

A NONSTANDARD VERSION OF DIRICHLET'S THEOREM

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ABSTRACT. We prove a nonstandard version of Dirichlet's Theorem on the pointwise convergence of Fourier Series, a uniform convergence result was shown in [2]. We apply this result in [5].

Definition 0.1. We let $f : (-1,1) \rightarrow \mathcal{R}$ be analytic, and suppose that $f(-1)$ and $f(1)$ are defined, with upper and lower limits existing, $f^- = \lim_{x \rightarrow -1} f(x)$ and $f^+ = \lim_{x \rightarrow 1} f(x)$. We let $C_f = \frac{f^+ + f^-}{2}$. We let $V(\bar{V}_\eta) = {}^*[-1, 1)$, with the $*$ -topology generated by $\{[\frac{i}{\eta}, \frac{i+1}{\eta}), -\eta \leq i \leq \eta-1\}$, with $\eta \in {}^*\mathcal{N}$ prime. If $g \in V(\bar{V}_\eta)$, we use the forward derivative;

$$g^D(x) = \eta(g(x + \frac{1}{\eta}) - g(x)), \text{ for } x \in V(\bar{V}_\eta) \setminus [\frac{\eta-1}{\eta}, 1)$$

$$g^D(\frac{\eta-1}{\eta}) = \eta(g(-1) - g(\frac{\eta-1}{\eta}))$$

We use the notation in [3].

Lemma 0.2. With f as in the above definition, we have that f_η^D is finitely bounded on $[\frac{1}{\eta}, \frac{\eta-1}{\eta})$

Proof. For $x \in (-1, 1)$ and $\epsilon > 0$, with $x + \epsilon \in (-1, 1)$, we have, using Taylor's Theorem, that;

$$f(x + \epsilon) = f(x) + \epsilon f'(x) + \frac{\epsilon^2 f''(c)}{2}$$

and $c \in (x, x + \epsilon)$. It follows that;

$$|\frac{f(x+\epsilon)-f(x)}{\epsilon} - f'(x)| \leq |\epsilon f''(c)| \leq D\epsilon$$

where $D \in \mathcal{R}$ as f'' is bounded on $(-1, 1)$. Taking ϵ to be the infinitesimal $\frac{1}{\eta}$, and transferring this result, we obtain that;

$$f^D(x) \simeq (f')_\eta(x)$$

for $x \in [\frac{1}{\eta}, \frac{\eta-1}{\eta})$. As $(f')_\eta$ is bounded by assumption, we obtain the result. \square

Definition 0.3. For $g \in V(\overline{V}_\eta)$, and $k \in \mathcal{Z}_\eta$, with $|k| \leq \eta - 1$ we define the nonstandard partial Fourier sum, by;

$$S(k, g) = \frac{1}{2} * \sum_{|m| \leq k} \mathcal{F}_\eta(f)(m) \exp_\eta(\pi i x m)$$

and the Dirichlet kernel by;

$$D_k(x) = * \sum_{|m| \leq k} \exp_\eta(\pi i x m)$$

Lemma 0.4. We have that;

$$D_k(x) = \frac{\sin_\eta(\pi x(k + \frac{1}{2}))}{\sin_\eta(\frac{\pi x}{2})}, \text{ for } x \notin [0, \frac{1}{\eta}]$$

$$D_k(0) = 2k + 1$$

Proof. We have, when $x \neq 0$;

$$\begin{aligned} D_k(x) &= * \sum_{|m| \leq k} \exp_\eta(\pi i x m) \\ &= \exp_\eta(-\pi i x k) (* \sum_{m=0}^{2k} \exp_\eta(\pi i x m)) \\ &= \exp_\eta(-\pi i x k) \left(\frac{\exp_\eta(\pi i x)^{2k+1} - 1}{\exp_\eta(\pi i x) - 1} \right) \\ &= \frac{\exp_\eta(\frac{\pi i x}{2})(\exp_\eta(-\pi i x(k + \frac{1}{2})) - \exp_\eta(\pi i x(k + \frac{1}{2})))}{\exp_\eta(\frac{\pi i x}{2})(\exp_\eta(-\frac{\pi i x}{2}) - \exp_\eta(\frac{\pi i x}{2}))} \\ &= \frac{-2i \sin_\eta(\pi x(k + \frac{1}{2}))}{-2i \sin_\eta(\frac{\pi x}{2})} \\ &= \frac{\sin_\eta(\pi x(k + \frac{1}{2}))}{\sin_\eta(\frac{\pi x}{2})} \end{aligned}$$

