations, in the rest frame. We let;*

$\overline{g} = (g_1, g_2, g_3) = \epsilon_0(\overline{E} \times \overline{B})$

For $1 \leq i, j \leq 3$, *we let;*

$p_{ij} = -\epsilon_0(e_i e_j + c^2 b_i b_j - \frac{1}{2}\delta_{ij}(e^2 + c^2 b^2))$

be Maxwell's stress tensor, where $\overline{E} = (e_1, e_2, e_3)$ *and* $\overline{B} = (b_1, b_2, b_3)$. *The stress energy tensor is given by* $\overline{M}$, *defined by;*

$(\overline{M})_{00} = \sigma$

$(\overline{M})_{i0} = cg_i$, *for* $1 \leq i \leq 3$

$(\overline{M})_{0j} = cg_j$, *for* $1 \leq j \leq 3$

$(\overline{M})_{ij} = p_{ij}$, *for* $1 \leq i, j \leq 3$

It transforms between inertial frames, using the summation rule, see [8]*;*

$(\overline{M}')_{i'j'} = (\overline{\Lambda})_{ii'}(\overline{\Lambda})_{jj'}(\overline{M})_{ij}$

Lemma 1.16. *Suppose that* $(\rho, \overline{J})$ *satisfies the continuity equation and is surface non-radiating in the sense of Definition 2.8 of* [6]*, then there exist* 3 *real families of electric and magnetic fields, indexed by* $v \in \mathcal{R}$, $(\overline{E}^1_v, \overline{B}^1_v)$, $(\overline{E}^2_v, \overline{B}^2_v)$, $(\overline{E}^3_v, \overline{B}^3_v)$, *satisfying Maxwell's equations in the rest frame* S *and the additional equations;*

$$\alpha_v \frac{\partial \sigma_v^1}{\partial x} + \beta_v \frac{\partial g_{1,v}^1}{\partial x} + \gamma_v \frac{\partial p_{11,v}^1}{\partial x} + \delta_v \frac{\partial \sigma_v^1}{\partial t} + \epsilon_v \frac{\partial g_{1,v}^1}{\partial t} + \xi_v \frac{\partial p_{11,v}^1}{\partial t} + \eta_v div(\overline{g}_v^1)$$

$$+\theta_v (f_{1,v}^1 + \frac{\partial g_{1,v}^1}{\partial t}) = 0$$

$$\alpha_v \frac{\partial \sigma_v^2}{\partial y} + \beta_v \frac{\partial g_{2,v}^2}{\partial y} + \gamma_v \frac{\partial p_{22,v}^2}{\partial y} + \delta_v \frac{\partial \sigma_v^2}{\partial t} + \epsilon_v \frac{\partial g_{2,v}^2}{\partial t} + \xi_v \frac{\partial p_{22,v}^2}{\partial t} + \eta_v div(\overline{g}_v^2)$$

$$+\theta_v (f_{2,v}^2 + \frac{\partial g_{2,v}^2}{\partial t}) = 0$$

$$\alpha_v \frac{\partial \sigma_v^3}{\partial z} + \beta_v \frac{\partial g_{3,v}^3}{\partial z} + \gamma_v \frac{\partial p_{33,v}^3}{\partial z} + \delta_v \frac{\partial \sigma_v^3}{\partial t} + \epsilon_v \frac{\partial g_{3,v}^3}{\partial t} + \xi_v \frac{\partial p_{33,v}^3}{\partial t} + \eta_v div(\overline{g}_v^3)$$

$$+\theta_v (f_{3,v}^3 + \frac{\partial g_{3,v}^3}{\partial t}) = 0 \ (\dagger)$$

where;

$$\alpha_v = \frac{-\beta\gamma^3}{c}, \beta_v = ((\beta^2+1)\gamma^3 - \gamma), \gamma_v = (\frac{\gamma\beta}{c} - \frac{\gamma^3\beta}{c}), \delta_v = \frac{-\beta\gamma^3 v}{c^3}$$

$$\epsilon_v = \frac{(\beta^2+1)\gamma^3 v}{c^2}, \xi_v = \frac{-\beta\gamma^3 v}{c^3}, \eta_v = \gamma, \theta_v = \frac{\gamma\beta}{c}$$

Moreover, we can take a fixed pair $(\overline{E}, \overline{B})$ *in the rest frame, with* $div(\overline{E}\times\overline{B}) = 0$, *such that* $\overline{E} = \overline{E}_0^1 = \overline{E}_0^2 = \overline{E}_0^3$ *and* $\overline{B} = \overline{B}_0^1 = \overline{B}_0^2 = \overline{B}_0^3$.

Proof. Transforming between frames, and corresponding fields $(\overline{E}', \overline{B}')$ in S' and $(\overline{E}_v^1, \overline{B}_v^1)$ in S, we have, dropping the index v throughout the proof, that;

$$\sigma' = (\overline{M}')_{00} = (\overline{\Lambda})_{i0}(\overline{\Lambda})_{j0}(\overline{M})_{ij}$$

$$= \gamma^2\sigma - \gamma^2\beta c g_1 - \gamma^2\beta c g_1 + \gamma^2\beta^2 p_{11}$$

$$= \gamma^2(\sigma - 2\beta c g_1 + \beta^2 p_{11})$$

$$c g_1' = (\overline{M}')_{10} = (\overline{\Lambda})_{i1}(\overline{\Lambda})_{j0}(\overline{M})_{ij}$$

$$= -\gamma^2\beta\sigma + \gamma^2\beta^2 c g_1 + \gamma^2 c g_1 - \gamma^2\beta p_{11}$$

$$= \gamma^2(-\beta\sigma + (\beta^2 c + c) g_1 - \beta p_{11})$$

$$c g_2' = (\overline{M}')_{20} = (\overline{\Lambda})_{i2}(\overline{\Lambda})_{j0}(\overline{M})_{ij}$$

$$= \gamma c g_2 - \gamma\beta p_{21}$$

$cg_3' = (\overline{M}')_{30} = (\overline{\Lambda})_{i3}(\overline{\Lambda})_{j0}(\overline{M})_{ij}$

$= \gamma c g_3 - \gamma\beta p_{31}$ $(*)$

The condition that $div'(\overline{E}' \times \overline{B}') = 0$ in the frame S' is equivalent to;

$\bigtriangledown' \centerdot (g_1', g_2', g_3') = 0$ $(**)$

We have the transformation rule for $\bigtriangledown'$, given in [1];

$\frac{\partial}{\partial x'} = \gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})$

$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$

$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$

Applying this to $(*), (**)$, we obtain;

$\gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})g_1' + \frac{\partial g_2'}{\partial y} + \frac{\partial g_3'}{\partial z}$

$= \frac{\gamma}{c}\frac{\partial}{\partial x}(\gamma^2(-\beta\sigma + (\beta^2 c + c)g_1 - \beta p_{11}))$

$+ \frac{\gamma v}{c^3}\frac{\partial}{\partial t}(\gamma^2(-\beta\sigma + (\beta^2 c + c)g_1 - \beta p_{11}))$

$+ \frac{1}{c}\frac{\partial}{\partial y}(\gamma c g_2 - \gamma\beta p_{21})$

$+ \frac{1}{c}\frac{\partial}{\partial z}(\gamma c g_3 - \gamma\beta p_{31}) = 0$, $(***)$

and, rearranging, we have that;

$\frac{-\beta\gamma^3}{c}\frac{\partial\sigma}{\partial x} + ((\beta^2+1)\gamma^3 - \gamma)\frac{\partial g_1}{\partial x} + \gamma div(\overline{g}) + (\frac{\gamma\beta}{c} - \frac{\gamma^3\beta}{c})\frac{\partial p_{11}}{\partial x}$

$-\frac{\gamma\beta}{c}div(\overline{T}_1) - \frac{\beta\gamma^3 v}{c^3}\frac{\partial\sigma}{\partial t} + \frac{(\beta^2+1)\gamma^3 v}{c^2}\frac{\partial g_1}{\partial t} - \frac{\beta\gamma^3 v}{c^3}\frac{\partial p_{11}}{\partial t} = 0$, $(****)$

where $\overline{T}$ is the Maxwell stress tensor. We have, see [3], that;

$div(\overline{T}_1) + \frac{\partial g_1}{\partial t} = -f_1$

where $\overline{f}=(f_1,f_2,f_3)$ is the force applied by the fields $\{\overline{E}_v^1,\overline{B}_v^1\}$ relative to the charge and current $(\rho,\overline{J})$ in the rest frame S.

Rearranging again, we obtain;

$$\frac{-\beta\gamma^3}{c}\frac{\partial\sigma}{\partial x}+((\beta^2+1)\gamma^3-\gamma)\frac{\partial g_1}{\partial x}+(\frac{\gamma\beta}{c}-\frac{\gamma^3\beta}{c})\frac{\partial p_{11}}{\partial x}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial\sigma}{\partial t}$$

$$+\frac{(\beta^2+1)\gamma^3 v}{c^2}\frac{\partial g_1}{\partial t}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial p_{11}}{\partial t}+\gamma div(\overline{g})+\frac{\gamma\beta}{c}(f_1+\frac{\partial g_1}{\partial t})=0\ (*****)$$

By symmetry, for boosts with velocity v in the y and z directions, we obtain the relations for $(\overline{E}_v^2,\overline{B}_v^2)$ and $(\overline{E}_v^3,\overline{B}_v^3)$ in the rest frame S;

$$\frac{-\beta\gamma^3}{c}\frac{\partial\sigma}{\partial y}+((\beta^2+1)\gamma^3-\gamma)\frac{\partial g_2}{\partial y}+(\frac{\gamma\beta}{c}-\frac{\gamma^3\beta}{c})\frac{\partial p_{22}}{\partial y}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial\sigma}{\partial t}$$

$$+\frac{(\beta^2+1)\gamma^3 v}{c^2}\frac{\partial g_2}{\partial t}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial p_{22}}{\partial t}+\gamma div(\overline{g})+\frac{\gamma\beta}{c}(f_2+\frac{\partial g_2}{\partial t})=0$$

and;

$$\frac{-\beta\gamma^3}{c}\frac{\partial\sigma}{\partial z}+((\beta^2+1)\gamma^3-\gamma)\frac{\partial g_3}{\partial z}+(\frac{\gamma\beta}{c}-\frac{\gamma^3\beta}{c})\frac{\partial p_{33}}{\partial z}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial\sigma}{\partial t}$$

$$+\frac{(\beta^2+1)\gamma^3 v}{c^2}\frac{\partial g_3}{\partial t}-\frac{\beta\gamma^3 v}{c^3}\frac{\partial p_{33}}{\partial t}+\gamma div(\overline{g})+\frac{\gamma\beta}{c}(f_3+\frac{\partial g_3}{\partial t})=0$$

The final claim is clear by the definition of surface non-radiating. □

Lemma 1.17. *For $(\overline{x_0},t_0)$ in the rest frame S, there exists, at $(\overline{x_0},t_0)$ a polynomial approximation of $(\overline{E}_v^1,\overline{B}_v^1)$ from Lemma 1.16, satisfying Maxwell's equations, and the additional equations there. Moreover, the conditions are algebraic.*

Proof. Fix $(\overline{x}_0,t_0)$ in the rest frame S and define the vector fields $(\overline{L}_v^1,\overline{M}_v^1)$ by;

$$l_{i,v}^1=\sum_{0\leq j+k+l+m\leq 1}l_{i,jklm,v}^1(x-x_0)^j(y-y_0)^k(z-z_0)^l(t-t_0)^m$$

$$m_{i,v}^1=\sum_{0\leq j+k+l+m\leq 1}m_{i,jklm,v}^1(x-x_0)^j(y-y_0)^k(z-z_0)^l(t-t_0)^m$$

for $1\leq i\leq 3$, where $\overline{L}_v^1=(l_{1,v}^1,l_{2,v}^1,l_{3,v}^1)$, $\overline{M}_v^1=(m_{1,v}^1,m_{2,v}^1,m_{3,v}^1)$ and;

$$l_{i,jklm,v}^1=\frac{\partial^{(j+k+l+m)}e_{i,v}^1}{\partial x^j\partial y^k\partial z^l\partial t^m}|_{(\overline{x}_0,t_0)}$$

$m^1_{i,jklm,v}=\frac{\partial^{(j+k+l+m)}b^1_{i,v}}{\partial x^j\partial y^k\partial z^l\partial t^m}|_{(\overline{x}_0,t_0)}$

Then, for $v\in\mathcal{R}$, at $(\overline{x}_0,t_0)$, $(\overline{L}^1_v,\overline{M}^1_v)$ satisfy Maxwell's equations, see the proof in [1], and the first equation of $(\dagger)$ in Lemma 1.16. The satisfaction of Maxwell's equations at (x_0,t_0), defines 8 linear conditions on the 30 coefficients $l^1_{i,jklm,v}$ and $m^1_{i,jklm,v}$ given by;

$l^1_{1,1000,v}+l^1_{2,0100,v}+l^1_{3,0010,v}-\frac{\rho(\overline{x}_0,t_0)}{\epsilon_0}=0\ (i)$

$l^1_{3,0100,v}-l^1_{2,0010,v}+m^1_{1,0001,v}=0$

$l^1_{3,1000,v}-l^1_{1,0010,v}+m^1_{2,0001,v}=0$

$l^1_{2,1000,v}-l^1_{1,0100,v}+m^1_{3,0001,v}=0\ (ii)$

$m^1_{1,1000,v}+m^1_{2,0100,v}+m^1_{3,0010,v}=0\ (iii)$

$m^1_{3,0100,v}-m^1_{2,0010,v}-\mu_0\epsilon_0 l^1_{1,0001,v}-\mu_0 j_1(\overline{x}_0,t_0)=0$

$m^1_{3,1000,v}-m^1_{1,0010,v}-\mu_0\epsilon_0 l^1_{2,0001,v}-\mu_0 j_2(\overline{x}_0,t_0)=0$

$m^1_{2,1000,v}-m^1_{1,0100,v}-\mu_0\epsilon_0 l^1_{3,0001,v}-\mu_0 j_3(\overline{x}_0,t_0)=0\ (iv)$

where $\overline{J}=(j_1,j_2,j_3)$. For the first equation of $(\dagger)$ in Lemma 1.16, a simple computation using the product rule and the formula given in [3];

$\overline{f}^1_v=\rho\overline{E}^1_v+\overline{J}\times\overline{B}^1_v$

we obtain that;

$\alpha_v\frac{\partial\sigma^1_v}{\partial x}=\epsilon_0\alpha_v(l^1_{1,0000,v}l^1_{1,1000,v}+l^1_{2,0000,v}l^1_{2,1000,v}+l^1_{3,0000,v}l^1_{3,1000,v}$

$+c^2m^1_{1,0000,v}m^1_{1,1000,v}+c^2m^1_{2,0000,v}m^1_{2,1000,v}+c^2m^1_{3,0000,v}m^1_{3,1000,v})$

$\beta_v\frac{\partial g^1_{1,v}}{\partial x}=\epsilon_0\beta_v(l^1_{2,1000,v}m^1_{3,0000,v}+l^1_{2,0000,v}m^1_{3,1000,v}-l^1_{3,1000,v}m^1_{2,0000,v}$

$-l^1_{3,0000,v}m^1_{2,1000,v})$

$\gamma_v\frac{\partial p^1_{11,v}}{\partial x}=-\epsilon_0\gamma_v(l^1_{1,0000,v}l^1_{1,1000,v}+c^2m^1_{1,0000,v}m^1_{1,1000,v}-l^1_{2,0000,v}l^1_{2,1000,v}$

$-l^1_{3,0000,v}l^1_{3,1000,v} - c^2m^1_{2,0000,v}m^1_{2,1000,v} - c^2m^1_{3,0000,v}m^1_{3,1000,v})$

$\delta_v\frac{\partial\sigma^1_v}{\partial t} = \epsilon_0\delta_v(l^1_{1,0000,v}l^1_{1,0001,v}+l^1_{2,0000,v}l^1_{2,0001,v}+l^1_{3,0000,v}l^1_{3,0001,v}+c^2m^1_{1,0000,v}m^1_{1,0001,v}$

$+c^2m^1_{2,0000,v}m^1_{2,0001,v} + c^2m^1_{3,0000,v}m^1_{3,0001,v})$

$\epsilon_v\frac{\partial g^1_{1,v}}{\partial t} = \epsilon_0\epsilon_v(l^1_{2,0001,v}m^1_{3,0000,v} + l^1_{2,0000,v}m^1_{3,0001,v} - l^1_{3,0001,v}m^1_{2,0000,v}$

$-l^1_{3,0000,v}m^1_{2,0001,v})$

$\xi_v\frac{\partial p^1_{11,v}}{\partial t} = -\epsilon_0\xi_v(l^1_{1,0000,v}l^1_{1,0001,v} + c^2m^1_{1,0000,v}m^1_{1,0001,v} - l^1_{2,0000,v}l^1_{2,0001,v}$

$-l^1_{3,0000,v}l^1_{3,0001,v} - c^2m^1_{2,0000,v}m^1_{2,0001,v} - c^2m^1_{3,0000,v}m^1_{3,0001,v})$

$\eta_v div(\overline{g}^1_v) = \epsilon_0\eta_v(l^1_{2,1000,v}m^1_{3,0000,v} + l^1_{2,0000,v}m^1_{3,1000,v} - l^1_{3,1000,v}m^1_{2,0000,v}$

$-l^1_{3,0000,v}m^1_{2,1000,v} + l^1_{3,0100,v}m^1_{1,0000,v} + l^1_{3,0000,v}m^1_{1,0100,v} - l^1_{1,0100,v}m^1_{3,0000,v}$

$-l^1_{1,0000,v}m^1_{3,0100,v} + l^1_{1,0010,v}m^1_{2,0000,v} + l^1_{1,0000,v}m^1_{2,0010,v} - l^1_{2,0010,v}m^1_{1,0000,v}$

$-l^1_{2,0000,v}m^1_{1,0010,v})$

$\theta_v f^1_{1,v} = \theta_v(p(\overline{x}_0,t_0)l^1_{1,0000,v} + j_2(\overline{x}_0,t_0)m^1_{3,0000,v} - j_3(\overline{x}_0,t_0)m^1_{2,0000,v})$

$\theta_v\frac{\partial g^1_{1,v}}{\partial t} = \epsilon_0\theta_v(l^1_{2,0001,v}m^1_{3,0000,v} + l^1_{2,0000,v}m^1_{3,0001,v} - l^1_{3,0001,v}m^1_{2,0000,v}$

$-l^1_{3,0000,v}m^1_{2,0001,v})$ $(\dagger\dagger)$

Combining $(\dagger)$ and $(\dagger\dagger)$ we obtain 1 condition on the coefficients. We introduce 30 new variables $\{x^1_{i,jklm}, y^1_{i,jklm}\}$, for $1 \leq i \leq 3$ and $0 \leq j+k+l+m \leq 1$. Substituting the variables for the corresponding $\{l^1_{i,jklm,v}, m^1_{i,jklm,v}\}$ in the equations $(\dagger), (\dagger\dagger)$ and $(i)-(iv)$, when $v \in \mathcal{C}\setminus\{-c,c\}$, we obtain algebraic conditions. □

Lemma 1.18. *Limit Equations We can take a limit as $v \rightarrow \infty$ of the equations $(\dagger)$ in Lemma 1.16, to obtain the limit relations;*

$\frac{\partial}{\partial t}(\overline{E}^1_\infty \times \overline{B}^1_\infty)_1 = -\frac{1}{\epsilon_0}(\frac{\partial p^1_{12,\infty}}{\partial y} + \frac{\partial p^1_{13,\infty}}{\partial z})$

$\frac{\partial}{\partial t}(\overline{E}^2_\infty \times \overline{B}^2_\infty)_2 = -\frac{1}{\epsilon_0}(\frac{\partial p^2_{12,\infty}}{\partial x} + \frac{\partial p^2_{23,\infty}}{\partial z})$

$\frac{\partial}{\partial t}(\overline{E}^3_\infty \times \overline{B}^3_\infty)_3 = -\frac{1}{\epsilon_0}(\frac{\partial p^3_{13,\infty}}{\partial x} + \frac{\partial p^3_{23,\infty}}{\partial y})$

for the transferred fields $(\overline{E}^i_\infty, \overline{B}^i_\infty)$, $1 \leq i \leq 3$, *see Lemma 1.14.*

Proof. Observe that, taking compatible square roots;

$lim_{v\rightarrow\infty}\gamma = lim_{v\rightarrow\infty}\alpha_v = lim_{v\rightarrow\infty}\beta_v = lim_{v\rightarrow\infty}\delta_v = lim_{v\rightarrow\infty}\xi_v$

$= lim_{v\rightarrow\infty}\eta_v = 0$

$lim_{v\rightarrow\infty}\gamma_v = lim_{v\rightarrow\infty}\theta_v = lim_{v\rightarrow\infty}\frac{\gamma\beta}{c}$

$= lim_{v\rightarrow\infty}\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}}$

$= \frac{1}{c^2}lim_{v\rightarrow\infty}\frac{1}{\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}}$

$= \frac{1}{c^2}\frac{c}{i} = -\frac{i}{c}$

$lim_{v\rightarrow\infty}\epsilon_v = lim_{v\rightarrow\infty}\frac{\beta^2 v\gamma^3}{c^2}$

$= lim_{v\rightarrow\infty}\frac{v^3}{c^4(1-\frac{v^2}{c^2})^{\frac{3}{2}}}$

$= \frac{1}{c^4}lim_{v\rightarrow\infty}\frac{1}{(\frac{1}{v^2}-\frac{1}{c^2})^{\frac{3}{2}}}$

$= \frac{1}{c^4}\frac{c^3}{i^3} = \frac{i}{c}$

Taking the limit of the first equation in (†) of Lemma 1.16, we obtain;

$-\frac{i}{c}\frac{\partial p^1_{11,\infty}}{\partial x} + \frac{i}{c}\frac{\partial g^1_{1,\infty}}{\partial t} - \frac{i}{c}(f^1_{1,\infty} + \frac{\partial g^1_{1,\infty}}{\partial t}) = 0$

which simplifies to;

$f^1_{1,\infty} = -\frac{\partial p^1_{11,\infty}}{\partial x}$ $(*)$

Similarly, taking limits of the second and third equations in (†) of Lemma 1.16, we obtain;

$f^2_{2,\infty} = -\frac{\partial p^2_{22,\infty}}{\partial y}$

$f^3_{3,\infty} = -\frac{\partial p^3_{33,\infty}}{\partial z}$ $(**)$

Using the definition of p_{ii}, for $1 \leq i \leq 3$, we can rearrange $(*), (**)$ to obtain;

$$f^1_{1,\infty} = (\epsilon_0 \tfrac{\partial}{\partial x}((e^1_{1,\infty})^2 + c^2(b^1_{1,\infty})^2) - \tfrac{\epsilon_0}{2} \tfrac{\partial}{\partial x}((e^1_{\infty})^2 + c^2(b^1_{\infty})^2))$$

$$f^2_{2,\infty} = (\epsilon_0 \tfrac{\partial}{\partial y}((e^2_{2,\infty})^2 + c^2(b^2_{2,\infty})^2) - \tfrac{\epsilon_0}{2} \tfrac{\partial}{\partial y}((e^2_{\infty})^2 + c^2(b^2_{\infty})^2))$$

$$f^3_{3,\infty} = (\epsilon_0 \tfrac{\partial}{\partial z}((e^3_{3,\infty})^2 + c^2(b^3_{3,\infty})^2) - \tfrac{\epsilon_0}{2} \tfrac{\partial}{\partial z}((e^3_{\infty})^2 + c^2(b^3_{\infty})^2)) \; (***)$$

Using the definition of force density $\overline{f}$, we have, see [3], the formula;

$$\overline{f} = -\tfrac{1}{2} \bigtriangledown (\epsilon_0 E^2 + \tfrac{1}{\mu_0} B^2) - \epsilon_0 \tfrac{\partial}{\partial t}(\overline{E} \times \overline{B}) + \epsilon_0((\bigtriangledown \cdot \overline{E})\overline{E} + (\overline{E} \cdot \bigtriangledown)\overline{E})$$

$$+\tfrac{1}{\mu_0}((\bigtriangledown \cdot \overline{B})\overline{B} + (\overline{B} \cdot \bigtriangledown)\overline{B})$$

and, substituting for the first equation in $(***)$, using the product rule, we obtain;

$$-\epsilon_0 \tfrac{\partial}{\partial t}(\overline{E}^1_\infty \times \overline{B}^1_\infty)_1 = \tfrac{\partial}{\partial x}(\epsilon_0(e^1_{1,\infty})^2 + \tfrac{1}{\mu_0}(b^1_{1,\infty})^2)$$

$$-\epsilon_0((\bigtriangledown \cdot \overline{E})\overline{E} + (\overline{E} \cdot \bigtriangledown)\overline{E})_1 - \tfrac{1}{\mu_0}((\bigtriangledown \cdot \overline{B})\overline{B} + (\overline{B} \cdot \bigtriangledown)\overline{B})_1$$

$$= \tfrac{\partial}{\partial x}(\epsilon_0(e^1_{1,\infty})^2 + \tfrac{1}{\mu_0}(b^1_{1,\infty})^2) - \epsilon_0(\tfrac{\partial e^1_{1,\infty}}{\partial x} + \tfrac{\partial e^1_{2,\infty}}{\partial y} + \tfrac{\partial e^1_{3,\infty}}{\partial z})e^1_{1,\infty}$$

$$-\epsilon_0(e^1_{1,\infty}\tfrac{\partial}{\partial x} + e^1_{2,\infty}\tfrac{\partial}{\partial y} + e^1_{3,\infty}\tfrac{\partial}{\partial z})e^1_{1,\infty} - \tfrac{1}{\mu_0}(\tfrac{\partial b^1_{1,\infty}}{\partial x} + \tfrac{\partial b^1_{2,\infty}}{\partial y} + \tfrac{\partial b^1_{3,\infty}}{\partial z})b^1_{1,\infty}$$

$$-\tfrac{1}{\mu_0}(b^1_{1,\infty}\tfrac{\partial}{\partial x} + b^1_{2,\infty}\tfrac{\partial}{\partial y} + b^1_{3,\infty}\tfrac{\partial}{\partial z})b^1_{1,\infty}$$

$$= -\epsilon_0(\tfrac{\partial(e^1_{1,\infty}e^1_{2,\infty})}{\partial y} + \tfrac{\partial(e^1_{1,\infty}e^1_{3,\infty})}{\partial z}) - \tfrac{1}{\mu_0}(\tfrac{\partial(b^1_{1,\infty}b^1_{2,\infty})}{\partial y} + \tfrac{\partial(b^1_{1,\infty}b^1_{3,\infty})}{\partial z})$$

and, rearranging;

$$\tfrac{\partial}{\partial t}(\overline{E}^1_\infty \times \overline{B}^1_\infty)_1 = (\tfrac{\partial(e^1_{1,\infty}e^1_{2,\infty} + c^2 b^1_{1,\infty}b^1_{2,\infty})}{\partial y} + \tfrac{\partial(e^1_{1,\infty}e^1_{3,\infty} + c^2 b^1_{1,\infty}b^1_{3,\infty})}{\partial z}) \; (\dagger)$$

Similarly, substituting into the second and third equations of $(***)$, we obtain;

$$\tfrac{\partial}{\partial t}(\overline{E}^2_\infty \times \overline{B}^2_\infty)_2 = (\tfrac{\partial(e^2_{1,\infty}e^2_{2,\infty} + c^2 b^2_{1,\infty}b^2_{2,\infty})}{\partial x} + \tfrac{\partial(e^2_{2,\infty}e^2_{3,\infty} + c^2 b^2_{2,\infty}b^2_{3,\infty})}{\partial z})$$

$$\tfrac{\partial}{\partial t}(\overline{E}^3_\infty \times \overline{B}^3_\infty)_3 = (\tfrac{\partial(e^3_{1,\infty}e^3_{3,\infty} + c^2 b^3_{1,\infty}b^3_{3,\infty})}{\partial x} + \tfrac{\partial(e^3_{2,\infty}e^3_{3,\infty} + c^2 b^3_{2,\infty}b^3_{3,\infty})}{\partial y}) \; (\dagger\dagger)$$

Using the definition of the stress tensor, we can write this as;

$\frac{\partial}{\partial t}(\overline{E}^1_\infty \times \overline{B}^1_\infty)_1 = -\frac{1}{\epsilon_0}(\frac{\partial p^1_{12,\infty}}{\partial y} + \frac{\partial p^1_{13,\infty}}{\partial z})$

$\frac{\partial}{\partial t}(\overline{E}^2_\infty \times \overline{B}^2_\infty)_2 = -\frac{1}{\epsilon_0}(\frac{\partial p^2_{12,\infty}}{\partial x} + \frac{\partial p^2_{23,\infty}}{\partial z})$

$\frac{\partial}{\partial t}(\overline{E}^3_\infty \times \overline{B}^3_\infty)_3 = -\frac{1}{\epsilon_0}(\frac{\partial p^3_{13,\infty}}{\partial x} + \frac{\partial p^3_{23,\infty}}{\partial y})$ (†††)

□

Lemma 1.19. *For the construction of $(\overline{E}, \overline{B})$ in the base frame S, transferred from the limit $(\overline{E}_\infty, \overline{B}_\infty)$ in the frame S_{infty}, see Lemmas 1.27,1.29,1.30 and 1.31, we obtain a set of equations $(*****\,*)$, as in the proof of the Lemma, for the quantities $\{\sigma, g_i, p_{jk}\}$, $1 \leq i \leq 3$, $1 \leq j \leq k \leq 3$, valid for $\overline{u}$ real, with $u_1 \neq 0$, $\overline{u} \neq u_1\overline{e}_1$, and with coefficients as defined in the proof of the Lemma.*

Proof. Fixing $\overline{u}$ real, with $u_1 \neq 0$ and $\overline{u} \neq u_1\overline{e}_1$, choose $g \in SO(3)$ with $g(\overline{e}_1) = \frac{1}{(u_2^2+u_3^2)^{\frac{1}{2}}}(0, u_2, u_3)$, so that, for $v \in \mathcal{R}$, $g(iv(u_2^2+u_3^2)^{\frac{1}{2}}\overline{e}_1) = iv(0, u_2, u_3)$. By the result of Lemma 1.13, we have that;

$B_{iv(0,u_2,u_3)} = R_g B_{iv(u_2^2+u_3^2)^{\frac{1}{2}}\overline{e}_1} R_g^{-1}$

and, by Lemma 1.11, there exists a unique $\overline{w}$, with $\overline{w} \rightarrow iv(0, u_2, u_3)$, such that;

$B_{\overline{w}} B_{\overline{u}} = R_h B_{v\overline{e}_1}$

where $h \in SG(3)$. Let $(\overline{E}_v, \overline{B}_v)$, be fields in the frames S_v, travelling with velocity $v\overline{e}_1$ relative to S, such that $div(\overline{E}_v \times \overline{B}_v) = 0$, then, in the rotated frames $R_h(S_v)$, we have, by Lemma 1.13, that $div(\overline{E}^h_v \times \overline{B}^h_v) = 0$, and this property is preserved in the limit frame $R_{h_\infty}(S_\infty)$, see Lemma 1.14 and Lemmas 1.27,1.29,1.30, 1.31, so that $div(\overline{E}^{h_\infty}_\infty \times \overline{B}^{h_\infty}_\infty) = 0$ as well. By the same argument, in the rotated frame, $R_{g^{-1}}R_{h_\infty}(S_\infty)$, we have that $div(\overline{E}^{g^{-1}h_\infty}_\infty \times \overline{B}^{g^{-1}h_\infty}_\infty) = 0$. Following the same argument as above, and taking compatible square roots, we have;

$lim_{v\rightarrow\infty}\gamma = lim_{v\rightarrow\infty}\alpha_{iv} = lim_{v\rightarrow\infty}\beta_{iv} = lim_{v\rightarrow\infty}\delta_{iv} = lim_{v\rightarrow\infty}\xi_{iv} = lim_{v\rightarrow\infty}\eta_{iv} = 0$

$lim_{v\rightarrow\infty}\gamma_{iv} = lim_{v\rightarrow\infty}\theta_{iv} = lim_{iv\rightarrow\infty}\frac{\gamma\beta}{c}$

$= lim_{v\rightarrow\infty}\frac{iv}{c^2\sqrt{1+\frac{v^2}{c^2}}}$

$= \frac{i}{c^2}lim_{v\rightarrow\infty}\frac{1}{\sqrt{\frac{1}{v^2}+\frac{1}{c^2}}}$

$= \frac{i}{c^2}c = \frac{i}{c}$

$lim_{v\rightarrow\infty}\epsilon_{iv} = lim_{v\rightarrow\infty}\frac{\beta^2 iv\gamma^3}{c^2}$

$= lim_{v\rightarrow\infty}\frac{-iv^3}{c^4(1+\frac{v^2}{c^2})^{\frac{3}{2}}}$

$= \frac{-i}{c^4}lim_{v\rightarrow\infty}\frac{1}{(\frac{1}{v^2}+\frac{1}{c^2})^{\frac{3}{2}}}$

$= \frac{-i}{c^4}c^3 = \frac{-i}{c}$

Taking the limit again of the first equation in (†) of Lemma 1.16, with iv replacing v, see Lemma 1.18, we obtain, for the transformed quantities in the limit frame S', connected to $R_{g^{-1}}R_{h_\infty}(S_\infty)$ as the limit of boosts with velocity vector $iv(0,u_2,u_3)$;

$\frac{i}{c}\frac{\partial p_{11,\infty}}{\partial x} - \frac{i}{c}\frac{\partial g_{1,\infty}}{\partial t} + \frac{i}{c}(f_{1,\infty} + \frac{\partial g_{1,\infty}}{\partial t}) = 0$

which simplifies again to;

$f_{1,\infty} = -\frac{\partial p_{11,\infty}}{\partial x}$ (†)

Following the same argument as Lemma 1.18, and using Maxwell's equations, we have;

$\frac{\partial}{\partial t}(\overline{E}_{\infty'} \times \overline{B}_{\infty'})_1 = -\frac{1}{\epsilon_0}(\frac{\partial p_{12,\infty}}{\partial y} + \frac{\partial p_{13,\infty}}{\partial z})$ (††)

for the transformed fields $(\overline{E}_{\infty'}, \overline{B}_{\infty'})$ in S'. Let S'' be the frame connected to S' by the relation $R_{g^{-1}}(S'') = S'$, where $R_{g^{-1}}(0,u_2,u_3) = w\overline{e}_1$ and $w = (u_2^2+u_3^2)^{\frac{1}{2}}$. Let $R_f(\overline{e}_1) = \frac{\overline{u}}{u}$, where $u = (u_1^2+u_2^2+u_3^2)^{\frac{1}{2}}$. By the same conjugation result, we have that;

$R_f B_{u\overline{e}_1} R_f^{-1} = B_{\overline{u}}$

Let S''' be the frame connected to S'' by the relation $R_f(S''')=S''$, the derivatives on S' transform to S''' by the relations;

$\frac{\partial}{\partial t}\mapsto\frac{\partial}{\partial t}$

$\frac{\partial}{\partial y}\mapsto(R_{g^{-1}}R_f)^{-1}(\frac{\partial}{\partial y})=R_f^{-1}R_g(\frac{\partial}{\partial y})$

$\frac{\partial}{\partial z}\mapsto(R_{g^{-1}}R_f)^{-1}(\frac{\partial}{\partial z})=R_f^{-1}R_g(\frac{\partial}{\partial z})$ $(\sharp)$

We have that;

$(\frac{(0,u_2,u_3)}{w},\overline{u})=\frac{w^2}{w}=w$

Let;

$T=\{\overline{\theta}:\overline{\theta}\centerdot u\overline{e}_1\}=w$

so that, as $f\in SO(3)$, $\theta_1 u=w$, where $\overline{\theta}=(\theta_1,\theta_2,\theta_3)$, and $R_f^{-1}R_g(\overline{e}_1)=\overline{\theta}$, with $|\overline{\theta}|=1$. We have that $R_g(\overline{e}_2)=\overline{v}$, with $\overline{v}\in(0,u_2,u_3)^{\perp}$ and $|\overline{v}|=1$. Observing that;

$(\overline{v},(u_1,u_2,u_3))=v_1u_1$

$(\overline{v},\frac{(0,u_2,u_3)}{w})=0$

Let;

$T'=\{\overline{\theta}':\overline{\theta}'\centerdot u\overline{e}_1=v_1u_1,\overline{\theta}'\centerdot\overline{\theta}=0\}$

so that, again as $f\in SO(3)$, $\theta_1'u=v_1u_1$, $\theta'=\frac{v_1u_1}{u}$, $\overline{\theta}'\centerdot\overline{\theta}=0$, and $R_f^{-1}R_g(\overline{e}_2)=\overline{\theta}'$, where $\overline{\theta}'=(\theta_1',\theta_2',\theta_3')$ and $|\overline{\theta}'|=1$. As $\{f,g\}\subset SO(3)$ and $\overline{e}_1\times\overline{e}_2=\overline{e}_3$, we have that $R_h^{-1}R_g(\overline{e}_3)=\theta''=\theta\times\theta'$. Transforming from S' to S''', we have, using $(\sharp)$, that;

$\frac{\partial}{\partial t}((\overline{E}_{\infty'}\times\overline{B}_{\infty'},\overline{e}_1))\mapsto\frac{\partial}{\partial t}((\overline{E}_{\infty'''}\times\overline{B}_{\infty'''},\overline{\theta}))$

$=\frac{\partial}{\partial t}(\frac{g_1}{\epsilon_0}\frac{w}{u}+\frac{g_2}{\epsilon_0}\alpha+\frac{g_3}{\epsilon_0}\beta)$

$=\frac{w}{\epsilon_0 u}\frac{\partial g_1}{\partial t}+\frac{\alpha}{\epsilon_0}\frac{\partial g_2}{\partial t}+\frac{\beta}{\epsilon_0}\frac{\partial g_3}{\partial t}$ $(\sharp\sharp)$

where $\overline{g}=(g_1,g_2,g_3)=\epsilon_0(\overline{E}_{\infty''}\times\overline{B}_{\infty''})$, for the transformed fields $\{\overline{E}_{\infty'''},\overline{B}_{\infty'''}\}$ in S''', and $\overline{\theta}=(\frac{w}{u},\alpha,\beta)$.

$(\overline{E}_{\infty'},\overline{e}_1)\mapsto(\overline{E}_{\infty'''},\overline{\theta})$

$(\overline{E}_{\infty'},\overline{e}_2)\mapsto(\overline{E}_{\infty'''},\overline{\theta}')$

$(\overline{E}_{\infty'},\overline{e}_3)\mapsto(\overline{E}_{\infty'''},\overline{\theta}\times\overline{\theta}')$

$(\overline{E}_{\infty'},\overline{e}_1)(\overline{E}_{\infty'},\overline{e}_2)\mapsto(\overline{E}_{\infty'''},\overline{\theta})(\overline{E}_{\infty'''},\overline{\theta}')$

$=(\frac{e_1w}{u}+e_2\alpha+e_3\beta)(\frac{e_1v_1u_1}{u}+e_2\gamma+e_3\delta)$

$=\frac{wv_1u_1}{u^2}e_1^2+(\frac{w\gamma}{u}+\frac{v_1u_1\alpha}{u})e_1e_2+(\frac{w\delta}{u}+\frac{v_1u_1\beta}{u})e_1e_3+\alpha\gamma e_2^2+(\alpha\delta+\beta\gamma)e_2e_3+\beta\delta e_3^2$

where $\overline{\theta}'=(\frac{v_1u_1}{u},\gamma,\delta)$, $\overline{E}_{\infty'''}=(e_1,e_2,e_3)$. Using the same reasoning for $\overline{B}_{\infty'}$, we have that;

$p_{12,S'}\mapsto\frac{wv_1u_1}{u^2}p_{11}+(\frac{w\gamma}{u}+\frac{v_1u_1\alpha}{u})p_{12}+(\frac{w\delta}{u}+\frac{v_1u_1\beta}{u})p_{13}+\alpha\gamma p_{22}+(\alpha\delta+\beta\gamma)p_{23}+\beta\delta p_{33}$

$-\frac{\epsilon_0}{2}\frac{wv_1u_1}{u^2}\frac{2\sigma}{\epsilon_0}-\frac{\epsilon_0}{2}\alpha\gamma\frac{2\sigma}{\epsilon_0}-\frac{\epsilon_0}{2}\beta\delta\frac{2\sigma}{\epsilon_0}$

$=\frac{wv_1u_1}{u^2}p_{11}+(\frac{w\gamma}{u}+\frac{v_1u_1\alpha}{u})p_{12}+(\frac{w\delta}{u}+\frac{v_1u_1\beta}{u})p_{13}+\alpha\gamma p_{22}+(\alpha\delta+\beta\gamma)p_{23}+\beta\delta p_{33}$ $(\sharp\sharp\sharp)$

as $\frac{wv_1u_1}{u^2}+\alpha\gamma+\beta\delta=0$, where $(p_{ij})_{1\leq i\leq j\leq 3}$ are the components of the stress tensor and σ is the energy term in S'''. We have that;

$(\overline{E}_{\infty'},\overline{e}_1)(\overline{E}_{\infty'},\overline{e}_3)\mapsto(\overline{E}_{\infty'''},\overline{\theta})(\overline{E}_{\infty'''},\overline{\theta}\times\overline{\theta}')$

$=(\frac{e_1w}{u}+e_2\alpha+e_3\beta)(e_1(\alpha\delta-\beta\gamma)+e_2(\frac{\beta v_1u_1}{u}-\frac{\delta w}{u})+e_3(\frac{\gamma w}{u}-\frac{\alpha v_1u_1}{u}))$

$=\frac{w}{u}(\alpha\delta-\beta\gamma)e_1^2$

$+(\frac{w}{u}(\frac{\beta v_1u_1}{u}-\frac{\delta w}{u})+\alpha(\alpha\delta-\beta\gamma))e_1e_2$

$+(\frac{w}{u}(\frac{\gamma w}{u}-\frac{\alpha v_1u_1}{u})+\beta(\alpha\delta-\beta\gamma))e_1e_3$

$+(\alpha(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))e_2^2$

$+(\alpha(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})+\beta(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))e_2e_3$

$+\beta(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})e_3^2$

Again, following the same reasoning as above;

$p_{13,S'}\mapsto \frac{w}{u}(\alpha\delta-\beta\gamma)p_{11}$

$+(\frac{w}{u}(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u})+\alpha(\alpha\delta-\beta\gamma))p_{12}$

$+(\frac{w}{u}(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})+\beta(\alpha\delta-\beta\gamma))p_{13}$

$+(\alpha(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))p_{22}$

$+(\alpha(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})+\beta(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))p_{23}$

$+\beta(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})p_{33}$

$-\frac{\epsilon_0}{2}\frac{w}{u}(\alpha\delta-\beta\gamma)\frac{2\sigma}{\epsilon_0}$

$-\frac{\epsilon_0}{2}\alpha(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u})\frac{2\sigma}{\epsilon_0}$

$-\frac{\epsilon_0}{2}\beta(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})\frac{2\sigma}{\epsilon_0}$

$=\frac{w}{u}(\alpha\delta-\beta\gamma)p_{11}$

$+(\frac{w}{u}(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u})+\alpha(\alpha\delta-\beta\gamma))p_{12}$

$+(\frac{w}{u}(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})+\beta(\alpha\delta-\beta\gamma))p_{13}$

$+(\alpha(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))p_{22}$

$+(\alpha(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})+\beta(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u}))p_{23}$

$+\beta(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})p_{33}$ $(\sharp\sharp\sharp\sharp)$

as $\frac{w}{u}(\alpha\delta-\beta\gamma)+\alpha(\frac{\beta v_1 u_1}{u}-\frac{\delta w}{u})+\beta(\frac{\gamma w}{u}-\frac{\alpha v_1 u_1}{u})=0$.