The calculation when $x = 0$ is clear. \square

Lemma 0.5. For $g \in V(\overline{V}_\eta)$;

$$S(k, g) \simeq \frac{1}{2} \int_{[0, 1)_\eta} D_k(v)(g(x + v) + g(x - v)) d\mu_\eta(v)$$

Proof. Using the definitions above, the nonstandard definition of the convolution, and the symmetry $D_k(v) = D_k(-v)$, for $v \in [-1, 1)_\eta \setminus [-1, \frac{1-\eta}{\eta})$,

and Lemma 0.4, we for $k \in {}^* \mathcal{Z}$, with $|k| \leq \eta - 1$, that;

$$\begin{aligned}
S(k, g) &= \frac{1}{2} {}^* \sum_{|m| \leq k} \mathcal{F}_\eta(g)(m) \exp_\eta(\pi i x m) \\
&= \frac{1}{2} {}^* \sum_{|m| \leq k} \left(\int_{[-1, 1)_\eta} g(x) \exp_\eta(-\pi i x m) d\mu_\eta(x) \right) \exp_\eta(\pi i x m) \\
&= \frac{1}{2} \int_{[-1, 1)_\eta} g(y) \left({}^* \sum_{|m| \leq k} \exp_\eta(\pi i x m) \exp_\eta(-\pi i m y) \right) d\mu_\eta(y) \\
&= \frac{1}{2} \int_{[-1, 1)_\eta} D_k(x - y) g(y) d\mu_\eta(y) \\
&= \frac{1}{2} (D_k * g) \\
&= \frac{1}{2} (g * D_k) \\
&= \frac{1}{2} \int_{[-1, 1)_\eta} g(x - v) D_k(v) d\mu_\eta(v) \\
&= \frac{1}{2\eta} g(x + 1) D_k(-1) + \frac{1}{2} \int_{[\frac{1-\eta}{\eta}, 1)_\eta} g(x - v) D_k(v) d\mu_\eta(v) \\
&= \frac{1}{2\eta} g(x + 1) D_k(-1) + \frac{1}{2} \int_{[\frac{1-\eta}{\eta}, 1)_\eta} g(x + v) D_k(-v) d\mu_\eta(v) \\
&= \frac{1}{2\eta} g(x + 1) D_k(-1) + \frac{1}{2} \int_{[\frac{1-\eta}{\eta}, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \\
&= \frac{1}{2\eta} g(x + 1) D_k(-1) - \frac{1}{2\eta} g(x - 1) D_k(-1) + \frac{1}{2} \int_{[-1, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \\
&= \frac{(-1)^k (g(x+1) - g(x-1))}{2\eta} + \frac{1}{2} \int_{[-1, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \\
&\simeq \frac{1}{2} \int_{[-1, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v)
\end{aligned}$$

as g is bounded. It follows that;

$$\begin{aligned}
S(k, g) &\simeq \frac{1}{2} \left(\int_{[-1, 0)_\eta} g(x + v) D_k(v) d\mu_\eta(v) + \int_{[0, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \right) \\
&= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2} \left(\int_{[\frac{1-\eta}{\eta}, 0)_\eta} g(x + v) D_k(v) d\mu_\eta(v) + \int_{[0, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \right) \\
&= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2} \left(\int_{(0, \frac{\eta-1}{\eta}]_\eta} g(x - v) D_k(-v) d\mu_\eta(v) + \int_{[0, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \right) \\
&= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2} \left(\int_{(0, 1)_\eta} g(x - v) D_k(v) d\mu_\eta(v) + \int_{[0, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \right) \\
&= \frac{D_k(-1)g(x-1)}{2\eta} - \frac{D_k(0)g(x)}{2\eta} + \frac{1}{2} \left(\int_{[0, 1)_\eta} g(x - v) D_k(v) d\mu_\eta(v) + \int_{[0, 1)_\eta} g(x + v) D_k(v) d\mu_\eta(v) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^k g(x-1)}{2\eta} - \frac{(2k+1)g(x)}{2\eta} + \frac{1}{2} \left(\int_{[0,1)_\eta} g(x-v) D_k(v) d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v) D_k(v) d\mu_\eta(v) \right) \\
&= \frac{(-1)^k g(x-1) - (2k+1)g(x)}{2\eta} + \frac{1}{2} \left(\int_{[0,1)_\eta} (g(x+v) + g(x-v)) D_k(v) d\mu_\eta(v) \right) \\
&\simeq \frac{1}{2} \left(\int_{[0,1)_\eta} (g(x+v) + g(x-v)) D_k(v) d\mu_\eta(v) \right)
\end{aligned}$$