By (♯), we have that;

$(\frac{\partial}{\partial y})_{S'} \mapsto \frac{v_1 u_1}{u}\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \delta\frac{\partial}{\partial z}$

$(\frac{\partial}{\partial z})_{S'} \mapsto (\alpha\delta - \beta\gamma)\frac{\partial}{\partial x} + (\frac{\beta v_1 u_1}{u} - \frac{\delta w}{u})\frac{\partial}{\partial y} + (\frac{\gamma w}{u} - \frac{\alpha v_1 u_1}{u})\frac{\partial}{\partial z}$ (♯♯♯♯♯)

Combining, (††), (♯♯),(♯♯♯),(♯♯♯♯),(♯♯♯♯♯), we obtain the relation in the frame S''';

$\frac{w}{\epsilon_0 u}\frac{\partial g_1}{\partial t} + \frac{\alpha}{\epsilon_0}\frac{\partial g_2}{\partial t} + \frac{\beta}{\epsilon_0}\frac{\partial g_3}{\partial t}$

$= -\frac{1}{\epsilon_0}[\frac{v_1 u_1}{u}\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \delta\frac{\partial}{\partial z}](\frac{w v_1 u_1}{u^2}p_{11} + (\frac{w\gamma}{u} + \frac{v_1 u_1 \alpha}{u})p_{12} + (\frac{w\delta}{u} + \frac{v_1 u_1 \beta}{u})p_{13} + \alpha\gamma p_{22} + (\alpha\delta + \beta\gamma)p_{23} + \beta\delta p_{33})$

$-\frac{1}{\epsilon_0}[(\alpha\delta - \beta\gamma)\frac{\partial}{\partial x} + (\frac{\beta v_1 u_1}{u} - \frac{\delta w}{u})\frac{\partial}{\partial y} + (\frac{\gamma w}{u} - \frac{\alpha v_1 u_1}{u})\frac{\partial}{\partial z}](\frac{w}{u}(\alpha\delta - \beta\gamma)p_{11}$

$+(\frac{w}{u}(\frac{\beta v_1 u_1}{u} - \frac{\delta w}{u}) + \alpha(\alpha\delta - \beta\gamma))p_{12}$

$+(\frac{w}{u}(\frac{\gamma w}{u} - \frac{\alpha v_1 u_1}{u}) + \beta(\alpha\delta - \beta\gamma))p_{13}$

$+(\alpha(\frac{\beta v_1 u_1}{u} - \frac{\delta w}{u}))p_{22}$

$+(\alpha(\frac{\gamma w}{u} - \frac{\alpha v_1 u_1}{u}) + \beta(\frac{\beta v_1 u_1}{u} - \frac{\delta w}{u}))p_{23}$

$+\beta(\frac{\gamma w}{u} - \frac{\alpha v_1 u_1}{u})p_{33})$ (†††)

Let S'''' be the frame connected to S''' by the relation $B_{u\bar{e}_1}(S'''') = S'''$. Using the formula for the boost matrix, we have that;

$(\frac{\partial}{\partial x})_{S'''} \mapsto \gamma_u(\frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t})$

$(\frac{\partial}{\partial y})_{S'''} \mapsto \frac{\partial}{\partial y}$

$(\frac{\partial}{\partial z})_{S'''} \mapsto \frac{\partial}{\partial z}$

$(\frac{\partial}{\partial t})_{S'''} \mapsto \gamma_u(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})$ (††††)

and, using the energy stress tensor;

$(cg_1)_{S'''} \mapsto (-\frac{u\gamma_u^2}{c})\sigma + (c\gamma_u^2 + \frac{u^2\gamma_u^2}{c})g_1 - (\frac{u\gamma_u^2}{c})p_{11}$

$(cg_2)_{S'''} \mapsto \gamma_u cg_2 - \frac{\gamma_u u}{c}p_{21}$

$(cg_3)_{S'''} \mapsto \gamma_u cg_3 - \frac{\gamma_u u}{c}p_{31}$

$(p_{11})_{S'''} \mapsto \gamma_u^2(\frac{u^2\sigma}{c^2} - 2ug_1 + p_{11})$

$(p_{12})_{S'''} \mapsto \gamma_u(-ug_2 + p_{12})$

$(p_{13})_{S'''} \mapsto \gamma_u(-ug_2 + p_{13})$

$(p_{22})_{S'''} \mapsto p_{22}$

$(p_{23})_{S'''} \mapsto p_{23}$

$(p_{33})_{S'''} \mapsto p_{33}$ (†††††)

Using the relations (†††), (††††), (†††††), we obtain the following relation in S'''';

$\frac{w}{\epsilon_0 u}\gamma_u(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})\frac{1}{c}((-\frac{u\gamma_u^2}{c})\sigma + (c\gamma_u^2 + \frac{u^2\gamma_u^2}{c})g_1 - (\frac{u\gamma_u^2}{c})p_{11})$

$+\frac{\alpha}{\epsilon_0 u}\gamma_u(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})\frac{1}{c}(\gamma_u cg_2 - \frac{\gamma_u u}{c}p_{21})$

$+\frac{\beta}{\epsilon_0 u}\gamma_u(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})\frac{1}{c}(\gamma_u cg_3 - \frac{\gamma_u u}{c}p_{31})$

$= -\frac{1}{\epsilon_0}[\frac{v_1u_1\gamma_u}{u}(\frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t}) + \gamma\frac{\partial}{\partial y} + \delta\frac{\partial}{\partial z}](\frac{wv_1u_1}{u^2}\gamma_u^2(\frac{u^2\sigma}{c^2} - 2ug_1 + p_{11}) + (\frac{w\gamma}{u} + \frac{v_1u_1\alpha}{u})\gamma_u(-ug_2 + p_{12}) + (\frac{w\delta}{u} + \frac{v_1u_1\beta}{u})\gamma_u(-ug_2 + p_{13}) + \alpha\gamma p_{22} + (\alpha\delta + \beta\gamma)p_{23} + \beta\delta p_{33})$

$-\frac{1}{\epsilon_0}[(\alpha\delta - \beta\gamma)\gamma_u(\frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t}) + (\frac{\beta v_1u_1}{u} - \frac{\delta w}{u})\frac{\partial}{\partial y} + (\frac{\gamma w}{u} - \frac{\alpha v_1u_1}{u})\frac{\partial}{\partial z}]$

$(\frac{w}{u}(\alpha\delta - \beta\gamma)\gamma_u^2(\frac{u^2\sigma}{c^2} - 2ug_1 + p_{11}) + (\frac{w}{u}(\frac{\beta v_1u_1}{u} - \frac{\delta w}{u}) + \alpha(\alpha\delta - \beta\gamma))\gamma_u(-ug_2 + p_{12})$

$+(\frac{w}{u}(\frac{\gamma w}{u} - \frac{\alpha v_1u_1}{u}) + \beta(\alpha\delta - \beta\gamma))\gamma_u(-ug_3 + p_{13}) + (\alpha(\frac{\beta v_1u_1}{u} - \frac{\delta w}{u}))p_{22}$

$+(\alpha(\frac{\gamma w}{u} - \frac{\alpha v_1u_1}{u}) + \beta(\frac{\beta v_1u_1}{u} - \frac{\delta w}{u}))p_{23} + \beta(\frac{\gamma w}{u} - \frac{\alpha v_1u_1}{u})p_{33})$ $(*****)$

Let;

$\lambda_{ij} = (\theta_i\theta'_j + \theta'_i\theta_j)$ for $1 \leq i < j \leq 3$

$\mu_{ij} = (\theta_i\theta''_j + \theta''_i\theta_j)$ for $1 \leq i < j \leq 3$

$\nu_{ij} = (\theta'_i\theta'_j + \theta''_i\theta''_j)$ for $1 \leq i \leq j \leq 3$

Then, using Lemma 1.20, we can rearrange $(*****)$, to obtain;

$$\frac{-w^3\gamma_u^3}{\epsilon_0 uc^2}\frac{\partial\sigma}{\partial x} + \frac{w\gamma_u^2 u\nu_{12}}{\epsilon_0 c^2}\frac{\partial\sigma}{\partial y} + \frac{w\gamma_u^2 u\nu_{13}}{\epsilon_0 c^2}\frac{\partial\sigma}{\partial z} + \frac{w\gamma_u^3(-1+\frac{1}{c^2}\frac{u_1^2}{u^2})}{\epsilon_0 c^2}\frac{\partial\sigma}{\partial t}$$

$$+\frac{w\gamma_u^3(u^2+\frac{2w^2c^2}{u^2}-c^2)}{\epsilon_0 c^2}\frac{\partial g_1}{\partial x} - \frac{2w\gamma_u^2\nu_{12}}{\epsilon_0}\frac{\partial g_1}{\partial y} - \frac{2w\gamma_u^2\nu_{13}}{\epsilon_0}\frac{\partial g_1}{\partial z} + \frac{w\gamma_u^3(2w^2-u^2+c^2)}{\epsilon_0 uc^2}\frac{\partial g_1}{\partial t}$$

$$+\frac{u\gamma_u^2(\theta_2-\theta'_1\lambda_{12}-\theta''_1\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial x} + \frac{u\gamma_u(-\theta'_2\lambda_{12}-\theta''_2\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial y} + \frac{u\gamma_u(-\theta'_3\lambda_{12}-\theta''_3\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial z}$$

$$+\frac{\gamma_u^2(\theta_2-\frac{u^2}{c^2}(\theta'_1\lambda_{12}+\theta''_1\mu_{12}))}{\epsilon_0}\frac{\partial g_2}{\partial t} + \frac{u\gamma_u^2(\theta_3-\theta'_1\lambda_{13}-\theta''_1\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial x} + \frac{u\gamma_u(-\theta'_2\lambda_{13}-\theta''_2\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial y}$$

$$+\frac{u\gamma_u(-\theta'_3\lambda_{13}-\theta''_3\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial z} + \frac{\gamma_u^2(\theta_3-\frac{u^2}{c^2}(\theta'_1\lambda_{13}+\theta''_1\mu_{13}))}{\epsilon_0}\frac{\partial g_3}{\partial t} + \frac{\gamma_u^3 w(1-\frac{w^2}{c^2}-\frac{u^2}{c^2})}{\epsilon_0 u}\frac{\partial p_{11}}{\partial x}$$

$$+\frac{\gamma_u^2 w\nu_{12}}{\epsilon_0 u}\frac{\partial p_{11}}{\partial y} + \frac{\gamma_u^2 w\nu_{13}}{\epsilon_0 u}\frac{\partial p_{11}}{\partial z} - \frac{\gamma_u^3 w^3}{\epsilon_0 c^2u^2}\frac{\partial p_{11}}{\partial t} + \frac{\gamma_u^2(-\theta_2u^2+c^2\theta'_1\lambda_{12}+c^2\theta''_1\mu_{12})}{c^2\epsilon_0}\frac{\partial p_{12}}{\partial x}$$

$$+\frac{\gamma_u(\theta'_2\lambda_{12}+\theta''_2\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial y} + \frac{\gamma_u(\theta'_3\lambda_{12}+\theta''_3\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial z} + \frac{\gamma_u^2 u(-\theta_2+\theta'_1\lambda_{12}+\theta''_1\mu_{12})}{c^2\epsilon_0}\frac{\partial p_{12}}{\partial t}$$

$$+\frac{\gamma_u^2(-\theta_3u^2+c^2\theta'_1\lambda_{13}+c^2\theta''_1\mu_{13})}{c^2\epsilon_0}\frac{\partial p_{13}}{\partial x} + \frac{\gamma_u(\theta'_2\lambda_{13}+\theta''_2\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial y} + \frac{\gamma_u(\theta'_3\lambda_{13}+\theta''_3\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial z}$$

$$+\frac{\gamma_u^2 u(-\theta_3+\theta'_1\lambda_{13}+\theta''_1\mu_{13})}{c^2\epsilon_0}\frac{\partial p_{13}}{\partial t} + \frac{\theta_2\gamma_u\nu_{12}}{\epsilon_0}\frac{\partial p_{22}}{\partial x} + \frac{\theta_2\nu_{22}}{\epsilon_0}\frac{\partial p_{22}}{\partial y} + \frac{\theta_2\nu_{23}}{\epsilon_0}\frac{\partial p_{22}}{\partial z} + \frac{\theta_2 u\gamma_u\nu_{12}}{\epsilon_0 c^2}\frac{\partial p_{22}}{\partial t}$$

$$+\frac{\gamma_u(\theta'_1\lambda_{23}+\theta''_1\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial x} + \frac{(\theta'_2\lambda_{23}+\theta''_2\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial y} + \frac{(\theta'_3\lambda_{23}+\theta''_3\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial z} + \frac{\gamma_u^2 u(\theta'_1\lambda_{23}+\theta''_1\mu_{23})}{c^2\epsilon_0}\frac{\partial p_{23}}{\partial t}$$

$$+\frac{\theta_3\gamma_u\nu_{13}}{\epsilon_0}\frac{\partial p_{33}}{\partial x} + \frac{\theta_3\nu_{23}}{\epsilon_0}\frac{\partial p_{33}}{\partial y} + \frac{\theta_3\nu_{33}}{\epsilon_0}\frac{\partial p_{33}}{\partial z} + \frac{\theta_3 u\gamma_u}{\epsilon_0 c^2}\frac{\partial p_{33}}{\partial t} = 0 \ (******)$$

□

Lemma 1.20. *With notation as in the previous Lemma 1.19, we have that;*

$$(\alpha\delta - \beta\gamma)^2 + \frac{(v_1u_1)^2}{u^2} + \frac{w^2}{u^2} = 1$$

Proof. By the fact that $|\theta''| = 1$, and the definition of $\theta \times \theta'$, we have that;

$(\alpha\delta-\beta\gamma)^2+(\frac{v_1u_1\beta}{u}-\frac{w\delta}{u})^2+(\frac{w\gamma}{u}-\frac{\alpha v_1u_1}{u})^2$

$=(\alpha\delta-\beta\gamma)^2+\frac{(v_1u_1)^2}{u^2}(1-\frac{w^2}{u^2})+\frac{w^2}{u^2}(1-\frac{u_1{v_2}^2}{u^2})-2\frac{w}{u^2}(u_1v_1)(\alpha\gamma+\beta\delta)=1$ $(*)$

As θ and θ' are orthogonal, we have that $\alpha\gamma+\beta\delta+\frac{wv_1u_1}{u^2}=0$. Substituting into $(*)$, we obtain that;

$(\alpha\delta-\beta\gamma)^2+(v_1u_1)^2(1-\frac{w^2}{u^2})+\frac{w^2}{u^2}(1-\frac{u_1{v_2}^2}{u^2})-2\frac{w}{u}(u_1v_1)(-\frac{wv_1u_1}{u^2})$

$=(\alpha\delta-\beta\gamma)^2+\frac{(v_1u_1)^2}{u^2}+\frac{w^2}{u^2}=1$

$\square$

Lemma 1.21. *With notation as in Lemma 1.18, for the equations* $(*****\!*)$*, we must have that;*

$\frac{\partial\sigma}{\partial y}=\frac{\partial\sigma}{\partial z}=\frac{\partial g_1}{\partial x}=\frac{\partial g_2}{\partial y}=\frac{\partial g_2}{\partial z}=\frac{\partial g_2}{\partial t}=\frac{\partial g_3}{\partial y}=\frac{\partial g_3}{\partial z}=\frac{\partial g_3}{\partial t}=\frac{\partial p_{12}}{\partial x}=\frac{\partial p_{13}}{\partial x}$

$=\frac{\partial p_{22}}{\partial y}=\frac{\partial p_{22}}{\partial z}=\frac{\partial p_{22}}{\partial t}=\frac{\partial p_{23}}{\partial y}=\frac{\partial p_{23}}{\partial z}=\frac{\partial p_{33}}{\partial y}=\frac{\partial p_{33}}{\partial z}=\frac{\partial p_{33}}{\partial t}=0$

Proof. Note that as $u\rightarrow\infty$, $\gamma_u\sim\frac{c}{iu}$. We let $\frac{u_2}{u_1}=\kappa$ and $\frac{u_3}{u_1}=\lambda$, with $u_1\neq 0$, so that $w=su$, with $s=(\frac{1+\kappa^2}{1+\kappa^2+\lambda^2})^{\frac{1}{2}}$. The equations $(******)$ in Lemma 1.18 hold for all u with $0<u<c$, and are algebraic, so letting $u\rightarrow\infty$, we obtain that;

$\frac{-\nu_{12}s}{\epsilon_0}\frac{\partial\sigma}{\partial y}-\frac{\nu_{13}s}{\epsilon_0}\frac{\partial\sigma}{\partial z}+\frac{sic}{\epsilon_0}\frac{\partial g_1}{\partial x}-\frac{ic(-\theta_2'\lambda_{12}-\theta_2''\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial y}-\frac{ic(-\theta_3'\lambda_{12}-\theta_3''\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial z}$

$+\frac{(\theta_1'\lambda_{12}+\theta_1''\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial t}-\frac{ic(-\theta_2'\lambda_{13}-\theta_2''\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial y}-\frac{ic(-\theta_3'\lambda_{13}-\theta_3''\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial z}+\frac{(\theta_1'\lambda_{13}+\theta_1''\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial t}$

$+\frac{\theta_2}{\epsilon_0}\frac{\partial p_{12}}{\partial x}+\frac{\theta_3}{\epsilon_0}\frac{\partial p_{13}}{\partial x}+\frac{\theta_2\nu_{22}}{\epsilon_0}\frac{\partial p_{22}}{\partial y}+\frac{\theta_2\nu_{23}}{\epsilon_0}\frac{\partial p_{22}}{\partial z}-\frac{i\theta_2\nu_{12}}{\epsilon_0 c}\frac{\partial p_{22}}{\partial t}+\frac{(\theta_2'\lambda_{23}+\theta_2''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial y}$

$+\frac{(\theta_3'\lambda_{23}+\theta_3''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial z}+\frac{\theta_3\nu_{23}}{\epsilon_0}\frac{\partial p_{33}}{\partial y}+\frac{\theta_3\nu_{33}}{\epsilon_0}\frac{\partial p_{33}}{\partial z}-\frac{i}{\epsilon_0 c}\frac{\partial p_{33}}{\partial t}=0$ $(*******)$

where;

$\theta_1=s$, $\theta_2=\alpha$, $\theta_3=\beta$

$\theta_1'=\frac{sv_1}{1+\kappa^2}$, $\theta_2'=\gamma$, $\theta_3'=\delta$

$\theta_1''=\alpha\delta-\beta\gamma$, $\theta_2''=\frac{s\beta v_1}{\sqrt{1+\kappa^2}}-s\delta$, $\theta_3''=s\gamma-\frac{s\alpha v_1}{\sqrt{1+\kappa^2}}$

$\lambda_{12} = s\gamma + \frac{s\alpha v_1}{\sqrt{1+\kappa^2}}$

$\lambda_{13} = s\delta + \frac{s\beta v_1}{\sqrt{1+\kappa^2}}$

$\lambda_{23} = \alpha\delta + \beta\gamma$

$\mu_{12} = s(\frac{s\beta v_1}{\sqrt{1+\kappa^2}} - s\delta) + \alpha(\alpha\delta - \beta\gamma)$

$\mu_{13} = s(s\gamma - \frac{s\alpha v_1}{\sqrt{1+\kappa^2}}) + \beta(\alpha\delta - \beta\gamma)$

$\mu_{23} = \alpha(s\gamma - \frac{s\alpha v_1}{\sqrt{1+\kappa^2}}) + \beta(\frac{s\beta v_1}{\sqrt{1+\kappa^2}} - s\delta)$

$\nu_{11} = \frac{v_1^2 s^2}{1+\kappa^2} + (\alpha\delta - \beta\gamma)^2$

$\nu_{12} = \frac{s\gamma v_1}{\sqrt{1+\kappa^2}} + (\alpha\delta - \beta\gamma)(\frac{s\beta v_1}{\sqrt{1+\kappa^2}} - s\delta)$

$\nu_{13} = \frac{s\delta v_1}{\sqrt{1+\kappa^2}} + (\alpha\delta - \beta\gamma)(s\gamma - \frac{s\alpha v_1}{\sqrt{1+\kappa^2}})$

$\nu_{22} = \gamma^2 + (\frac{s\beta v_1}{\sqrt{1+\kappa^2}} - s\delta)^2$

$\nu_{23} = \gamma\delta + (\frac{s\beta v_1}{\sqrt{1+\kappa^2}} - s\delta)(s\gamma - \frac{s\alpha v_1}{\sqrt{1+\kappa^2}})$

$\nu_{33} = \delta^2 + (s\gamma - \frac{s\alpha v_1}{\sqrt{1+\kappa^2}})^2$

and, by the orthonormality relations between $\overline{\theta}$ and $\overline{\theta}'$;

$s^2 + \alpha^2 + \beta^2 = 1$

$\frac{v_1^2}{1+\kappa^2} + \gamma^2 + \delta^2 = 1$

$\frac{s v_1}{\sqrt{1+\kappa^2}} + \alpha\gamma + \beta\delta = 0$

$|v_1| \leq 1$ $(\dagger')$

Now, take $\kappa = 0$, $v_1 = 0$, $0 \leq s \leq 1$, $\tau = (1-s^2)^{\frac{1}{2}}$, $\alpha = \tau cos(\theta)$, $\beta = \tau sin(\theta)$, $\gamma = -sin(\theta)$, $\delta = cos(\theta)$, $0 \leq \theta < 2\pi$, then it is easily verified that the conditions of $(\dagger')$ are satisfied. Substituting into the equations $(*******)$, and taking the power series expansions of the functions involving θ, we can equate coefficients in $\{1, \theta, \theta^2\}$ respectively, to obtain the following set of equations;

$(i)\ \frac{s^2\tau}{\epsilon_0}\frac{\partial\sigma}{\partial y}+\frac{sic}{\epsilon_0}\frac{\partial g_1}{\partial x}-\frac{sic(\tau^2-s^2)}{\epsilon_0}\frac{\partial g_2}{\partial y}+\frac{\tau(\tau^2-s^2)}{\epsilon_0}\frac{\partial g_2}{\partial t}+\frac{sic}{\epsilon_0}\frac{\partial g_3}{\partial z}+\frac{\tau}{\epsilon_0}\frac{\partial p_{12}}{\partial x}+\frac{s^2\tau}{\epsilon_0}\frac{\partial p_{22}}{\partial y}$

$+\frac{is\tau^2}{c\epsilon_0}\frac{\partial p_{22}}{\partial t}+\frac{\tau}{\epsilon_0}\frac{\partial p_{23}}{\partial z}-\frac{i}{c\epsilon_0}\frac{\partial p_{33}}{\partial t}=0$

$(ii)\ \frac{s^2\tau}{\epsilon_0}\frac{\partial\sigma}{\partial z}-\frac{ics(1+(\tau^2-s^2))}{\epsilon_0}\frac{\partial g_2}{\partial z}-\frac{ics(1+(\tau^2-s^2))}{\epsilon_0}\frac{\partial g_3}{\partial y}+\frac{\tau(\tau^2-s^2)}{\epsilon_0}\frac{\partial g_3}{\partial t}+\frac{\tau}{\epsilon_0}\frac{\partial p_{13}}{\partial x}$

$+\frac{\tau(s^2-1)}{\epsilon_0}\frac{\partial p_{22}}{\partial z}+\frac{\tau(2s^2-1)}{\epsilon_0}\frac{\partial p_{23}}{\partial y}+\frac{\tau}{\epsilon_0}\frac{\partial p_{33}}{\partial z}=0$

$(iii)\ \frac{-s^2\tau}{2\epsilon_0}\frac{\partial\sigma}{\partial y}+\frac{sic(1+(\tau^2-s^2))}{\epsilon_0}\frac{\partial g_2}{\partial y}-\frac{\tau(\tau^2-s^2)}{2\epsilon_0}\frac{\partial g_2}{\partial t}-\frac{sic(1+(\tau^2-s^2))}{\epsilon_0}\frac{\partial g_3}{\partial z}-\frac{\tau}{2\epsilon_0}\frac{\partial p_{12}}{\partial x}$

$+\frac{2\tau-3\tau s^2}{2\epsilon_0}\frac{\partial p_{22}}{\partial y}-\frac{si\tau^2}{c\epsilon_0}\frac{\partial p_{22}}{\partial t}+\frac{\tau(4s^2-1)}{2\epsilon_0}\frac{\partial p_{23}}{\partial z}+\frac{\tau(s^2-1)}{\epsilon_0}\frac{\partial p_{33}}{\partial y}=0$

Using $\tau^2=1-s^2$, we can simplify (i) to obtain;

$\frac{s^2\tau}{\epsilon_0}(\frac{\partial\sigma}{\partial y}+\frac{\partial p_{22}}{\partial y})+\frac{sic}{\epsilon_0}(\frac{\partial g_1}{\partial x}+\frac{\partial g_3}{\partial z})-\frac{sic(1-2s^2)}{\epsilon_0}\frac{\partial g_2}{\partial y}+\frac{\tau(1-2s^2)}{\epsilon_0}\frac{\partial g_2}{\partial t}$

$+\frac{\tau}{\epsilon_0}(\frac{\partial p_{12}}{\partial x}+\frac{\partial p_{23}}{\partial z})+\frac{is(1-s^2)}{c\epsilon_0}\frac{\partial p_{22}}{\partial t}-\frac{i}{c\epsilon_0}\frac{\partial p_{33}}{\partial t}=0\ (*)$

We can write $(*)$ in the form $\sum_{1=1}^{7}\lambda_i\mu_i=0$, where;

$\mu_1=(\frac{\partial\sigma}{\partial y}+\frac{\partial p_{22}}{\partial y})$, $\mu_2=\frac{\partial g_1}{\partial x}+\frac{\partial g_3}{\partial z}$, $\mu_3=\frac{\partial g_2}{\partial y}$, $\mu_4=\frac{\partial g_2}{\partial t}$, $\mu_5=\frac{\partial p_{12}}{\partial x}+\frac{\partial p_{23}}{\partial z}$

$\mu_6=\frac{\partial p_{22}}{\partial t}$, $\mu_7=\frac{\partial p_{33}}{\partial t}$

Using Newton's expansion of;

$\tau=(1-s^2)^{\frac{1}{2}}=1-\frac{s^2}{2}-\frac{s^4}{8}-\frac{s^6}{16}+O(s^8)$

and equating coefficients up to s^6 to zero, we obtain the following equations;

$a(1).\ \mu_4+\mu_5-\frac{i}{c}\mu_7=0$

$b(s).\ ic(\mu_2+\mu_3)+\frac{i}{c}\mu_6=0$

$c(s^2).\ \mu_1-\frac{5}{2}\mu_4-\mu_5=0$

$d(s^3).\ -2ic\mu_3-\frac{i}{c}\mu_6=0$

$e(s^4).\ \frac{-1}{2}\mu_1+\frac{7}{8}\mu_4-\frac{1}{8}\mu_5=0$

$f(s^6)$. $\frac{-1}{8}\mu_1+\frac{3}{16}\mu_4=0$

Using c,e,f, and solving the three simultaneous equations, we obtain that $\mu_1=\mu_4=\mu_5=0$. From a, we then obtain that $\mu_7=0$. Using b,d and eliminating μ_6, we obtain $\mu_2=\mu_3$ and $\mu_6=-2c^2\mu_3$, so we obtain;

$$\frac{\partial\sigma}{\partial y}=-\frac{\partial p_{22}}{\partial y},\ \frac{\partial p_{12}}{\partial x}=-\frac{\partial p_{23}}{\partial z},\ \frac{\partial p_{22}}{\partial t}=-2c^2\frac{\partial g_2}{\partial y}$$

$$\frac{\partial g_2}{\partial t}=\frac{\partial p_{33}}{\partial t}=0$$

$$\frac{\partial g_1}{\partial x}+\frac{\partial g_3}{\partial z}=\frac{\partial g_2}{\partial y}\ (\sharp)$$

Using $\tau^2=1-s^2$ again, we can simplify (ii) to obtain;

$$(ii)\ \frac{s^2\tau}{\epsilon_0}\frac{\partial\sigma}{\partial z}-\frac{ics(2-2s^2))}{\epsilon_0}\left(\frac{\partial g_2}{\partial z}+\frac{\partial g_3}{\partial y}\right)+\frac{\tau(1-2s^2)}{\epsilon_0}\left(\frac{\partial g_3}{\partial t}-\frac{\partial p_{23}}{\partial y}\right)+\frac{\tau}{\epsilon_0}\left(\frac{\partial p_{13}}{\partial x}-\frac{\partial p_{33}}{\partial z}\right)$$

$$+\frac{\tau(s^2-1)}{\epsilon_0}\frac{\partial p_{22}}{\partial z}=0\ (\dagger)$$

We can write $(\dagger)$ in the form $\sum_{i=1}^5\lambda_i\mu_i=0$, $(\dagger\dagger)$ where;

$$\mu_1=\frac{\partial\sigma}{\partial z},\ \mu_2=\frac{\partial g_2}{\partial z}+\frac{\partial g_3}{\partial y},\ \mu_3=\frac{\partial g_3}{\partial t}-\frac{\partial p_{23}}{\partial y},\ \mu_4=\frac{\partial p_{13}}{\partial x}-\frac{\partial p_{33}}{\partial z},\ \mu_5=\frac{\partial p_{22}}{\partial z}$$

where $\lambda_1+\lambda_3+\lambda_5=0$ and $\lambda_1-\lambda_4-\lambda_5=0$. Using the first relation in $(\sharp)$, we can write $(\dagger\dagger)$ in the form;

$$\lambda_1(\mu_1-\mu_5)+\lambda_2\mu_2+\lambda_3(\mu_3-\mu_5)+\lambda_4\mu_4=0,\ (\dagger\dagger\dagger)$$

Using the second relation in $(\sharp)$, we have that $2\lambda_1=\lambda_4-\lambda_3$, so that multiplying $(\dagger\dagger\dagger)$ by 2, and substituting, we obtain the relation;

$$2\lambda_2\mu_2+\lambda_3(2(\mu_3-\mu_5)-(\mu_1-\mu_5))+\lambda_4(2\mu_4+(\mu_1-\mu_5))=0\ (\dagger\dagger\dagger\dagger)$$

By taking a power series expansion of $(1-s^2)^{\frac{1}{2}}$ or otherwise, it is easily checked that varying the coefficients $(2\lambda_2,\lambda_3,\lambda_4)$, with $0\leq s\leq 1$, gives that;

$$\mu_2=2(\mu_3-\mu_5)-(\mu_1-\mu_5)=2\mu_4+(\mu_1-\mu_5)=0$$

so we obtain;

$\frac{\partial g_2}{\partial z} = -\frac{\partial g_3}{\partial y}$

$\frac{\partial \sigma}{\partial z} = 2(\frac{\partial g_3}{\partial t} - \frac{\partial p_{23}}{\partial y}) - \frac{\partial p_{22}}{\partial z} = -(2(\frac{\partial p_{13}}{\partial x} - \frac{\partial p_{33}}{\partial z}) - \frac{\partial p_{22}}{\partial z})$

$\frac{\partial p_{22}}{\partial z} = (\frac{\partial g_3}{\partial t} - \frac{\partial p_{23}}{\partial y}) + (\frac{\partial p_{13}}{\partial x} - \frac{\partial p_{33}}{\partial z})$

$\frac{\partial \sigma}{\partial z} = (\frac{\partial g_3}{\partial t} - \frac{\partial p_{23}}{\partial y}) - (\frac{\partial p_{13}}{\partial x} - \frac{\partial p_{33}}{\partial z})$ $(\sharp\sharp)$

Using $\tau^2 = 1 - s^2$ again, we can simplify (iii) to obtain;

$\frac{-s^2\tau}{2\epsilon_0}\frac{\partial \sigma}{\partial y} + \frac{sic(2-2s^2))}{\epsilon_0}(\frac{\partial g_2}{\partial y} - \frac{\partial g_3}{\partial z}) - \frac{\tau(1-2s^2)}{2\epsilon_0}\frac{\partial g_2}{\partial t} - \frac{\tau}{2\epsilon_0}\frac{\partial p_{12}}{\partial x}$

$+\frac{2\tau-3\tau s^2}{2\epsilon_0}\frac{\partial p_{22}}{\partial y} - \frac{si(1-s^2)}{c\epsilon_0}\frac{\partial p_{22}}{\partial t} + \frac{\tau(4s^2-1)}{2\epsilon_0}\frac{\partial p_{23}}{\partial z} + \frac{\tau(s^2-1)}{\epsilon_0}\frac{\partial p_{33}}{\partial y} = 0$ $(\dagger\dagger\dagger')$

We can write $(\dagger\dagger\dagger')$ in the form $\sum_{i=1}^{8}\lambda_i\mu_i = 0$, $(\dagger\dagger\dagger\dagger)$ where;

$\mu_1 = \frac{\partial \sigma}{\partial y}$, $\mu_2 = \frac{\partial g_2}{\partial y} - \frac{\partial g_3}{\partial z}$, $\mu_3 = \frac{\partial g_2}{\partial t}$, $\mu_4 = \frac{\partial p_{12}}{\partial x}$, $\mu_5 = \frac{\partial p_{22}}{\partial y}$, $\mu_6 = \frac{\partial p_{22}}{\partial t}$

$\mu_7 = \frac{\partial p_{23}}{\partial z}$, $\mu_8 = \frac{\partial p_{23}}{\partial y}$

where $4\lambda_1 - 4\lambda_4 + 2\lambda_8 = 0$, $6\lambda_1 - 4\lambda_4 - 2\lambda_5 = 0$, $4\lambda_1 - \lambda_4 + \lambda_7 = 0$ and $2\lambda_2 + \lambda_3 - \lambda_4 = 0$. Using the first relation of $(\sharp\sharp)$, we can write $(\dagger\dagger\dagger\dagger)$ in the form;

$\lambda_1(\mu_1 - 2\mu_8) + \sum_{i=2}^{3}\lambda_i\mu_i + \lambda_4(\mu_4 + 2\mu_8) + \sum_{i=5}^{7}\lambda_i\mu_i = 0$

Then, using the second relation of $(\sharp\sharp)$, we can write $(\dagger\dagger\dagger\dagger)$ in the form;

$\lambda_1(\mu_1 + 3\mu_5 - 2\mu_8) + \sum_{i=2}^{3}\lambda_i\mu_i + \lambda_4(\mu_4 - 2\mu_5 + 2\mu_8) + \sum_{i=6}^{7}\lambda_i\mu_i = 0$

Using the third relation of $(\sharp\sharp)$, we obtain;

$\lambda_1(\mu_1 + 3\mu_5 - 4\mu_7 - 2\mu_8) + \sum_{i=2}^{3}\lambda_i\mu_i + \lambda_4(\mu_4 - 2\mu_5 + \mu_7 + \mu_7 + \mu_8)$

$+\lambda_6\mu_6 = 0$

and, using the fourth relation of $(\sharp\sharp)$, we obtain;

$\lambda_1(\mu_1 - 2\mu_3 + 3\mu_5 - 4\mu_7 - 2\mu_8) + \lambda_2\mu_2 + \lambda_4(\mu_3 + \mu_4 - 2\mu_5 + \mu_7$

$+\mu_8)+\lambda_6\mu_6=0$

Using Newton's expansion of τ again, and equating coefficients up to s^2, we obtain the equations;

$a(1).\ -\frac{\delta_4}{2}=0.$

$b(s).\ 2ic\delta_2-\frac{i}{c}\delta_6=0$

$c(s^2).\ -\frac{\delta_1}{2}+\frac{\delta_4}{4}=0$

where $\delta_1=\mu_1-2\mu_3+3\mu_5-4\mu_7-2\mu_8$, $\delta_2=\mu_2$, $\delta_4=\mu_3+\mu_4-2\mu_5+\mu_7+\mu_8$, $\delta_6=\mu_6$

From b, we obtain that $\mu_2=\frac{1}{2c^2}\mu_6$, so that;

$\frac{\partial g_2}{\partial y}-\frac{\partial g_3}{\partial z}=\frac{1}{2c^2}\frac{\partial p_{22}}{\partial t}$, $(****)$

From the rearrangement of (i), we had that $\frac{\partial p_{22}}{\partial t}=-2c^2\frac{\partial g_2}{\partial y}$, so that, using $(****)$, we obtain;

$\frac{\partial g_2}{\partial y}-\frac{\partial g_3}{\partial z}=-\frac{\partial g_2}{\partial y}$ and $2\frac{\partial g_2}{\partial y}=\frac{\partial g_3}{\partial z}$ (!)

It follows by symmetry that $2\frac{\partial g_3}{\partial z}=\frac{\partial g_2}{\partial y}$, so that by (!), $4\frac{\partial g_2}{\partial y}=\frac{\partial g_2}{\partial y}$ and, using $(****)$, $\frac{\partial g_2}{\partial y}=\frac{\partial g_3}{\partial z}=\frac{\partial p_{22}}{\partial t}=0$. It follows from the second equation in $(\sharp)$ that $\frac{\partial g_1}{\partial x}=0$ as well, (E).

From a,c, we obtain that $\delta_1=\delta_4=0$, so that $\mu_1-2\mu_3+3\mu_5-4\mu_7-2\mu_8=0$ and $\mu_3+\mu_4-2\mu_5+\mu_7+\mu_8=0$. It follows that;

$\frac{\partial\sigma}{\partial y}-2\frac{\partial g_2}{\partial t}+3\frac{\partial p_{22}}{\partial y}-4\frac{\partial p_{23}}{\partial z}-2\frac{\partial p_{23}}{\partial y}=0$

$\frac{\partial g_2}{\partial t}+\frac{\partial p_{12}}{\partial x}-2\frac{\partial p_{22}}{\partial y}+\frac{\partial p_{23}}{\partial z}+\frac{\partial p_{23}}{\partial y}=0$ (!!!!)

From the third and fourth equations in $(\sharp\sharp)$, we have;

$\frac{\partial g_3}{\partial t}=\frac{\partial p_{22}}{\partial z}+\frac{\partial p_{22}}{\partial y}-\frac{\partial p_{13}}{\partial x}+\frac{\partial p_{33}}{\partial z}$

$\frac{\partial\sigma}{\partial z}=\frac{\partial g_3}{\partial t}-\frac{\partial p_{23}}{\partial y}-\frac{\partial p_{13}}{\partial x}+\frac{\partial p_{33}}{\partial z}$

$= \frac{\partial p_{22}}{\partial z} + \frac{\partial p_{22}}{\partial y} - \frac{\partial p_{13}}{\partial x} + \frac{\partial p_{33}}{\partial z} - \frac{\partial p_{23}}{\partial y} - \frac{\partial p_{13}}{\partial x} + \frac{\partial p_{33}}{\partial z}$

$= -2\frac{\partial p_{13}}{\partial x} + \frac{\partial p_{22}}{\partial y} + \frac{\partial p_{22}}{\partial z} - \frac{\partial p_{23}}{\partial y} + 2\frac{\partial p_{33}}{\partial z}$, (!!!)

and, from (!!!!), second equation of ($\sharp$), we have that;

$\frac{\partial g_2}{\partial t} = -\frac{\partial p_{12}}{\partial x} + 2\frac{\partial p_{22}}{\partial y} - \frac{\partial p_{23}}{\partial y} - \frac{\partial p_{23}}{\partial z} = 0$

$\frac{\partial \sigma}{\partial y} = 2\frac{\partial g_2}{\partial t} - 3\frac{\partial p_{22}}{\partial y} + 4\frac{\partial p_{23}}{\partial z} + 2\frac{\partial p_{23}}{\partial y}$
$= 2(-\frac{\partial p_{12}}{\partial x} + 2\frac{\partial p_{22}}{\partial y} - \frac{\partial p_{23}}{\partial y} - \frac{\partial p_{23}}{\partial z}) - 3\frac{\partial p_{22}}{\partial y} + 4\frac{\partial p_{23}}{\partial z} + 2\frac{\partial p_{23}}{\partial y}$

$= -2\frac{\partial p_{12}}{\partial x} - 7\frac{\partial p_{22}}{\partial y} + 2\frac{\partial p_{23}}{\partial z}$, (!!!!!)