□

Lemma 0.6. *If f as above is analytic on $(-1, 1)$, then for $r \in \mathcal{R}_{>0}$, there exists $N_r \in {}^*\mathcal{N}$, with $N_r = \eta - 1$, such that;*

$$\int_{[r,1)_\eta} f(x) D_k(x) d\mu_\eta(x) \simeq 0, \text{ for } 0 < k \leq N_r, k \text{ infinite.}$$

Proof. We have that;

$$\begin{aligned}
&\int_{[r,1)_\eta} f(x) D_k(x) d\mu_\eta(x) \\
&= \int_{[r,1)_\eta} \frac{f(x) \sin(\pi(k + \frac{1}{2})x)}{\sin \frac{\pi x}{2}} d\mu_\eta(x)
\end{aligned}$$

A simple calculation shows that $\frac{1}{\sin(\frac{\pi x}{2})}$ is analytic on $(r, 1)$ for $0 < r < 1$, hence $\frac{f}{\sin(\frac{\pi x}{2})}$ is analytic on $(r, 1)$. Using the alternation method and Lemma 6 from the paper [4], we obtain the result, noting the spacing of the zeroes of $\sin(\pi(k + \frac{1}{2})x)$. Namely, we have that;

$$\sin(\pi(k + \frac{1}{2})x) = 0$$

$$\text{iff } \pi(k + \frac{1}{2})x = \frac{\pi}{2} + n\pi$$

$$\text{iff } x = \frac{n + \frac{1}{2}}{k + \frac{1}{2}}$$

for $n \in \mathcal{Z}$. The spacing is then;

$$\begin{aligned}
&\frac{n+1+\frac{1}{2}}{k+\frac{1}{2}} - \frac{n+\frac{1}{2}}{k+\frac{1}{2}} \\
&= \frac{2}{2k+1}
\end{aligned}$$

$$\text{We have that } \frac{2}{2k+1} > \frac{1}{\eta}$$

$$\text{iff } k < \eta - \frac{1}{2}$$

so we can take $N_r = \eta - 1$, as required.

□

Definition 0.7. We define the Fourier kernel by;

$$F_k(x) = \frac{2\sin_\eta(\pi x(k + \frac{1}{2}))}{\pi x_\eta}, \text{ for } x \in [-1, 1)_\eta \setminus [0, \frac{1}{\eta})$$

$$F_k(0) = 2k + 1$$

Lemma 0.8. For $g \in V([0, 1)_\eta)$, with g analytic, there exists $N \in {}^*\mathcal{N}$, infinite, $N = \eta - 1$, such that for all $k \in {}^*\mathcal{N}$, with k infinite and $0 < k \leq N$, we have that;

$$\int_{[0,1)_\eta} g(x)F_k(x)d\mu_\eta(x) \simeq \int_{[0,1)_\eta} g(x)D_k(x)d\mu_\eta(x)$$

Proof. We have that $\frac{1}{\sin_\eta(\frac{\pi x}{2})} - \frac{2}{\pi x}$ is analytic on $(0, 1)$, hence, so is $h = \frac{g}{\sin_\eta(\frac{\pi x}{2})} - \frac{2g}{\pi x}$. Moreover;

$$\begin{aligned} & \int_{[0,1)_\eta} g(x)D_k(x)d\mu_\eta(x) \\ &= \int_{[0,1)_\eta} g(x)(D_k - F_k + F_k)d\mu_\eta(x) \\ &= \int_{[0,1)_\eta} g(x)(D_k - F_k)d\mu_\eta(x) \\ &+ \int_{[0,1)_\eta} g(x)F_kd\mu_\eta(x) \\ &= \int_{[0,1)_\eta} h(x)\sin_\eta(\pi(k + \frac{1}{2})x)d\mu_\eta(x) \\ &+ \int_{[0,1)_\eta} g(x)F_kd\mu_\eta(x) \\ &\simeq \int_{[0,1)_\eta} g(x)F_kd\mu_\eta(x) \end{aligned}$$

using the argument of Lemma 0.6, and with the same choice of N .