We also note, from the fact that $\frac{\partial g_2}{\partial t} = 0$ in ($\sharp$), that $\frac{\partial g_3}{\partial t} = 0$ by symmetry. Rewriting the equations ($*******$) of the previous lemma in terms of the stress tensor, using the above relations, we obtain;

$\frac{-\nu_{12}s}{\epsilon_0}(-2\frac{\partial p_{12}}{\partial x} - 7\frac{\partial p_{22}}{\partial y} + 2\frac{\partial p_{23}}{\partial z})$

$-\frac{\nu_{13}s}{\epsilon_0}(-2\frac{\partial p_{13}}{\partial x} + \frac{\partial p_{22}}{\partial y} + \frac{\partial p_{22}}{\partial z} - \frac{\partial p_{23}}{\partial y} + 2\frac{\partial p_{33}}{\partial z})$

$-\frac{ic(-\theta_3'\lambda_{12}-\theta_3''\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial z} - \frac{ic(-\theta_2'\lambda_{13}-\theta_2''\mu_{13})}{\epsilon_0}\frac{-\partial g_2}{\partial z}$

$+\frac{\theta_2}{\epsilon_0}\frac{\partial p_{12}}{\partial x} + \frac{\theta_3}{\epsilon_0}\frac{\partial p_{13}}{\partial x} + \frac{\theta_2\nu_{22}}{\epsilon_0}\frac{\partial p_{22}}{\partial y} + \frac{\theta_2\nu_{23}}{\epsilon_0}\frac{\partial p_{22}}{\partial z} + \frac{(\theta_2'\lambda_{23}+\theta_2''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial y}$

$+\frac{(\theta_3'\lambda_{23}+\theta_3''\mu_{23})}{\epsilon_0}\frac{-\partial p_{12}}{\partial x} + \frac{\theta_3\nu_{23}}{\epsilon_0}\frac{\partial p_{33}}{\partial y} + \frac{\theta_3\nu_{33}}{\epsilon_0}\frac{\partial p_{33}}{\partial z} = 0$ ($********$)

with;

$-\frac{\partial p_{12}}{\partial x} + 2\frac{\partial p_{22}}{\partial y} - \frac{\partial p_{23}}{\partial y} - \frac{\partial p_{23}}{\partial z} = 0$

$-2\frac{\partial p_{12}}{\partial x} - 7\frac{\partial p_{22}}{\partial y} + 2\frac{\partial p_{23}}{\partial z} = -\frac{\partial p_{22}}{\partial y}$ (A)

by (!!!!!) and the fact that $\frac{\partial \sigma}{\partial y} = -\frac{\partial p_{22}}{\partial y}$ in ($\sharp$), and the additional relations;

$-2\frac{\partial p_{13}}{\partial x} + \frac{\partial p_{22}}{\partial y} + \frac{\partial p_{22}}{\partial z} - \frac{\partial p_{23}}{\partial y} + 2\frac{\partial p_{33}}{\partial z} = -\frac{\partial p_{33}}{\partial z}$

$-\frac{\partial p_{13}}{\partial x} + 2\frac{\partial p_{33}}{\partial z} - \frac{\partial p_{23}}{\partial z} - \frac{\partial p_{23}}{\partial y} = 0$ (B)

which we obtain by symmetry, see Lemma 1.26, $\frac{\partial\sigma}{\partial z}=-\frac{\partial p_{33}}{\partial z}$, from $\frac{\partial\sigma}{\partial y}=-\frac{\partial p_{22}}{\partial y}$, and from $-\frac{\partial p_{12}}{\partial x}+2\frac{\partial p_{22}}{\partial y}-\frac{\partial p_{23}}{\partial y}-\frac{\partial p_{23}}{\partial z}=0$.

Eliminating $\{\frac{\partial p_{12}}{\partial x},\frac{\partial p_{13}}{\partial x},\frac{\partial p_{22}}{\partial y},\frac{\partial p_{23}}{\partial z}\}$, from $(A),(B)$, we obtain the equations;

$$\frac{\partial p_{12}}{\partial x}=\frac{1}{12}(\frac{-\partial p_{33}}{\partial z}+\frac{\partial p_{22}}{\partial z}-8\frac{\partial p_{23}}{\partial y})$$

$$\frac{\partial p_{13}}{\partial x}=\frac{1}{12}(19\frac{\partial p_{33}}{\partial z}+5\frac{\partial p_{22}}{\partial z}-4\frac{\partial p_{23}}{\partial y})$$

$$\frac{\partial p_{22}}{\partial y}=\frac{1}{12}(2\frac{\partial p_{33}}{\partial z}-2\frac{\partial p_{22}}{\partial z}+4\frac{\partial p_{23}}{\partial y})$$

$$\frac{\partial p_{23}}{\partial z}=\frac{1}{12}(5\frac{\partial p_{33}}{\partial z}-5\frac{\partial p_{22}}{\partial z}+4\frac{\partial p_{23}}{\partial y})\ (C)$$

Substituting into $(********)$, we obtain;

$$\frac{ic(\theta_3'\lambda_{12}+\theta_3''\mu_{12}-\theta_2'\lambda_{13}-\theta_2''\mu_{13})}{\epsilon_0}\frac{\partial g_2}{\partial z}$$

$$+\frac{\theta_3\nu_{23}}{\epsilon_0}\frac{\partial p_{33}}{\partial y}$$

$$+\frac{(\frac{\nu_{12}s}{6}+\nu_{13}s-\frac{\theta_2}{12}+\frac{19\theta_3}{12}+\frac{\theta_3'\lambda_{23}}{12}+\frac{\theta_3''\mu_{23}}{12}+\frac{\theta_2\nu_{22}}{6}+\theta_3\nu_{33})}{\epsilon_0}\frac{\partial p_{33}}{\partial z}$$

$$+\frac{(-\frac{\nu_{12}s}{6}+\frac{\theta_2}{12}+\frac{5\theta_3}{12}-\frac{\theta_3'\lambda_{23}}{12}-\frac{\theta_3''\mu_{23}}{12}-\frac{\theta_2\nu_{22}}{6}+\theta_2\nu_{23})}{\epsilon_0}\frac{\partial p_{22}}{\partial z}$$

$$+\frac{(\frac{-2\nu_{12}s}{3}-\frac{2\theta_2}{3}-\frac{\theta_3}{3}+\frac{2\theta_3'\lambda_{23}}{3}+\frac{2\theta_3''\mu_{23}}{3}+\frac{\theta_2\nu_{22}}{3}+\theta_2'\lambda_{23}+\theta_2''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial y}=0$$

Taking $\kappa=0$, $v_1=0$, $s=0$, $\tau=(1-s^2)^{\frac{1}{2}}=1$, $\alpha=\tau cos(\theta)$, $\beta=\tau sin(\theta)$, $\gamma=-sin(\theta)$, $\delta=cos(\theta)$, for $\theta\in\{0,\frac{\pi}{4},\frac{3\pi}{4},\frac{\pi}{2}\}$ respectively, noting that the first coefficient in $\frac{\partial g_2}{\partial z}$ is zero, we obtain the coefficient matrix $(A)_{ij}$, for $1\leq i\leq j\leq 4$ for the remaining 4 variables $\{\frac{\partial p_{33}}{\partial y},\frac{\partial p_{33}}{\partial z},\frac{\partial p_{22}}{\partial z},\frac{\partial p_{23}}{\partial y}\}$, where;

$$a_{11}=0,a_{12}=0,a_{13}=0,a_{14}=\frac{2}{3\epsilon_0}$$

$$a_{21}=\frac{-1}{2\sqrt{2}\epsilon_0},a_{22}=\frac{13}{6\sqrt{2}\epsilon_0},a_{23}=\frac{-7}{12\sqrt{2}\epsilon_0},a_{24}=\frac{1}{6\sqrt{2}\epsilon_0}$$

$$a_{31}=\frac{1}{2\sqrt{2}\epsilon_0},a_{32}=\frac{2}{\sqrt{2}\epsilon_0},a_{33}=\frac{5}{12\sqrt{2}\epsilon_0},a_{34}=\frac{-13}{6\sqrt{2}\epsilon_0}$$

$$a_{41}=0,a_{42}=\frac{19}{12\epsilon_0},a_{43}=0,a_{44}=\frac{-1}{\epsilon_0}$$

By a straightforward calculation, we have that $det(A) = \frac{19}{72\epsilon_0^4} \neq 0$, so that;

$\frac{\partial p_{33}}{\partial y} = \frac{\partial p_{33}}{\partial z} = \frac{\partial p_{22}}{\partial z} = \frac{\partial p_{23}}{\partial y} = 0$

By (C), we have that;

$\frac{\partial p_{12}}{\partial x} = \frac{\partial p_{13}}{\partial x} = \frac{\partial p_{22}}{\partial y} = \frac{\partial p_{23}}{\partial z} = 0$

By (!!!!!), we obtain that;

$\frac{\partial g_2}{\partial t} = \frac{\partial \sigma}{\partial y} = 0$

By (!!!), we have;

$\frac{\partial g_3}{\partial t} = \frac{\partial \sigma}{\partial z} = 0$

By (E), we have;

$\frac{\partial g_2}{\partial y} = \frac{\partial g_3}{\partial z} = \frac{\partial g_1}{\partial x} = \frac{\partial p_{22}}{\partial t} = 0$

Substituting these values into $(******)$ and using the relation;

$\frac{\partial g_2}{\partial z} = -\frac{\partial g_3}{\partial y}$ (F)

we obtain;

$\frac{ic(\theta_3'\lambda_{12}+\theta_3''\mu_{12}-\theta_2'\lambda_{13}-\theta_2''\mu_{13})}{\epsilon_0}\frac{\partial g_2}{\partial z} = 0$, (G)

Let $s = \frac{1}{2}$, $\tau = (1-s^2)^{\frac{1}{2}} = \frac{\sqrt{3}}{2}$, $\nu_1 = 1$, $1+\kappa^2 = \frac{5}{3}$, $\rho = (1-\frac{v_1^2}{1+\kappa^2}) = \frac{\sqrt{2}}{\sqrt{5}}$, $\lambda^2 = 3+3\kappa^2$, $\alpha = \tau cos(\frac{\pi}{8})$, $\beta = \tau sin(\frac{\pi}{8})$, $\gamma = \rho cos(\frac{\pi}{8})$, $\delta = -\rho sin(\frac{\pi}{8})$, then $\alpha\delta - \beta\gamma = -\frac{\sqrt{3}}{2\sqrt{5}}$ and $\frac{s\nu_1}{\sqrt{1+\kappa^2}} = \frac{\sqrt{3}}{2\sqrt{5}}$. It is easily verified that the conditions of (†) in the Lemma are met. Calculating the coefficient in (G), we obtain that;

$\frac{-3ic}{80\epsilon_0}\frac{\partial g_2}{\partial z} = 0$

so that, using (F), $\frac{\partial g_2}{\partial z} = \frac{\partial g_3}{\partial y} = 0$. This proves the Lemma.

□

Lemma 1.22. *Using the notation of Lemma 1.19, we have that;*

$$\frac{\partial\sigma}{\partial x}+3\frac{\partial p_{11}}{\partial x}=\frac{\partial g_1}{\partial t}+\frac{\partial p_{11}}{\partial x}=\frac{\partial p_{12}}{\partial t}-c^2\frac{\partial g_2}{\partial x}=\frac{\partial p_{13}}{\partial t}-c^2\frac{\partial g_3}{\partial x}=ic\frac{\partial p_{23}}{\partial x}+\frac{\partial p_{23}}{\partial t}$$

$$=\frac{\partial g_1}{\partial y}=\frac{\partial g_1}{\partial z}=\frac{\partial p_{12}}{\partial y}=\frac{\partial p_{12}}{\partial z}=\frac{\partial p_{13}}{\partial y}=\frac{\partial p_{13}}{\partial z}=\frac{\partial p_{22}}{\partial x}=\frac{\partial p_{33}}{\partial x}=0$$

Proof. Using the result of Lemma 1.21, multiplying the equations $(*****)$ in Lemma 1.19 by u, taking the limit as $u\rightarrow\infty$, and again noting that the equations $(******)$ in Lemma 1.19 hold for all u with $0<u<c$, and are algebraic, we obtain that;

$$\frac{-ics^3}{\epsilon_0}\frac{\partial\sigma}{\partial x}+\frac{2c^2\nu_{12}s}{\epsilon_0}\frac{\partial g_1}{\partial y}+\frac{2c^2\nu_{13}s}{\epsilon_0}\frac{\partial g_1}{\partial z}+\frac{sic(2s^2-1)}{\epsilon_0}\frac{\partial g_1}{\partial t}-\frac{c^2(\theta_2-\theta_1'\lambda_{12}-\theta_1''\mu_{12})}{\epsilon_0}\frac{\partial g_2}{\partial x}$$

$$-\frac{c^2(\theta_3-\theta_1'\lambda_{13}-\theta_1''\mu_{13})}{\epsilon_0}\frac{\partial g_3}{\partial x}-\frac{s(s^2+1)ci}{\epsilon_0}\frac{\partial p_{11}}{\partial x}-\frac{ic(\theta_2'\lambda_{12}+\theta_2''\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial y}-\frac{ic(\theta_3'\lambda_{12}+\theta_3''\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial z}$$

$$-\frac{(-\theta_2+\theta_1'\lambda_{12}+\theta_1''\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial t}-\frac{ic(\theta_2'\lambda_{13}+\theta_2''\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial y}-\frac{ic(\theta_3'\lambda_{13}+\theta_3''\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial z}-\frac{(-\theta_3+\theta_1'\lambda_{13}+\theta_1''\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial t}$$

$$-\frac{ic\theta_2\nu_{12}}{\epsilon_0}\frac{\partial p_{22}}{\partial x}-\frac{ic(\theta_1'\lambda_{23}+\theta_1''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial x}-\frac{(\theta_1'\lambda_{23}+\theta_1''\mu_{23})}{\epsilon_0}\frac{\partial p_{23}}{\partial t}-\frac{ic\theta_3\nu_{13}}{\epsilon_0}\frac{\partial p_{33}}{\partial x}=0$$

Rearranging, we can write this as;

$$\frac{-ics^3}{\epsilon_0}\left(\frac{\partial\sigma}{\partial x}+3\frac{\partial p_{11}}{\partial x}\right)+\frac{2c^2\nu_{12}s}{\epsilon_0}\frac{\partial g_1}{\partial y}+\frac{2c^2\nu_{13}s}{\epsilon_0}\frac{\partial g_1}{\partial z}+\frac{sic(2s^2-1)}{\epsilon_0}\left(\frac{\partial g_1}{\partial t}+\frac{\partial p_{11}}{\partial x}\right)$$

$$-\frac{ic(\theta_2'\lambda_{12}+\theta_2''\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial y}-\frac{ic(\theta_3'\lambda_{12}+\theta_3''\mu_{12})}{\epsilon_0}\frac{\partial p_{12}}{\partial z}-\frac{(-\theta_2+\theta_1'\lambda_{12}+\theta_1''\mu_{12})}{\epsilon_0}\left(\frac{\partial p_{12}}{\partial t}-c^2\frac{\partial g_2}{\partial x}\right)$$

$$-\frac{ic(\theta_2'\lambda_{13}+\theta_2''\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial y}-\frac{ic(\theta_3'\lambda_{13}+\theta_3''\mu_{13})}{\epsilon_0}\frac{\partial p_{13}}{\partial z}-\frac{(-\theta_3+\theta_1'\lambda_{13}+\theta_1''\mu_{13})}{\epsilon_0}\left(\frac{\partial p_{13}}{\partial t}-c^2\frac{\partial g_3}{\partial x}\right)$$

$$-\frac{ic\theta_2\nu_{12}}{\epsilon_0}\frac{\partial p_{22}}{\partial x}-\frac{(\theta_1'\lambda_{23}+\theta_1''\mu_{23})}{\epsilon_0}\left(ic\frac{\partial p_{23}}{\partial x}+\frac{\partial p_{23}}{\partial t}\right)-\frac{ic\theta_3\nu_{13}}{\epsilon_0}\frac{\partial p_{33}}{\partial x}=0\ (*)$$

Now follow the method of the proof in Lemma 1.21 to obtain the result.

□

Lemma 1.23. *Let S be surface non-radiating, for real charge and current $(\rho,\overline{J})$, then there exists a complex solution $(\overline{E},\overline{B})$ to Maxwell's equations with $\overline{E}\times\overline{B}=0$.*

Proof. By Lemma 1.21, there exists a complex solution $(\overline{E},\overline{B})$ to Maxwell's equations, with, in components, $\overline{E}\times\overline{B}=\frac{1}{\epsilon_0}(g_1,g_2,g_3)$, such that;

$$\frac{\partial g_1}{\partial x}=\frac{\partial g_2}{\partial y}=\frac{\partial g_2}{\partial z}=\frac{\partial g_2}{\partial t}=\frac{\partial g_3}{\partial y}=\frac{\partial g_3}{\partial z}=\frac{\partial g_3}{\partial t}=0$$

By Lemma 1.22, we also have that;

$$\frac{\partial g_1}{\partial y}=\frac{\partial g_1}{\partial z}=\frac{\partial p_{12}}{\partial t}-c^2\frac{\partial g_2}{\partial x}=\frac{\partial p_{13}}{\partial t}-c^2\frac{\partial g_3}{\partial x}=\frac{\partial g_1}{\partial t}+\frac{\partial p_{11}}{\partial x}=0$$

It follows that g_1 is independent of $\overline{x}\in\mathcal{R}^3$. By the boundary condition that $lim_{|\overline{x}|\rightarrow\infty}g_1(\overline{x},t)=0$, see Remarks 1.32, for $t\in\mathcal{R}_{\geq 0}$, we have that $g_1=0$. Similarly, g_2 is independent of the coordinates (y,z,t), and we have that;

$$\frac{\partial g_2}{\partial x}=\frac{1}{c^2}\frac{\partial p_{12}}{\partial t}$$

By Lemma 1.21, we have that $\frac{\partial p_{12}}{\partial x}=0$. It follows that $\frac{\partial p_{12}}{\partial t}$ is independent of x, and, therefore, g_2 is constant. Again, using the boundary condition that $lim_{|\overline{x}|\rightarrow\infty}g_2(\overline{x},t)=0$, for $t\in\mathcal{R}_{\geq 0}$, we have that $g_2=0$. Finally, g_3 is again independent of the coordinates (y,z,t), and we have that;

$$\frac{\partial g_3}{\partial x}=\frac{1}{c^2}\frac{\partial p_{13}}{\partial t}$$

Again, by Lemma 1.21, we have that $\frac{\partial p_{13}}{\partial x}=0$. By the same argument, g_3 is constant, and, using the boundary condition, that $g_3=0$. □

Lemma 1.24. *Let $(\overline{E},\overline{B})$ be a complex solution to Maxwell's equations for a real pair $(\rho,\overline{J})$, with $\overline{E}\times\overline{B}=0$, then on the open set $U\subset\mathcal{R}^4$, for which $\overline{B}\neq 0$, we have that $\overline{E}=\lambda\overline{B}$, and $\rho|U=0$.*

Proof. Writing $\overline{E}$ and $\overline{B}$ in components (e_1,e_2,e_3) and (b_1,b_2,b_3), the condition that $\overline{E}\times\overline{B}=0$ amounts to the equations;

$$e_2b_3=e_3b_2\ e_3b_1=e_1b_3\ e_1b_2=e_2b_1$$

Without loss of generality, we can assume that $\{b_1,b_2,b_3\}$ are non-vanishing on U, to obtain that;

$$\frac{e_2}{b_2}=\frac{e_3}{b_3}=\lambda$$

$$\frac{e_1}{b_1}=\frac{e_3}{b_3}=\mu$$

$$\frac{e_1}{b_1}=\frac{e_2}{b_2}=\nu$$

and, clearly then, $\lambda = \mu = \nu$, so that $\overline{E} = \lambda\overline{B}$, $(*)$. We have that the pairs $(Re(\overline{E}), Re(\overline{B}))$ and $(Im(\overline{E}), Im(\overline{B}))$ satisfy Maxwell's equations for the pair $(\rho, \overline{J})$, and in free space. In particularly, $div(Re(\overline{B})) = div(Im(\overline{B})) = 0$, $(**)$. From $(*)$, we have that;

$$Re(\overline{E}) = fRe(\overline{B}) + gIm(\overline{B})$$

and, from $(**)$;

$$\frac{\rho}{\epsilon_0} = div(Re(\overline{E})) = grad(f) \cdot Re(\overline{B}) + grad(g) \cdot Im(\overline{B}) \ (***)$$

Let γ be a level surface for f, with interior U_γ, then by the divergence theorem;

$$\int_{U_\gamma} \frac{\rho}{\epsilon_0} d\overline{x}$$

$$= \int_{U_\gamma} div(Re(\overline{E}))d\overline{x}$$

$$= \int_\gamma Re(\overline{E}) \cdot d\overline{S}$$

$$= \int_\gamma (fRe(\overline{B}) + gIm(\overline{B})) \cdot d\overline{S}$$

$$= f\int_\gamma Re(\overline{B}) \cdot d\overline{S} + \int_\gamma gIm(\overline{B}) \cdot d\overline{S}$$

$$= \int_\gamma gIm(\overline{B}) \cdot d\overline{S}$$

$$= \int_{U_\gamma} div(gIm(\overline{B}))d\overline{x}$$

$$= \int_{U_\gamma} grad(g) \cdot Im(\overline{B})d\overline{x}$$

By continuity of f, we can cover U with open sets of the form U_γ contained within any ball $B(x_0, \epsilon)$ of radius $\epsilon > 0$, so that we conclude;

$$\frac{\rho}{\epsilon_0} = grad(g) \cdot Im(\overline{B}) \ (****)$$

Similarly, we can conclude that;

$$\frac{\rho}{\epsilon_0} = grad(f) \cdot Re(\overline{B}) \ (*****)$$

From $(***), (****), (*****)$, we have that $\rho = 0$ on U.

□

Lemma 1.25. *Let the frame S be surface non-radiating in the sense of* [6]*, for charge and current $(\rho,\overline{J})$, then in any inertial frame S' connected to S by a boost with velocity vector $\overline{v}$, with $|\overline{v}|<c$, we have that S' is surface non-radiating, for the transformed current and charge $(\rho',\overline{J}')$. Moreover, if the frame S is surface non-radiating for charge and current $(\rho,\overline{J})$, then for any $g\in O(3)$, if $(\rho^g,\overline{J}^g)$, are the transformed current and charge in the rotated or reflected frame S', then S' is surface non-radiating.*

Proof. Let S'' be a frame connected to S' by a velocity vector $\overline{w}$. By Lemma 1.9, we have that $B_{\overline{w}}B_{\overline{v}}=R_gB_{\overline{v}*\overline{w}}$, where $g\in SO(3)$. Let S''' be connected to S by the velocity vector $\overline{v}*\overline{w}$, then as S is surface non-radiating, there exist $(\overline{E}''',\overline{B}''')$ satisfying Maxwell's equations in S''' for the transformed charge and current $(\rho''',\overline{J}''')$, with $div'''(\overline{E}'''\times\overline{B}''')=0$. By Lemma 1.4, we have that in S'', $(\overline{E}'',\overline{B}'')$ satisfy Maxwell's equations for the transformed current and charge $(\rho'',\overline{J}'')$ with $div(\overline{E}''\times\overline{B}'')=0$, where $\overline{E}''=\overline{E}'''^g$ and $\overline{B}''=\overline{B}'''^g$. As $(\rho,\overline{J})$ transforms as a 4-vector between inertial frames, and using the result of Lemma 1.7, we have verified the surface non-radiating condition for S', with the transformed current $(\rho',\overline{J}')$. For the last part, let S'' be a frame connected to S' by a velocity vector $\overline{v}$ and let $\overline{w}=g^{-1}(\overline{w})$. In the frame S''' connected to S by the velocity vector $\overline{w}$, by the definition of surface non radiating, there exist fields $(\overline{E}''',\overline{B}''')$, satisfying Maxwell's equations in S''', with $\bigtriangledown'''(\overline{E}'''\times\overline{B}''')=0$. Using Lemmas 1.4, 1.7 and 1.8, if $(\overline{E}'',\overline{B}'')$ are the fields in S'' corresponding to $(\overline{E}^g,sign(g)\overline{B}^g)$, where $(\overline{E},\overline{B})$ are the fields in S corresponding to $(\overline{E}''',\overline{B}''')$ in S''', then $\bigtriangledown''\cdot(\overline{E}''\times\overline{B}'')=0$. Moreover, $(\rho'',\overline{J}'',\overline{E}'',\overline{B}'')$ satisfy Maxwell's equations in S'' for the current and charge $(\rho'',\overline{J}'')$ in S'' corresponding to $(\rho^g,\overline{J}^g)$. . □

Lemma 1.26. *Let τ be a permutation of $(1,2,3)$, and let $(\rho,\overline{J},\overline{E},\overline{B})$ satisfy Maxwell's equations in S. Let $(\overline{E}^\tau,sign(\tau)\overline{B}^\tau)$ be the corresponding fields in the reflected frame S'. Let $\{h_i,h_i'\}$, for $1\leq i\leq 3$, $\{p_{ij},p_{ij}'\}$, for $1\leq i\leq j\leq 3$, and $\{\sigma,\sigma'\}$ be the components of the Poynting vector, stress tensor and energies for $(\overline{E},\overline{B})$ and $(\overline{E}^\tau,sign(\tau)\overline{B}^\tau)$ respectively, then;*

$$\sigma'=\sigma^\tau\ \ h_i'=h^\tau_{\tau(i)}\ \ p_{ij}'=p^\tau_{\tau(i)\tau(j)}$$

Let hypotheses be as in Lemma 1.19, with the assumption that S is surface non-radiating, for charge and current $(\rho,\overline{J})$, and $(\overline{E}_\infty,\overline{B}_\infty)$ are *the fields constructed in the limit frame S_∞, then the conclusion is satisfied by $(\overline{E}_\infty^{\tau_{23}}, -\overline{B}_\infty^{\tau_{23}})$ in the reflected frame $S_\infty^{\tau_{23}}$ for charge and current $(\rho^{\tau_{23}}, \overline{J}^{\tau_{23}})$. Moreover, for the permutation τ_{23} of $(1,2,3,4)$, if, in the context of Lemmas 1.21 and 1.22, we derive a relation of the form;*

$$\sum_{1\leq k\leq 4}\alpha_k\frac{\partial\sigma}{\partial x_k}+\sum_{1\leq i\leq 3,1\leq k\leq 4}\beta_{ik}\frac{\partial g_i}{\partial x_k}+\sum_{1\leq i\leq j\leq 3,1\leq k\leq 4}\gamma_{ijk}\frac{\partial p_{ij}}{\partial x_k}=0\ (*)$$

where $\{\alpha_k,\beta_{ik},\gamma_{ijk}\}\subset\mathcal{C}$, then we can derive the relation;

$$\sum_{1\leq k\leq 4}\alpha_k\frac{\partial\sigma}{\partial x_{\tau_{23}(k)}}+\sum_{1\leq i\leq 3,1\leq k\leq 4}\beta_{ik}\frac{\partial g_{\tau_{23}(i)}}{\partial x_{\tau_{23}(k)}}$$

$$+\sum_{1\leq i\leq j\leq 3,1\leq k\leq 4}\gamma_{ijk}\frac{\partial p_{\tau_{23}(i)\tau_{23}(j)}}{\partial x_{\tau_{23}(k)}}=0\ (**)$$

Proof. For the first claim, it is sufficient to prove the result for the elementary permutation τ_{23}. We have that, in components, $(e_1',e_2',e_3')=(e_1^{\tau_{23}},e_2^{\tau_{23}},e_3^{\tau_{23}})$ and $(b_1',b_2',b_3')=(-b_1^{\tau_{23}},-b_3^{\tau_{23}},-b_2^{\tau_{23}})$. Then, by a straightforward calculation;

$$e'^2=e_1'^2+e_2'^2+e_3'^2=(e_1^{\tau_{23}})^2+(e_3^{\tau_{23}})^2+(e_2^{\tau_{23}})^2=(e^2)^{\tau_{23}}$$

$$b'^2=b_1'^2+b_2'^2+b_3'^2=(-b_1^{\tau_{23}})^2+(-b_3^{\tau_{23}})^2+(-b_2^{\tau_{23}})^2=(b^2)^{\tau_{23}}$$

$$\sigma'=\tfrac{1}{\epsilon_0}(e'^2+c^2b'^2)=\tfrac{1}{\epsilon_0}((e^2)^{\tau_{23}}+c^2(b^2)^{\tau_{23}})=\sigma^{\tau_{23}}$$

$$h_1'=(e_2b_3-e_3b_2)^{\tau_{23}}=h_1^{\tau_{23}}$$

$$h_2'=(e_1b_2-e_2b_1)^{\tau_{23}}=h_3^{\tau_{23}}$$

$$h_3'=(e_3b_1-e_1b_3)^{\tau_{23}}=h_2^{\tau_{23}}$$

$$p_{ij}'=-\epsilon_0(e_i'e_j'+c^2b_i'b_j')+\delta_{ij}\sigma'$$

$$=-\epsilon_0(e_{\tau_{23}(i)}^{\tau_{23}}e_{\tau_{23}(j)}^{\tau_{23}}+c^2(-b_{\tau_{23}(i)}^{\tau_{23}})(-b_{\tau_{23}(j)}^{\tau_{23}}))+\delta_{ij}\sigma^{\tau_{23}}$$

$$=-\epsilon_0(e_{\tau_{23}(i)}^{\tau_{23}}e_{\tau_{23}(j)}^{\tau_{23}}+c^2b_{\tau_{23}(i)}^{\tau_{23}})b_{\tau_{23}(j)}^{\tau_{23}}))+\delta_{\tau_{23}(i)\tau_{23}(j)}\sigma^{\tau_{23}}$$

$$=p_{\tau_{23}(i)\tau_{23}(j)}^{\tau_{23}}$$

For the second claim, we have by both parts of Lemma 1.25, that the reflected frame $S'^{\tau_{23}}$ corresponding to S' is surface non-radiating for the transformed current and charge $(\rho'^{\tau_{23}},\overline{J}'^{\tau_{23}})$, where $(\rho',\overline{J}')$ correspond to $(\rho,\overline{J})$ in S. Let $\{\overline{E}_r,\overline{B}_r\}$ be the fields constructed in Lemma 1.27, with corresponding fields $\{\overline{E}_r^{\tau_{23}},-\overline{B}_r^{\tau_{23}}\}$ in the reflected frames $S_r''^{\tau_{23}}$ corresponding to S_r'', then $\bigtriangledown'''(\overline{E}_r^{\tau_{23}}\times-\overline{B}_r^{\tau_{23}})=0$ by Lemma 1.4 and, as τ_{23} fixes $\overline{e}_1$, $S_r''^{\tau_{23}}$ is connected to S'' by the velocity vectors $-r\overline{e}_1$. Moreover, in the notation of Lemma 1.19, $\overline{u}$ is fixed, so that when we construct $(\overline{E}'_\infty,\overline{B}'_\infty)$ from the fields $\{\overline{E}_r^{\tau_{23}},-\overline{B}_r^{\tau_{23}}\}$, it is clear, using the fact that the transformations connecting the frames $\{S,S',S_r''\}$ with S_∞ are the same as those between $\{S^{\tau_{23}},S'^{\tau_{23}},S_r''^{\tau_{23}}\}$ and $S_\infty^{\tau_{23}}$, that $\overline{E}'_\infty=\overline{E}_\infty^{\tau_{23}}$ and $\overline{B}'_\infty=-\overline{B}_\infty^{\tau_{23}}$. For the final claim, let $(\rho,\overline{J},\overline{E},\overline{B})$, be the tuple, satisfying Maxwell's equations in the base frame S, for which we derive the relation $(*)$, then $(\rho^{\tau_{23}},\overline{J}^{\tau_{23}},\overline{E}^{\tau_{23}},-\overline{B}^{\tau_{23}})$ satisfies Maxwell's equations in the reflected frame $S^{\tau_{23}}$, and, by Lemma 1.25, $S^{\tau_{23}}$ is surface non-radiating for the reflected charge and current $(\rho^{\tau_{23}},\overline{J}^{\tau_{23}})$. Using the fact that $(\overline{E},\overline{B})$ corresponds to the fields $(\overline{E}_\infty,\overline{B}_\infty)$ in the limit frame S_∞, by the proof of the second claim, we have that $(\overline{E}_\infty^{\tau_{23}},-\overline{B}_\infty^{\tau_{23}})$ corresponds to the fields $(\overline{E}_\infty^{\tau_{23}},-\overline{B}_\infty^{\tau_{23}})$ in the reflected frame $S_\infty^{\tau_{23}}$. We can then follow the proof of Lemma 1.18, to obtain the same relation $(*)$ for the quantities $\{\sigma',(g_i')_{1\leq i\leq 3},(p_{ij}')_{1\leq i\leq j\leq 3}\}$, corresponding to $(\overline{E}^{\tau_{23}},-\overline{B}^{\tau_{23}})$. By the first part of the lemma, we obtain the relation;

$$\sum_{1\leq k\leq 4}\alpha_k\frac{\partial\sigma^{\tau_{23}}}{\partial x_k}+\sum_{1\leq i\leq 3,1\leq k\leq 4}\beta_{ik}\frac{\partial g^{\tau_{23}}_{\tau_{23}(i)}}{\partial x_k}$$

$$+\sum_{1\leq i\leq j\leq 3,1\leq k\leq 4}\gamma_{ijk}\frac{\partial p^{\tau_{23}}_{\tau_{23}(i)\tau_{23}(j)}}{\partial x_k}=0$$

Using the chain rule, we then obtain that;

$$\sum_{1\leq k\leq 4}\alpha_k\frac{\partial\sigma}{\partial x_{\tau_{23}(k)}}+\sum_{1\leq i\leq 3,1\leq k\leq 4}\beta_{ik}\frac{\partial g_{\tau_{23}(i)}}{\partial x_{\tau_{23}(k)}}$$

$$+\sum_{1\leq i\leq j\leq 3,1\leq k\leq 4}\gamma_{ijk}\frac{\partial p_{\tau_{23}(i)\tau_{23}(j)}}{\partial x_{\tau_{23}(k)}}=0$$

as required.

□

Lemma 1.27. *We can construct limit fields $(\overline{E}_\infty,\overline{B}_\infty)$, in the limit frame S_∞, with $div_\infty(\overline{E}_\infty\times\overline{E}_\infty)=0$.*

Proof. Let $\epsilon>0$,$\delta<0$, and let S' travel with velocity vector $\overline{u}=(c-\epsilon)\overline{e}_1$, relative to S, and let S'' travel with velocity vector $\overline{w}=(-\frac{c^2}{c-\epsilon}-\delta)\overline{e}_1$ relative to S'. By Lemma 1.9, there exists $g\in SO(3)$ with;

$B_{\overline{w}}B_{\overline{u}} = R_g B_{\overline{u}*\overline{w}}$

with $\overline{u}*\overline{w} = \frac{\overline{u}+\overline{w}}{1+\frac{\overline{u}\cdot\overline{w}}{c^2}} + \frac{\gamma_u}{c^2(\gamma_u+1)}\frac{\overline{u}\times(\overline{u}\times\overline{w})}{1+\frac{\overline{u}\cdot\overline{w}}{c^2}}$

$= \frac{\overline{u}+\overline{w}}{1+\frac{\overline{u}\cdot\overline{w}}{c^2}}$

$= \frac{((c-\epsilon)-\frac{c^2}{c-\epsilon}-\delta)\overline{e}_1}{\frac{-\delta(c-\epsilon)}{c^2}}$

$= \frac{c^2((c-\epsilon)-\frac{c^2}{c-\epsilon}-\delta)}{-\delta(c-\epsilon)}\overline{e}_1$ $(*)$

By inspection of $(*)$, for given $0<\epsilon<c$, we can see that, as $\delta\rightarrow 0$ from below, $\overline{u}*\overline{w}\rightarrow\infty$ along the direction $\overline{e}_1$. By Lemma 1.25, we can assume that S' is surface non radiating, and there exist a family of electric and magnetic fields $\{\overline{E}_r,\overline{B}_r\}$, with $0\leq r<c$, such that $div(\overline{E}_r\times\overline{B}_r)=0$ in the inertial frames S''_r, travelling at velocity $-r\overline{e}_1$ relative to S'. Assume that there exists a uniform polynomial approximation $\{\overline{E}_r^\gamma,\overline{B}_r^\gamma\}$, with error term $\gamma>0$, to the fields, transferred back to the base frame S. By continuity, for sufficiently small δ, we can find a polynomial family $\{\overline{E}_r^{\gamma'},\overline{B}_r^{\gamma'}\}$, for $0<r<|\overline{w}|$, with error term γ', and $0<\gamma'<2\gamma$, such that $|div_\infty(\overline{E}_r\times\overline{B}_r)|<\gamma'$. In the limit frame S_∞, using Lemma 1.3, Lemma 1.9 and Definition 1.12, we obtain that $|div(\overline{E}_\infty\times\overline{B}_\infty)|<\gamma'$ as well. □

Remarks 1.28. *A more rigorous proof of the final claim in the previous lemma is given below.*

Lemma 1.29. *Let S be a frame with bounded current $(\rho,\overline{J})$ and let S'_ϵ be connected to S by the velocity vector $(c-\epsilon)\overline{e}_1$, for $0<\epsilon\leq c$, and $S''_{\epsilon,\delta}$ be connected to S'_ϵ by the velocity vector $(-c+\delta)\overline{e}_1$, where $\delta=(1+\tau)\epsilon$, for;*

$|\tau|\leq\frac{1}{2}\leq\frac{c}{c-1}\leq\frac{\frac{2}{c}-\frac{\epsilon}{c^2}}{1-\frac{1}{c}+\frac{\epsilon}{c^2}}$

then the transfers $(\rho_{\epsilon,\delta},\overline{J}_{\epsilon,\delta})$ to $S''_{\epsilon,\delta}$ are uniformly bounded in the frames $S''_{\epsilon,\delta}$. Moreover, we can assume there exists a constant F independent of ϵ, and, for any given ϵ, a family of tuples $(\rho_{\epsilon,\delta},\overline{J}_{\epsilon,\delta},\overline{E}_{\epsilon,\delta},\overline{B}_{\epsilon,\delta})$, satisfying Maxwell's equations, with $div(\overline{E}_{\epsilon,\delta}\times\overline{B}_{\epsilon,\delta})=0$ in the frame $S''_{\epsilon,\delta}$, and $max(|\overline{E}_{\epsilon,\delta}|,|\overline{E}_{\epsilon,\delta}|)\leq F$.

Proof. We have that $S''_{\epsilon,\delta}$ is connected to S by the boost matrix $B_{\overline{u}*\overline{v}}$ where $\overline{u}=(c-\epsilon)\overline{e}_1$, $\overline{v}=(-c+\delta)\overline{e}_1$ and;

$|\overline{u}*\overline{v}|=|\frac{\overline{u}+\overline{v}}{1+\frac{\overline{u}.\overline{v}}{c^2}}+\frac{\gamma_u}{c^2(\gamma_u+1)}\frac{\overline{u}\times(\overline{u}\times\overline{v})}{1+\frac{\overline{u}.\overline{v}}{c^2}}|$

$=|\frac{\overline{u}+\overline{v}}{1+\frac{\overline{u}.\overline{v}}{c^2}}|$

$=|\frac{(\delta-\epsilon)\overline{e}_1}{\frac{\delta+\epsilon}{c}-\frac{\epsilon\delta}{c^2}}|$

$=|\frac{\epsilon(\theta-1)\overline{e}_1}{\epsilon(\frac{\theta+1}{c})-\frac{\theta\epsilon^2}{c^2}}\ (\theta=1+\tau)|$

$=|\frac{(\theta-1)\overline{e}_1}{(\frac{\theta+1}{c})-\frac{\theta\epsilon}{c^2}}|\leq 1$

A straightforward calculation using the transfer rules for $(\rho,\overline{J})$ to frames $S_{\overline{w}}$, connected to S by a velocity vector $\overline{w}$, with $|\overline{w}|\leq 1$, shows that the transfers $(\rho_{\epsilon,\delta},\overline{J}_{\epsilon,\delta})$ are uniformly bounded. For the last part, we can, using the conjecture (ii) in Remark 1.32, assume there exists a family $(\overline{E}_{s\overline{e}_1},\overline{B}_{s\overline{e}_1})$ on the frames $S_{s\overline{e}_1}$, connected to S by the velocity vector $s\overline{e}_1$, for $|s|<1$, with $div_{s\overline{e}_1}(\overline{E}_{s\overline{e}_1}\times\overline{B}_{s\overline{e}_1})=0$, such that the transfers $(\overline{E}'_{s\overline{e}_1},\overline{B}'_{s\overline{e}_1})$ to S form a smooth family on $B(\overline{0},r_0)\times(0,t_0)$. By continuity, the transfers are bounded by some constant F as required.

□

Lemma 1.30. *Polynomial Approximation*
For any $\gamma>0$, with F as in Lemma 1.29, there exists a sequence of pairs $(\overline{E}_{n,\infty},\overline{B}_{n,\infty})$ for $n\geq n_\gamma$, in the limit frame S_∞, with $div_{S_\infty}|(\overline{E}_{n,\infty}\times\overline{B}_{n,\infty})|<\gamma$ and $max(|\overline{E}_{n,\infty}|,|\overline{B}_{n,\infty}|)<F+1$ on some $B(\overline{0},r_\infty)\times(0,t_\infty)$.

Proof. Let S'_ϵ be as in Lemma 1.29, and let $\{\overline{E}'_{\epsilon,\delta},\overline{B}'_{\epsilon,\delta}\}$ be the transfers of the fields $\{\overline{E}_{\epsilon,\delta},\overline{B}_{\epsilon,\delta}\}$, guaranteed by Lemma 1.29, to S'_ϵ. By the proof of Lemma 1.29, restricted to some $B(\overline{0},r_\epsilon)\times(0,t_\epsilon)$, they form a smooth bounded family on S'_ϵ,indexed by $-c+\delta\in(-c+\frac{\epsilon}{2},-c+\frac{3\epsilon}{2})$. By the Stone-Weierstrass approximation theorem, there exists a uniformly convergent sequence of polynomial approximations $\{\overline{E}'_{n,\epsilon,\delta},\overline{B}'_{n,\epsilon,\delta}\}$ to $\{\overline{E}'_{\epsilon,\delta},\overline{B}'_{\epsilon,\delta}\}$. By choosing the approximating polynomials in the frames $S_{s\overline{e}_1}$ from the previous Lemma, using continuity, the fact that the transfer of polynomial fields are polynomial, and formulating the fact that $div_{S''_{\epsilon,\delta}}(\overline{E}'_{\epsilon,\delta}\times\overline{B}'_{\epsilon,\delta})=0$ algebraically in the base frame S, see Lemma 1.17, we can assume that, for any $\gamma>0$, there exists $\{n_\gamma,\epsilon_\gamma\}$, such that $|div_{S''_{\epsilon,\delta'}}(\overline{E}'_{n,\epsilon,\delta'}\times\overline{B}'_{n,\epsilon,\delta'})|<\gamma$, for $-c+\delta'\in(-c-\frac{\epsilon}{2},-c)$, and $n\geq n_\gamma$, restricted to some $B(\overline{0},r_{\epsilon,\delta})\times(0,t_{\epsilon,\delta})$, where $S''_{\epsilon,\delta'}$ is connected to S_ϵ by the velocity vector $(-c+\delta')\overline{e}_1$. Taking the limit as $\epsilon\to 0$, using Lemma

1.27, the constant F from Lemma 1.29, we obtain a sequence of pairs $(\overline{E}_{n,\infty}, \overline{B}_{n,\infty})$ in the limit frame S_∞, with $div_{S_\infty}|(\overline{E}_{n,\infty} \times \overline{B}_{n,\infty})| < \gamma$ and $max(|\overline{E}_{n,\infty}|, |\overline{B}_{n,\infty}|) < F + 1$ on some $B(\overline{0}, r_\infty) \times (0, t_\infty)$.