□

Lemma 0.9. For $g \in V(\overline{V}_\eta)$ analytic on $[-1, 1)_\eta$ and $x \in [-1, 1)_\eta$, there exists an infinite $N = \eta - 1$, as in Lemma 0.8, independent of x , such that, for infinite $0 < k \leq N$

$$S(k, g)(x) \simeq \frac{1}{2} \int_{[0,1)_\eta} F_k(v)(g(x + v) + g(x - v))d\mu_\eta(v)$$

Proof. Using Lemma 0.5, we have that;

$$S(k, g)(x) \simeq \frac{1}{2} \int_{[0,1]_\eta} D_k(v)(g(x+v) + g(x-v))d\mu_\eta(v)$$

where $(g(x+v) + g(x-v))$ is the transfer of an analytic function on $[0, 1)_\eta$. Using Lemma 0.8, there exists $N \in {}^*\mathcal{N}$, $N = \eta - 1$, such that;

$$\begin{aligned} & \frac{1}{2} \int_{[0,1]_\eta} D_k(v)(g(x+v) + g(x-v))d\mu_\eta(v) \\ & \simeq \frac{1}{2} \int_{[0,1]_\eta} F_k(v)(g(x+v) + g(x-v))d\mu_\eta(v) \end{aligned}$$

for all k infinite, with $0 < k \leq N$. □

Lemma 0.10. *For infinite k , with $0 < k \leq N$, we have that*

$$\begin{aligned} & \frac{1}{2} \int_{[0,1]_\eta} F_k(v)(g(x+v) + g(x-v))d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v) - g_x^+)}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^+}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x-v) - g_x^-)}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^-}{v} d\mu_\eta(v) \end{aligned}$$

Proof. Using the definition of $F_k(v)$ and rearranging, we have that;

$$\begin{aligned} & \frac{1}{2} \int_{[0,1]_\eta} F_k(v)(g(x+v) + g(x-v))d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v) - g_x^+ + g_x^+ + g(x-v) - g_x^- + g_x^-)}{v} d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v) - g_x^+)}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^+}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x-v) - g_x^-)}{v} d\mu_\eta(v) \\ & \quad + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^-}{v} d\mu_\eta(v) \end{aligned}$$

as required. □

Lemma 0.11. *Suppose that $g_x^+ = g_x^-$, then there exists an infinite $M \in {}^*\mathcal{R}$, $M = \sqrt{\eta}$, such that for infinite $k \in {}^*\mathcal{N}$, with $0 < k \leq M$;*

$$\begin{aligned} & \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^+}{v} d\mu_\eta(v) \\ & + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^-}{v} d\mu_\eta(v) \\ & \simeq \frac{(g_x^+ + g_x^-)}{2} \end{aligned}$$

Proof. Using the alternation argument, with this choice of M , noting $\sqrt{\eta} < \eta - 1$, for η infinite, we have;

$$\begin{aligned} & \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^+}{v} d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{g_x^+}{2} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{v} d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\frac{\pi v}{2}} d\mu_\eta(v) \\ & \simeq \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \end{aligned}$$

Similarly;

$$\begin{aligned} & \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^-}{v} d\mu_\eta(v) \\ & \simeq \frac{2}{\pi} \frac{\pi g_x^-}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \end{aligned}$$

It follows that, by the choice of M , noting that $\frac{2M+1}{\eta} \simeq 0$;

$$\begin{aligned} & \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^+}{v} d\mu_\eta(v) \\ & + \frac{2}{\pi} \frac{1}{2} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{g_x^-}{v} d\mu_\eta(v) \\ & \simeq \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \\ & + \frac{2}{\pi} \frac{\pi g_x^-}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \\ & + \frac{2}{\pi} \frac{\pi g_x^-}{4} \int_{(-\frac{\eta-1}{\eta}, 0]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})-v)}{\sin_\eta(\frac{\pi-v}{2})} d\mu_\eta(v) \\ & = \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[0,1]_\eta} \frac{\sin_\eta(\pi(k + \frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\pi} \frac{\pi g_x^-}{4} \int_{[-1,0)_\eta} \frac{\sin_\eta(\pi(k+\frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) + \frac{2}{\pi} \frac{\pi g_x^-}{4} \frac{2k+1}{\eta} - \frac{(-1)^k}{\eta} \\
& \simeq \frac{2}{\pi} \frac{\pi g_x^+}{4} \int_{[-1,1)_\eta} \frac{\sin_\eta(\pi(k+\frac{1}{2})v)}{\sin_\eta(\frac{\pi v}{2})} d\mu_\eta(v) \\
& = \frac{2}{\pi} \frac{2\pi g_x^+}{4} \\
& = \frac{2}{\pi} \frac{\pi g_x^+}{2} \\
& = g_x^+ = \frac{g_x^+ + g_x^-}{2}
\end{aligned}$$

as $\int_{[-1,1)_\eta} F_k(x) d\mu_\eta(x) = 2$, for all k .

□

Lemma 0.12. *For infinite k , with $k \leq M$, and $g_x^+ = g_x^-$, we have that;*

$$\begin{aligned}
S_k(g) & \simeq g_x^+ \\
& + \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v) - g_x^+)}{v} d\mu_\eta(v) \\
& + \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x-v) - g_x^-)}{v} d\mu_\eta(v)
\end{aligned}$$

Proof. This follows by combining the previous Lemmas 0.9, 0.10 and 0.11. □

Lemma 0.13. *If g is analytic on $(-1, 1)$, with g right differentiable at -1 , and g left differentiable at 1 , and $g_1^- = g_{-1}^+$, then, there exists N infinite, $N = \eta - 1$, such that for all $0 < k \leq M$, if $x \in [-1, 1)_\eta$, with ${}^\circ x \subset \{-1, 1\}$ and k infinite;*

$$\begin{aligned}
& \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v) - g_x^+)}{v} d\mu_\eta(v) \\
& \simeq \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x-v) - g_x^-)}{v} d\mu_\eta(v) \\
& \simeq 0
\end{aligned}$$

Proof. Suppose that $x = 1$, then, as g is left differentiable at 1 , for $v \in [0, 1)$ the standard function $\frac{g(x+v) - g_x^+}{v}$ is continuous with $g'^+, (0)$ being the definition at 0 . It follows that the transfer, where v can range over $[0, 1)_\eta$ is S -continuous, $(*)$. This is due to the usual argument, for a continuous function f . If $f(a) = b$, then if $a' \in \mathcal{V}_a$, $|a - a'| < \epsilon$ for

all real $\epsilon > 0$. By transfer of continuity, $|f(a) - f(a')| < \delta$, for all real $\delta > 0$, in particular $f(a') \simeq b$. Now if $a' \simeq a''$, then, as $\{a', a''\}$ are finite $a' \simeq a'' \simeq st(a') = a$, and the result follows by transitivity of \simeq . By (*) it follows that, for $v \in \mathcal{V}_0$, $\frac{g(x+v)-g_x^+}{v} \simeq g'^+(0)$, so bounded, (**). For $v_0 \in \mathcal{R}_{>0}$, we have that, $\frac{(g(x+v)-g_x^+)}{v}$ is analytic on $[0, 1)_\eta \setminus ([0, v_0)_\eta$. It follows, by (**), that, for infinite κ ;

$$\begin{aligned} & \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \\ & \simeq \frac{1}{\pi} \int_{[0,1)_\eta \setminus ([0, \frac{1}{\kappa})} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \end{aligned}$$

In particular, for a given $\epsilon_r = \frac{1}{r}$, with $r \in \mathcal{N}$, using underflow, uniformly in k , as $|\sin_\eta(\pi(k + \frac{1}{2})v) \leq 1|$, there exists $n_r \in \mathcal{N}$, for which;

$$\begin{aligned} & \left| \frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \right. \\ & \left. - \frac{1}{\pi} \int_{[0,1)_\eta \setminus ([0, \frac{1}{n_r})} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \right| < \epsilon_r \end{aligned}$$

Then, using the alternation argument above, we can find an infinite $N_{1,r} = \eta - 1$, such that for $0 < k \leq N_{1,r}$, k infinite;

$$\frac{1}{\pi} \int_{[\frac{1}{n_r}, 1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \simeq 0$$

Extending the sequence $\{N_{1,r} : r \in \mathcal{N}\}$ by countable comprehension, we can for infinite $r' \in {}^*\mathcal{N}$, find an infinite $N'_1 = \eta - 1$ such that;

(i). For $0 < k \leq N'_1 = \eta - 1$;

$$\frac{1}{\pi} \int_{[\frac{1}{n_{r'}}, 1 - \frac{1}{n_{r'}})_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \simeq 0$$

(ii). $|\frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v)$

$$- \frac{1}{\pi} \int_{[0,1)_\eta \setminus ([0, \frac{1}{n_{r'}})} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v)| < \epsilon_{r'} \simeq 0$$

It follows that;

$$\frac{1}{\pi} \int_{[0,1)_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x+v)-g_x^+)}{v} d\mu_\eta(v) \simeq 0$$

A similar argument produces $N'_2 = \eta - 1$ to show that;

$$\frac{1}{\pi} \int_{[0,1]_\eta} \sin_\eta(\pi(k + \frac{1}{2})v) \frac{(g(x-v) - g_x^-)}{v} d\mu_\eta(v) \simeq 0$$

We can then take $N = \min(N'_1, N'_2)$, $N = \eta - 1$. The proofs for $x = -1$, and ${}^\circ x \subset \{-1, 1\}$ are similar and left to the reader. \square

Lemma 0.14. *For g analytic on $(0, 1)$, g finite, left and right differentiable at $\{1, -1\}$, with the limits $g_x^+ = g_x^-$ and ${}^\circ x \subset \{-1, 1\}$ there exists $P \in {}^*\mathcal{N}$, with P infinite, $P = \sqrt{\eta}$, such that for all $0 < k \leq P$, k infinite;*

$$S_k(g) \simeq g_x^+ = g_x^- = g_x$$

If g is continuous at 1, then $S_k(g) \simeq g(x)$.

Proof. The proof is clear by the previous Lemmas 0.12 and 0.13, \square

Definition 0.15. *We define the standard sawtooth function on $[-1, 1]$ by;*

$$S(x) = x$$

and the Fourier series $S(S)$ by;

$$\lim_{k \rightarrow \infty} S_m(S)$$

$$\text{where } S_k(S) = \frac{1}{2} \sum_{|m| \leq k} \mathcal{F}(S)(k) e^{\pi i k x}$$

and the nonstandard sawtooth function on $[-1, 1]_\eta$ by;

$$S(x) = x_\eta$$

Lemma 0.16. *For $m \in \mathcal{Z}$, we have that;*

$$\mathcal{F}(S)(m) = \frac{2i(-1)^{|m|}}{\pi m}$$

$$\text{and } S(S) = \frac{2}{\pi} \sum_{m>0} \frac{(-1)^{m+1} \sin(\pi m x)}{m}$$

Proof. We compute, for $m \neq 0$;

$$\mathcal{F}(S)(m) = \int_{-1}^1 S(x) e^{-\pi i m x} dx$$

$$\begin{aligned}
&= \int_{-1}^1 x e^{-\pi i m x} dx \\
&= \left[\frac{x e^{-\pi i m x}}{-\pi i m} \right]_{-1}^1 + \frac{1}{\pi i m} \int_{-1}^1 e^{-\pi i m x} dx \\
&= \frac{e^{-\pi i m} + e^{\pi i m}}{-\pi i m} + \frac{1}{\pi i m} \left[e^{-\frac{\pi i m x}{-1}} \right]_{-1}^1 \\
&= \frac{2 \cos(\pi m)}{-\pi m i} + \frac{1}{\pi^2 m^2} (e^{\pi i m} - e^{-\pi i m}) \\
&= \frac{2(-1)^m}{-\pi m i} \\
&= \frac{2i(-1)^m}{\pi m}
\end{aligned}$$

$$\text{We have } \mathcal{F}(S)(0) = \int_{-1}^1 x dx = 0$$

Then;

$$\begin{aligned}
S_k(S) &= \frac{1}{2} \sum_{|m| \leq k, m \neq 0} \frac{2i(-1)^m}{\pi m} e^{\pi i m x} \\
&= \frac{i}{\pi} \left(\sum_{|m| \leq k, m \neq 0} \frac{\cos(\pi m x)(-1)^m}{m} + \frac{i \sin(\pi m x)(-1)^m}{m} \right) \\
&= \frac{2}{\pi} \sum_{0 < m \leq k} \frac{\sin(\pi m x)(-1)^{m+1}}{m}
\end{aligned}$$

so that;

$$S(S) = \frac{2}{\pi} \sum_{m > 0} \frac{(-1)^{m+1} \sin(\pi m x)}{m}$$

□

Lemma 0.17. *Let f be analytic on $(-1, 1)$, such that the left and right derivatives exist at -1 and 1 respectively. Then there exists g analytic on $(-1, 1)$ and constants λ, μ, ν such that;*

$$g = \lambda f + \mu S + \nu$$

$$\text{with } g_