□

Lemma 1.31. *Choose a sequence of error terms $\gamma_m > 0$ with $lim_{m\rightarrow\infty}\gamma_m = 0$, and pairs $(\overline{E}_{n_m,\infty}, \overline{B}_{n_m,\infty})$, satisfying the conclusion of Lemma 1.30. Then, as $m \rightarrow \infty$ the sequence $(\overline{E}_{n_m,\infty}, \overline{B}_{n_m,\infty})$ converges to a pair $(\overline{E}_\infty, \overline{B}_\infty)$ satisfying Maxwell's equations in S_∞ for the transferred charge and current $(\rho_\infty, \overline{J}_\infty)$ with $div_\infty(\overline{E}_\infty \times \overline{B}_\infty) = 0$. Moreover, we obtain the conclusion of Lemma 1.23 for the transfer $(\overline{E}, \overline{B})$ back to the base frame S.*

Proof. By the construction of Lemma 1.30, we have that the sequence $(\overline{E}_{n_m,\infty}, \overline{B}_{n_m,\infty})$ is Cauchy and uniformly bounded, and converges to a bounded limit on S_∞. If $(\rho', \overline{J}', \overline{E}', \overline{B})'$ is a tuple, satisfying Maxwell's equations in a base frame S', then, by the proof in [1], $(\rho'', \overline{J}'', \overline{E}'', \overline{B}'')$ satisfies Maxwell's equations at corresponding points of S'', connected to S by a real velocity vector $\overline{v}$, with $|\overline{v}| < c$, $(*)$. By the generalisation of the rules for transforming derivatives, the algebraic formulation of the connecting relations at corresponding points, complex linearity of the transformed derivative, and the generic formulation of $(*)$, $(\rho''', \overline{J}''', \overline{E}''', \overline{B}''')$ satisfies Maxwell's equations at corresponding points of S''', connected to S by a complex velocity vector $\overline{v}$, with $\overline{v}^2 \neq c^2$. Taking limits, this also holds for a transformation to a limit frame S_∞, if the limit exists. It follows that the transformation of the fields $(\overline{E}_{\epsilon,\delta}, \overline{B}_{\epsilon,\delta})$ to fields $(\overline{E}_{\epsilon,\delta,\infty}, \overline{B}_{\epsilon,\delta,\infty})$ in the limit frame S_∞ satisfy Maxwell's equations. and so the transformations $(\overline{E}_{n,\epsilon,\delta',\infty}, \overline{B}_{n,\epsilon,\delta',\infty})$ to the limit frame S_∞ satisfy Maxwell's equations up to a constant $\epsilon(n)$, which converges to 0 as $n \rightarrow \infty$, so that $(\overline{E}_{n_m,\infty}, \overline{B}_{n_m,\infty})$ satisfy Maxwell's equations up to a constant $\epsilon'(n_m)$, which again converges to zero as $m \rightarrow \infty$, both in S_∞ and the frame $S_{\epsilon,\delta'}$, for sufficiently small $\{\epsilon, \delta'\}$. In particularly, $(\overline{E}_\infty, \overline{B}_\infty)$ satisfying Maxwell's equations in S_∞ for the transferred charge and current $(\rho_\infty, \overline{J}_\infty)$, and, so does the transfer $(\overline{E}, \overline{B})$ of $(\overline{E}_\infty, \overline{B}_\infty)$ back to the base frame S, for the original current and charge $(\rho, \overline{J})$. The claim that $div_\infty(\overline{E}_\infty \times \overline{B}_\infty) = 0$ follows from the transformation rules for derivatives, back to the frames $S_{\epsilon,\delta'}$, the fact that the polynomial approximations $(\overline{E}_{n,\epsilon,\delta'}, \overline{B}_{n,\epsilon,\delta'})$ converge to smooth fields $((\overline{E}_{\epsilon,\delta'}, \overline{B}_{\epsilon,\delta'}))$ in the frame $S_{\epsilon,\delta'}$, interchanging limits with derivatives in $S_{\epsilon,\delta'}$ and the construction that $|div_\infty(\overline{E}_{n_m,\infty} \times \overline{B}_{n_m,\infty})| < \gamma_m$, with $\gamma_m \rightarrow 0$, as $m \rightarrow \infty$. For the final claim, we can approximate the fields $(\overline{E}_\infty, \overline{B}_\infty)$

by the polynomial fields $(\overline{E}_{n_m,\infty}, \overline{B}_{n_m,\infty})$, in S_∞, and follow through the argument of Lemma 1.19, to obtain the conclusion of Lemma 1.23 for the fields $(\overline{E}_{n_m}, \overline{B}_{n_m})$ transferred back to the base frame S, up to a constant $\epsilon''(n_m)$, which converges to 0 as $m \rightarrow \infty$. As the fields $(\overline{E}_{n_m}, \overline{B}_{n_m})$ and their derivatives converge to $(\overline{E}, \overline{B})$ in the base frame S, we obtain the conclusion of Lemma 1.23 for $(\overline{E}, \overline{B})$. □

Remarks 1.32. *In the definition of surface non-radiating, for the frame S, with charge and current $(\rho, \overline{J})$, we can impose the additional requirement, that, for any given velocity $\overline{v}$, with $|\overline{v}| = 1$, there exist pairs $\{\overline{E}_{s\overline{v}}, \overline{B}_{s\overline{v}}\}$, with $0 \leq s < c$, such that the condition of surface non-radiating is fulfilled, and the series is smooth and decaying at infinity, that is the fields $\{\overline{E}_{s\overline{v}}, \overline{B}_{s\overline{v}}\}$ are smooth and ;*

(i). $lim_{|\overline{x}_{s\overline{v}}| \rightarrow \infty} max(|\overline{E}_{s\overline{v}}|, |\overline{B}_{s\overline{v}}|) = 0$

in the coordinates $(\overline{x}_{s\overline{v}}, t_{s\overline{v}})$ of the frame $S_{s\overline{v}}$. With these extra assumptions, we conjecture, using polynomial approximations, that it possible to choose $\{\overline{E}_{s\overline{v}}, \overline{B}_{s\overline{v}}\}$ such that the above conditions hold, and also;

(ii) *The families defined by;*

$E(\overline{x}, t, s) = \overline{E}'_{s\overline{v}}(\overline{x}, t)$, $B(\overline{x}, t, s) = \overline{B}'_{s\overline{v}}(\overline{x}, t)$

are smooth on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0} \times (0, c)$. In particularly, for finite $\{t_0, r_0, c_0\} \subset \mathcal{R}$, with $0 \leq c_0 < c$, $max_{0 \leq s \leq c_0}(|\overline{E}'_{s\overline{v}}|_{B(\overline{0}, r_0) \times (0, t_0)}|, |\overline{B}'_{s\overline{v}}|_{B(\overline{0}, r_0) \times (0, t_0)}|) \leq G_{r, t_0, c_0}$ for some constant $G_{r_0, t_0, c_0} \in \mathcal{R}_{\geq 0}$, where $\{\overline{E}_{s\overline{v}}, \overline{B}_{s\overline{v}}\}$ are the fields transferred back to the base frame S.

Lemma 1.33. *If the frame S is decaying surface non-radiating, in the sense of Definition 1.1, then if $(\rho, \overline{J})$ is real analytic, either $\rho = 0$ and $\overline{J} = \overline{0}$, or S is non-radiating, in the sense of* [6].

Proof. The transfers of $(\rho, \overline{J})$ to any frame $S_{\overline{v}}$, connected to S by a velocity vector $\overline{v}$ with $|\overline{v}| < c$ is also real analytic. By the proof of Lemma 1.25, the frames $S_{\overline{v}}$ are also decaying surface non-radiating. By Lemma 1.24, and using continuity, in each frame $S_{\overline{v}}$, either the transfers $\rho_{\overline{v}}$ are identically zero or there exists a real solution $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ to Maxwell's equations with $\overline{B}_{\overline{v}} = 0$. If S is not non-radiating, then, without loss of generality, we can assume that $\rho = 0$ in the base frame S. By the transformation rules for $(\rho, \overline{J})$, we have that;

$\rho_{\overline{v}} = -\frac{\gamma_v <\overline{v},\overline{J}>}{c^2}$

$\overline{J}_{\overline{v}} = \gamma_v \overline{J}_{||,\overline{v}} + \overline{J}_{\perp,\overline{v}}$

As $\overline{J}$ is analytic, $\{\overline{v} :< \overline{v}, \overline{J} >= 0\}$ includes $\overline{0}$ and, if infinite, is both open and closed inside the ball $B_{|\overline{v}|<c}$, so that $\overline{J}_{||,\overline{v}} = 0$, for every $\overline{v} \in B_{|\overline{v}|<c}$ with $\overline{v} \neq 0$, in particular, $\overline{J} = 0$. We can, therefore assume that $\rho_{\overline{v}} \neq 0$ in all but finitely many frames $S_{\overline{v}}$, and there exist real solutions $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ to Maxwell's equations with $\overline{B}_{\overline{v}} = 0$. Now, we can use the proof of Lemma 2.7 in [6], to derive the equation;

$\overline{v} \times (\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}) = \overline{0}$ $(*)$

valid for all but finitely many $\overline{v} \in B_{|\overline{v}|<c}$. Using continuity, we can conclude that $(*)$ holds for all $\overline{v} \in B_{|\overline{v}|<c}$ and that;

$(\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t}) = \overline{0}$

Then follow through the rest of the proof of Lemma 2.7 in [6] to conclude that $\square^2(p) = 0$ and $\square^2(\overline{E}) = \overline{0}$. Now use the proof of Lemma 2.4 in [6] to get $\square^2(\overline{J}) = \overline{0}$, and Lemma 2.5 in [6] to conclude that S is non-radiating.

$\square$

2. Some Thermodynamic Arguments

Definition 2.1. *Given $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfying Maxwell's equations, and $t_0 \in \mathcal{R}_{>0}$, we define the reversed process $(\rho', \overline{J}', \overline{E}', \overline{B}')$ on $\mathcal{R}^3 \times (0, t_0)$ by;*

$\rho'(\overline{x}, t) = \rho(\overline{x}, t_0 - t)$

$\overline{J}'(\overline{x}, t) = -\overline{J}(\overline{x}, t_0 - t)$

$\overline{E}'(\overline{x}, t) = \overline{E}(\overline{x}, t_0 - t)$

$\overline{B}'(\overline{x}, t) = -\overline{B}(\overline{x}, t_0 - t)$

Lemma 2.2. *For the reversed process, $(\rho', \overline{J}', \overline{E}', \overline{B}')$, we have that $(\rho', \overline{J}')$ satisfies the continuity equation and $(\rho', \overline{J}', \overline{E}', \overline{B}')$ satisfies Maxwell's equations on $\mathcal{R}^3 \times (0, t_0)$. Moreover $div(\overline{E}' \times \overline{B}') = -div(\overline{E} \times \overline{B})$*

Proof. For the first part, we have, using the chain rule, the definitions and the continuity equation for $(\rho, \overline{J})$, that;

$\frac{\partial\rho'}{\partial t}|_(\overline{x}, t) = -\frac{\partial\rho}{\partial t}|_(\overline{x}, t_0 - t)$

$= -div(\overline{J})|_{(\overline{x},t_0-t)}$

$= div(\overline{J}')_{(\overline{x},t)}$

For the second part, we have, using the chain rule again, the definitions, and Maxwell's equations for $(\rho, \overline{J}, \overline{E}, \overline{B})$, that;

$(i).\ div(\overline{E}')|_{(\overline{x},t)} = div(\overline{E})|_{(\overline{x},t_0-t)} = \frac{\rho}{\epsilon_0}|_{(\overline{x},t_0-t)} = \frac{\rho}{\epsilon_0}|_{(\overline{x},t)}$

$(ii).\ (\bigtriangledown \times \overline{E}')|_{(\overline{x},t)} = (\bigtriangledown \times \overline{E})|_{(\overline{x},t_0-t)} = -\frac{\partial\overline{B}}{\partial t}|_{(\overline{x},t_0-t)} = -\frac{\partial\overline{B}'}{\partial t}|_{(\overline{x},t)}$

$(iii).\ div(\overline{B}')|_{(\overline{x},t)} = -div(\overline{B})_{(\overline{x},t_0-t)} = 0$

$(iv).\ (\bigtriangledown\times\overline{B}')_{(\overline{x},t)} = (\bigtriangledown\times-\overline{B})_{(\overline{x},t_0-t)} = -(\epsilon_0\overline{J})|_{(\overline{x},t_0-t)} - (\mu_0\epsilon_0\frac{\partial\overline{E}}{\partial t})|_{(\overline{x},t_0-t)}$

$= (\epsilon_0\overline{J}')|_{(\overline{x},t)} + (\mu_0\epsilon_0\frac{\partial\overline{B}'}{\partial t})|_{(\overline{x},t)}$

as required. The last claim follows easily from the definitions of $\{\overline{E}', \overline{B}'\}$

□

Definition 2.3. *Given a solution* $(\rho, \overline{J}, \overline{E}, \overline{B})$ *to Maxwell's equation, we say that* $(\overline{E}, \overline{B})$ *is classically non-radiating if, uniformly in* $t \in \mathcal{R}_{>0}$, *we have that;*

$lim_{r\rightarrow\infty} \int_{B(0,r)} div(\overline{E}_t \times \overline{B}_t) d\overline{x} = 0$

Lemma 2.4. *Given a smooth solution* $(\rho, \overline{J}, \overline{E}, \overline{B})$ *to Maxwell's equations, and* $t_0 \in \mathcal{R}_{>0}$ *with* $div(\overline{E} \times \overline{B})|_{t_0} \neq 0$ *and* $(\overline{E}, \overline{J})|_{t_0} \neq 0$, *there exists a smooth volume* $S \subset \mathcal{R}^3$ *and* $\epsilon > 0$, *with;*

$\int_S div(\overline{E}_t \times \overline{B}_t)|d\overline{x} \neq 0$

$\int_S (\overline{E}_t, \overline{J}_t) d\overline{x} \neq 0$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Proof. Choose $\{\overline{x}_0, \overline{x}_1\} \subset \mathcal{R}^3$, with $div(\overline{E} \times \overline{B})(\overline{x}_0, t_0) \neq 0$ and $(\overline{E}, \overline{J})(\overline{x}_1, t_0) \neq 0$. As $div(\overline{E} \times \overline{B})$ and $(\overline{E}, \overline{J})$ are smooth, there exist disjoint balls $B(\overline{x}_0, r_0)$ and $B(\overline{x}_0, r_1)$ with;

$\int_{B(\overline{x}_0, r_0)} div(\overline{E}_{t_0} \times \overline{B}_{t_0}))d\overline{x} \neq 0$

$\int_{B(\overline{x}_1, r_1)} (\overline{E}_{t_0}, \overline{J}_{t_0})d\overline{x} \neq 0$

Shrinking the ball $B(\overline{x}_0, r_0)$ if necessary to avoid cancellations, we can assume that;

$\int_{B(\overline{x}_0, r_0) \cup B(\overline{x}_1, r_1)} div(\overline{E}_{t_0} \times \overline{B}_{t_0}))d\overline{x} \neq 0$

$\int_{B(\overline{x}_0, r_0) \cup B(\overline{x}_1, r_1)} (\overline{E}_{t_0}, \overline{J}_{t_0})d\overline{x} \neq 0$

As $div(\overline{E}_{t_0} \times \overline{B}_{t_0})$ and $(\overline{E}_{t_0}, \overline{J}_{t_0})$ are smooth, they are bounded on a ball $B(0, r)$ with $B(0, r) \supset B(\overline{x}_0, r_0)$ and $B(0, r) \supset B(\overline{x}_1, r_1)$. Choosing a sufficiently small strip S' connecting the balls $B(\overline{x}_0, r_0)$ and $B(\overline{x}_1, r_1)$, and letting $S = B(\overline{x}_0, r_0) \cup B(\overline{x}_1, r_1) \cup S'$ be a smooth volume, we can assume that;

$\int_S div(\overline{E}_{t_0} \times \overline{B}_{t_0}))d\overline{x} \neq 0$

$\int_S (\overline{E}_{t_0}, \overline{J}_{t_0})d\overline{x} \neq 0$

Using smoothness again, we can assume that there exists $\epsilon > 0$ such that;

$\int_S div(\overline{E}_t \times \overline{B}_t d\overline{x} \neq 0$

$\int_S (\overline{E}_t, \overline{J}_t)d\overline{x} \neq 0$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, as required.

□

Lemma 2.5. *Given* $(\rho, \overline{J}, \overline{E}, \overline{B})$, *with* $(\overline{E}, \overline{B})$ *classically non-radiating, such that the hypotheses of Lemma 2.4 are satisfied. Let* T *be the surface boundary of* S, *then for any* $\kappa > 0$, *there exists volumes* $\{S, S_\kappa\}$ *with* $S \cap S_\kappa = \emptyset$, $T \subset \overline{S_\kappa}$, $\{\overline{T}, \overline{T}', \overline{T}''\}$ *outward normals to the volumes*

$\{S, S_\kappa, B(\overline{0}, r_\kappa)\}$, *such that;*

$\int_T (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T} = -\int_T (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T'}$

$\int_S div(\overline{E}_t \times \overline{B}_t) d\overline{x} \neq 0$

$\int_S (\overline{E}_t, \overline{J}_t) d\overline{x} \neq 0$

$|\int_{\delta B(\overline{0}, r_\kappa)} (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T}''| < \kappa$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$

Proof. By the definition of classically non-radiating, for any $\kappa > 0$, there exists $r_\kappa > 0$, with $S \subset B(\overline{0}, r_\kappa)$ such that;

$|\int_{B(\overline{0}, r_\kappa)} div(\overline{E}_t \times \overline{B}_t) d\overline{x}| < \kappa \ (*)$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Let $S_\kappa = B^0(\overline{0}, r_\kappa) \setminus \overline{S}$, then, as $\overline{T}'$ reverses the direction of $\overline{T}$;

$\int_T (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T} + \int_T (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T'} = 0$

and by $(*)$ and the divergence theorem;

$|\int_{\delta B(\overline{0}, r_\kappa)} (\overline{E}_t \times \overline{B}_t) \centerdot d\overline{T}''| = |\int_{B(\overline{0}, r_\kappa)} div(\overline{E}_t \times \overline{B}_t) d\overline{x}| < \kappa$

as required.

□

Lemma 2.6. *Let notation be as above, then, assuming thermal equilibrium for electrons, we cannot have, in a classically non radiating system, that;*

$\int_S div(\overline{E}_t \times \overline{B}_t) d\overline{x} > 0$

$\int_S (\overline{E}_t, \overline{J}_t) d\overline{x} < 0 \ (\dagger)$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Proof. Suppose that;

$\int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x} > 0$

$\int_S(\overline{E}_t, \overline{J}_t)d\overline{x} < 0$ (†)

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Shrinking the interval $(t_0-\epsilon, t_0+\epsilon)$ if necessary, we can assume that;

$\int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x} > \delta > 0$

$\int_S(\overline{E}_t, \overline{J}_t)d\overline{x} < -\delta < 0$ (†′)

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, some $\delta > 0$.

Choose $\kappa > 0$ with $\kappa < \delta$, so that;

$\kappa < min(\int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x}, -\int_S(\overline{E}_t, \overline{J}_t)d\overline{x})$, (†††)

for $t \in (t_0-\epsilon, t_0+\epsilon)$, and $\{r_\kappa, S_\kappa, B(\overline{0}, r_\kappa)\}$, such that the conclusion of Lemma 2.5 holds. Using Lemmas 2.4 and 2.5, we claim that;

$\int_{S_\kappa}(\overline{E}_t, \overline{J}_t)d\overline{x} \geq min(-\int_S(\overline{E}_t, \overline{J}_t)d\overline{x}, \int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x})$ (††)

uniformly for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Suppose not, then, for some $t_1 \in (t_0 - \epsilon, t_0 + \epsilon)$;

$\int_{S_\kappa}(\overline{E}_{t_1}, \overline{J}_{t_1})d\overline{x} < -\int_S(\overline{E}_{t_1}, \overline{J}_{t_1})d\overline{x}$

$\int_{S_\kappa}(\overline{E}_{t_1}, \overline{J}_{t_1})d\overline{x} < \int_S div(\overline{E}_{t_1} \times \overline{B}_{t_1})d\overline{x}$

and, by continuity, we can assume that;

$\int_{S_\kappa}(\overline{E}_t, \overline{J}_t)d\overline{x} < -\int_S(\overline{E}_t, \overline{J}_t)d\overline{x}$

$\int_{S_\kappa}(\overline{E}_t, \overline{J}_t)d\overline{x} < \int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x}$

for $t \in (t_1 - \delta', t_1 + \delta')$ with $(t_1 - \delta', t_1 + \delta') \subset (t_0 - \epsilon, t_0 + \epsilon)$

Let $\{E_{S_\kappa}, E_S, E_{S,el}, E_{S_\kappa,el}, E_{S,field}, E_{S_\kappa,field}\}$ denote the total energies in $\{S, S_\kappa\}$, the energies stored in the electrons contained in $\{S, S_\kappa\}$ and the electromagnetic energies restricted to $\{S, S_\kappa\}$, see [3]. By Poynting's theorem and the divergence theorem;

(i) $\frac{dE_{S_\kappa,el}}{dt'}|_t < -\frac{dE_{S,el}}{dt'}|_t$

(ii) $\frac{dE_{S_\kappa,el}}{dt'}|_t < (\int_S(\overline{E}_t \times \overline{B}_t) \cdot d\overline{T})_t$

for $t \in (t_1 - \delta', t_1 + \delta')$.

By $(\dagger'), (\dagger\dagger\dagger)$, there is an energy flux from the electrons in S to the total energies in S_κ, not all of which can leak from the boundary $\delta B(\overline{0}, r_\kappa)$. By (i) and Poynting's Theorem, there is some energy transferred into the electromagnetic energy of S_κ. By (ii), not all the energy transferred from the electrons in S is transferred to the energy of the electrons in S_κ. It follows that some of the energy flux from the electrons in S is transferred into the electromagnetic energy of S_κ. By the last claim in Lemma 2.2, the process is reversible, which contradicts Kelvin's formulation of the second law of thermodynamics, see [5], that it is impossible to devise an engine which, working in a cycle, shall produce no effect other than the extraction of heat from a reservoir and the performance of an equal amount of mechanical work. Given that $(\dagger\dagger)$ holds, we have;

$\int_{S_\kappa}(\overline{E}_t, \overline{J}_t)d\overline{x} \geq -\int_S(\overline{E}_t, \overline{J}_t)d\overline{x}$

$\int_{S_\kappa}(\overline{E}_t, \overline{J}_t)d\overline{x} \geq \int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x}$

uniformly in $t \in (t_0 - \epsilon, t_0 + \epsilon)$, so that;

(iii) $\frac{dE_{S_\kappa,el}}{dt'}|_t \geq -\frac{dE_{S,el}}{dt'}|_t$

(iv) $\frac{dE_{S_\kappa,el}}{dt'}|_t \geq (\int_S(\overline{E}_t \times \overline{B}_t) \cdot d\overline{T})_t$

uniformly in $t \in (t_0 - \epsilon, t_0 + \epsilon)$. By (iii), and $(\dagger')$, we have that $E_{S_\kappa,el}$ is increasing. Without loss of generality, we have 2 cases;

Case 1. $\int_{B(\overline{0},r_\kappa)} div(\overline{E}_t \times \overline{B}_t)d\overline{x} > 0$, for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Then, by (iv), the energy in the field from S_κ is constant or decreasing and transferring to the electrons in S_κ, $(*)$. As there is a net energy flux from S to S_κ, and using $(\dagger')$ again, there is an energy transfer from electrons in S to electrons in S_κ. Assuming thermal equilibrium and raising the temperature of the electrons in S_κ by a small amount, see [5], noting again that the process is reversible, this contradicts Clausius's formulation of the second law of thermodynamics; that it is impossible to devise an engine which, working in a cycle, shall produce no effect other than the transfer of heat from a colder to a hotter body.

Case 2. $\int_{B(\overline{0},r_\kappa)} div(\overline{E}_t \times \overline{B}_t)d\overline{x} < 0$, for $t \in (t_0-\epsilon, t_0+\epsilon)$. If $(*)$ holds, repeat the argument, otherwise, as there is a net energy flux from S to S_κ, and none of the energy is leaking through the boundary $\delta B(\overline{0}, r_\kappa)$, either there is an energy transfer from electrons in S to electrons in S_κ, from which we obtain the same contradiction as in Case 1, or there is an energy transfer from electrons in S to the electromagnetic energy in S_κ, in which case we can use the previous argument, based on Kelvin's formulation.

□

Remarks 2.7. *We can assume that an atomic system is classically non-radiating in all inertial frames, as, otherwise, by Rutherford's observation, the system would lose energy and collapse. Thermal equilibrium for electrons in all frames also seems a reasonable criterion for such systems, though there are difficulties in finding the correct definition of temperature in electromagnetism. We leave as a conjecture whether the above lemma holds with just the assumption that;*

$\int_S div(\overline{E}_t \times \overline{B}_t)d\overline{x} \neq 0$

$\int_S (\overline{E}_t, \overline{J}_t)d\overline{x} \neq 0$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Given this, we can probably obtain the conclusion, from Lemma 2.4, that either;

$div(\overline{E}_t \times \overline{B}_t) = 0$

in all frames, or;

$(\overline{E}_t, \overline{J}_t) = 0$

in all frames. In which case, we can either use the main result of the paper to conclude that the system is non-radiating and, by [6], that *the charge and current $\rho, \overline{J}$ obey certain wave equations, or classify the case where $(\overline{E}, \overline{J}) = 0$ in all frames. This is done in the final section.*

3. Force Invariance

Definition 3.1. *Suppose that in the base frame S, $(\rho, \overline{J}, \overline{E}, \overline{B})$, satisfy Maxwell's equations and $S_{\overline{v}}$ is an inertial frame, with velocity vector $\overline{v}$, $|\overline{v}| < c$. Let $(\rho_{\overline{v}}, \overline{J}_{\overline{v}}, \overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$ be the transformed quantities. We define the associated force;*

$$\overline{f}^{\overline{v}} = \rho_{\overline{v}} \overline{E}_{\overline{v}} + \overline{J}_{\overline{v}} \times \overline{B}_{\overline{v}}$$

We say that $(\rho, \overline{J}, \overline{E}, \overline{B})$ is electric current orthogonal in S if for every inertial frame $S_{\overline{v}}$, we have that $(\overline{E}_{\overline{v}}, \overline{J}_{\overline{v}}) = 0$. We say that $(\rho, \overline{J}, \overline{E}, \overline{B})$ is electromagnetic orthogonal in S if $(\overline{E}, \overline{B}) = 0$.

Lemma 3.2. *If $(\rho, \overline{J}, \overline{E}, \overline{B})$ is electromagnetic orthogonal in S, then for every inertial frame $S_{\overline{v}}$, we have that $(\overline{E}_{\overline{v}}, \overline{J}_{\overline{v}}) = 0$.*

Proof. This is well known.

□

Lemma 3.3. *Suppose that in the base frame S, the tuple $(\rho, \overline{J}, \overline{E}, \overline{B})$, satisfying Maxwell's equations, is electric current orthogonal, then, for every $S_{\overline{v}}$, $\overline{f}^{\overline{v}} = \overline{0}$.*

Proof. We have that;

$$\overline{E}_{\overline{v}} = \overline{E}_{||,\overline{v}} + \gamma_{\overline{v}}(\overline{E}_{\perp,\overline{v}} + \overline{v} \times \overline{B})$$

$$\overline{J}_{\overline{v}} = \gamma_{\overline{v}}(\overline{J}_{||,\overline{v}} - \rho\overline{v}) + \overline{J}_{\perp,\overline{v}}$$

so, using the fact that;

$$(\overline{E}_{||,\overline{v}}, \overline{J}_{\perp,\overline{v}}) = (\overline{E}_{\perp,\overline{v}} + \overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{||,\overline{v}} - \rho\overline{v}) = 0$$

we have;

$$(\overline{E}_{\overline{v}}, \overline{J}_{\overline{v}}) = (\overline{E}_{||,\overline{v}} + \gamma_{\overline{v}}(\overline{E}_{\perp,\overline{v}} + \overline{v} \times \overline{B}), \gamma_{\overline{v}}(\overline{J}_{||,\overline{v}} - \rho\overline{v}) + \overline{J}_{\perp,\overline{v}})$$

$$= \gamma_{\overline{v}}(\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}}) - \gamma_{\overline{v}}\rho(\overline{E}_{\overline{v}}, \overline{v}) + \gamma_{\overline{v}}(\overline{E}_{\perp,\overline{v}}, \overline{J}_{\perp,\overline{v}}) + \gamma_{\overline{v}}(\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}}) = 0$$

so that;

$(\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v}) + (\overline{E}_{\perp,\overline{v}}, \overline{J}_{\perp,\overline{v}}) + (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}})$

$= (\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v}) + (\overline{E}_{\overline{v}} - \overline{E}_{||,\overline{v}}, \overline{J}_{\overline{v}} - \overline{J}_{||,\overline{v}}) + (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}})$

$= 2(\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v}) + (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}}) - (\overline{E}_{\overline{v}}, \overline{J}_{||,\overline{v}}) - (\overline{E}_{||,\overline{v}}, \overline{J}_{\overline{v}})$

$= 2(\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v}) + (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}}) - (\overline{E}_{||,\overline{v}} + \overline{E}_{\perp,\overline{v}}, \overline{J}_{||,\overline{v}})$

$-(\overline{E}_{||,\overline{v}}, \overline{J}_{||,\overline{v}} + \overline{J}_{\perp,\overline{v}})$

$= (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\perp,\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v})$

$= (\overline{v} \times \overline{B}_{\overline{v}}, \overline{J}_{\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v})$

$= (\overline{v}, \overline{B}_{\overline{v}} \times \overline{J}_{\overline{v}}) - \rho(\overline{E}_{\overline{v}}, \overline{v}) = 0$

As $\overline{v}$, with $|\overline{v}| < c$ is arbitrary, we conclude that;

$\overline{B}_{\overline{v}} \times \overline{J}_{\overline{v}} - \rho\overline{E}_{\overline{v}}$

$= -\overline{J}_{\overline{v}} \times \overline{B}_{\overline{v}} - \rho\overline{E}_{\overline{v}} = \overline{0}$

so that $\overline{f} = \overline{0}$. By symmetry, we can conclude that $\overline{f}^{\overline{v}} = \overline{0}$ in any inertial frame $S_{\overline{v}}$ as required.

□

Lemma 3.4. *Suppose that in the base frame S, the tuple $(\rho, \overline{J}, \overline{E}, \overline{B})$, is electric current orthogonal, but not electromagnetic orthogonal, with $\{\rho, \overline{J}, \overline{E}, \overline{B}\}$ real analytic, then $\rho = 0$, $\overline{J} = \overline{0}$.*

Proof. By Lemma 3.3, Lemma 3.2 and symmetry, we have that $\overline{f}^{\overline{v}} = \overline{0}$ and $(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}}) \neq 0$, in every inertial frame $S_{\overline{v}}$, with $|\overline{v}| < c$, $(*)$. It follows , using the transformation rules, and Lemma 3.2 again, that;

$(\overline{f}^{\overline{v}}, \overline{E}_{\overline{v}}) = \rho_{\overline{v}}(\overline{E}_{\overline{v}}, \overline{B}_{\overline{v}})$

$= \gamma_{\overline{v}}(\rho - \frac{(\overline{v}, \overline{J})}{c^2})(\overline{E}, \overline{B}) = 0 \ (**)$

for every inertial frame $S_{\overline{v}}$, with $|\overline{v}| < c$, and, identifying coordinates, $(**)$ holds in S, for every $\overline{v}$, with $|\overline{v}| < c$, $(***)$. From $(*), (***)$, taking $\overline{v} = \overline{0}$, in the base frame S, we have that $\rho(\overline{E}, \overline{B}) = 0$, and $(\overline{E}, \overline{B}) \neq 0$, $(****)$. As $\{\overline{E}, \overline{B}\}$ are analytic, we have that $(\overline{E}, \overline{B})$ is analytic, so that, by $(****)$, there exists an open set $U \subset \mathcal{R}^3 \times \mathcal{R}_{>0}$, for which $(\overline{E},\overline{B})|_U\neq 0$. From $(****)$ again, we obtain that $\rho|_U=0$, and, as ρ is analytic, that $\rho=0$, $(\dagger)$. From $(**)$, $(***)$, $(\overline{E},\overline{B})$ analytic, and $(\dagger)$, we obtain that;

$(\rho-\frac{(\overline{v},\overline{J})}{c^2})|_U=-\frac{(\overline{v},\overline{J})}{c^2}|_U=0$

for every $\overline{v}$, with $|\overline{v}|<c$. It follows that $\overline{J}|_U=\overline{0}$, and, as $\overline{J}$ is analytic, that $\overline{J}=\overline{0}$, as required.

□

Lemma 3.5. *Suppose that in the base frame S, the tuple $(\rho,\overline{J},\overline{E},\overline{B})$, is electric current orthogonal and electromagnetic orthogonal, with $\{\rho,\overline{J},\overline{E},\overline{B}\}$ real analytic, then we can obtain a complete classification of cases except when in the base frame S, we have that $\rho\geq 0$, and $\overline{E}=-\frac{1}{\rho}(\overline{J}\times\overline{B})$, for which Remark 3.6 is relevant.*

Proof. By Definition 3.1 and Lemma 3.2, we have that in every inertial frame $S_{\overline{v}}$, with $|\overline{v}|<c$;

$\overline{E}_{\overline{v}}\times(\overline{J}_{\overline{v}}\times\overline{B}_{\overline{v}})$

$=\overline{J}_{\overline{v}}(\overline{E}_{\overline{v}},\overline{B}_{\overline{v}})-\overline{B}_{\overline{v}}(\overline{E}_{\overline{v}},\overline{J}_{\overline{v}})=\overline{0}$

In particularly, in the base frame S, $\overline{E}\times(\overline{J}\times\overline{B})=\overline{0}$. As $\{\overline{E},\overline{B},\overline{J}\}$ are analytic, it follows that, on an open subset $U\subset\mathcal{R}^3\times\mathcal{R}_{>0}$, either;

(i). $\overline{E}|_U=\overline{0}$

(ii). $\overline{J}\times\overline{B}|_U=\overline{0}$

(iii). There exists an analytic λ on U, with $\overline{E}|_U=\lambda(\overline{J}\times\overline{B})|U$

In case (i), we have, as $\overline{E}$ is analytic, that $\overline{E}=\overline{0}$, and using Maxwell's equations and the continuity equation, that $\rho=0$, $\overline{B}$ is time independent, $\overline{J}=\frac{1}{\mu_0}(\bigtriangledown\times\overline{B})$ is time independent and $\bigtriangledown\cdot\overline{J}=\bigtriangledown\cdot\overline{B}=0$. In case (ii), we can assume, without loss of generality, that, either;

$(i)'$. $\overline{B}|_U=\overline{0}$

$(ii)'$. $\overline{J}|_U=\overline{0}$

$(iii)'$. There exists an analytic λ on U, with $\overline{B}|_U = \lambda\overline{J}$

In case $(i)'$, we have, as $\overline{B}$ is analytic, that $\overline{B} = \overline{0}$. In case $(ii)'$, we have, as $\overline{J}$ is analytic, that $\overline{J} = \overline{0}$, and, using the continuity equation, that ρ is time independent. In case $(iii)'$, we have, using the divergence theorem, Maxwell's equations, and integrating over the level surfaces $\{S_\kappa : \kappa \in \mathcal{R}\}$ for λ, that;

$\int_{B_\kappa} \bigtriangledown \centerdot \overline{B} dB_\kappa = \int_{S_\kappa} \overline{B} \centerdot d\overline{S_\lambda}$

$= \int_{S_\kappa} \lambda\overline{J} \centerdot d\overline{S_\kappa}$

$= \lambda \int_{S_\kappa} \overline{J} \centerdot d\overline{S_\kappa}$

$= \lambda \int_{B_\kappa} \bigtriangledown \centerdot \overline{J} dB_\kappa = 0$

so that $\bigtriangledown \centerdot \overline{J}|_U = 0$, and, as $\bigtriangledown \centerdot \overline{J}$ is analytic, that $\bigtriangledown \centerdot \overline{J} = 0$. By the continuity equation, we obtain that $\frac{\partial \rho}{\partial t} = -\bigtriangledown \centerdot \overline{J} = 0$, so that ρ is time independent.

In case (iii), using the fact that $\overline{f} = \rho\overline{E} + \overline{J} \times \overline{B} = \overline{0}$, we have that;

$(\overline{E}, \rho\overline{E} + \overline{J} \times \overline{B})|_U$

$= (\rho|\overline{E}|^2| + (\overline{E}, \overline{J} \times \overline{B}))|_U$

$= (\rho|\overline{E}|^2| + (\lambda(\overline{J} \times \overline{B}), \overline{J} \times \overline{B}))|_U$

$= (\rho|\overline{E}|^2| + \lambda|\overline{J} \times \overline{B}|^2)|_U = 0$, $(\sharp)$

$(i)''$. It is not the case that $\rho|_U \leq 0$, then, shrinking U if necessary to W, and using continuity, we can assume that $\rho|_W > 0$.

$(i)''(a)$. It is not the case that $\lambda|_W \leq 0$, then, again shrinking W if necessary to V, and using continuity, we can assume that $\lambda|_V > 0$. By $(\sharp)$, we obtain that $\overline{E}|_V = (\overline{J} \times \overline{B})|_V = \overline{0}$. As $\{\overline{E}, \overline{J}, \overline{B}\}$ are analytic, we have that $\overline{E} = \overline{J} \times \overline{B} = \overline{0}$. We then have the following cases, similarly to the above;

$(i)''(a)(i)$. $\overline{E} = \overline{B} = \overline{0}$, so that $\rho = 0$, $\overline{J} = \overline{0}$ as well.

$(i)''(a)(ii)$. $\overline{E} = \overline{J} = \overline{0}$, so that $\rho = 0$, $\overline{B}$ is time independent, $\bigtriangledown \cdot \overline{B} = 0$, $\bigtriangledown \times \overline{B} = \overline{0}$.

$(i)''(a)(iii)$. $\overline{E} = \overline{0}$, $\overline{B}|_{V'} = \lambda \overline{J}|_{V'}$, for some analytic $\lambda|_{V'}$, $V' \subset \mathcal{R}^3 \times \mathcal{R}_{>0}$ open. Combining cases (i) and $(iii)'$, we obtain that $\rho = 0$, $\{\overline{B}, \overline{J}\}$ are time independent, $\overline{J} = \frac{1}{\mu_0}(\bigtriangledown \times \overline{B})$, $\bigtriangledown \cdot \overline{J} = 0$.

$(ii)''(b)$. $\lambda|_W \leq 0$. As $\rho|_W > 0$, we must have that $(\lambda + \frac{1}{\rho})(\overline{J} \times \overline{B}) = 0$, then either $\overline{J} \times \overline{B}|_W = \overline{0}$, and reduce to case (ii), or, using continuity, we can find $W'' \subset W$, W'' open, with $(\lambda + \frac{1}{\rho})|_{W''} = 0$.

$(i)'''$. $\rho|_U \leq 0$. If $\rho|_U = 0$, then, as ρ is analytic, $\rho = 0$, otherwise, shrinking U to W', we can assume that $\rho|_{W'} < 0$, then consider the tuple, $(-\rho', -\overline{J'}, -\overline{E'}, \overline{B'})$, for which (†) holds with $\lambda' = -\lambda$, and reduce to Case $(i)''$.

□

Remarks 3.6. *The case $(ii)'(b)$ requires further investigation to obtain a complete classification. As $\rho|_W > 0$ and ρ is analytic, $\rho \geq 0$ and is supported on W. We have that $\overline{f} = \overline{0}$, so a test particle in this field would move in a straight line at constant velocity. The electric and magnetic fields of such a particle are known, see* [3]*, but are not defined at the position of the particle. However, it seems reasonable to assert that the force exerted on the particle by its own field is zero. Moreover, removing a particle from the ensemble of moving charges would not significantly effect the total field. One might, therefore conjecture that the description of $(\rho, \overline{J})$ can be found by consideration of diffusions for the continuity equation, using the intuitive idea that $\overline{J} = \rho \overline{v}$. This is work in progress, see* [7]*.*

References

[1] Electromagnetic Fields and Waves, Dale Corson, Francoise Lorrain, Paul Lorrain, Freeman and Company, (1988).

[2] The Meaning of Rotation in the Special Theory of Relativity, Philip Franklin, Proceedings of the National Academy of Sciences, Volume 8, Number 9, (1922).

[3] Introduction to Electrodynamics, David Griffiths, Pearson, (2008).

[4] Fun with Fields, William Johnson, PhD thesis, University of California, Berkeley, (2016).

[5] The Elements of Classical Thermodynamics, A.B.Pippard, CUP, (1957).

[6] Some Arguments for the Wave Equation in Quantum Theory, Tristram de Piro, Open Research Journal of Mathematical Sciences, (2021).

[7] The Continuity Equation and Particle Motion, Tristram de Piro, work in progress, (2022).

[8] Introduction to Special Relativity, Wolfgang Rindler, Oxford Science Publications, (1991).

[9] The Relativistic Velocity Composition Paradox and the Thomas Rotation, Abraham Ungar, Foundations of Physics, Vol 19, No.11, (1989).

[10] Thomas Rotation and the Parametrization of the Lorentz Transformation Group, Abraham Ungar, Foundations of Physics Letters, (1988).

[11] Zariski Geometries, Geometry from the Logician's Point of View, Boris Zilber, London Mathematical Society, (2010).

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP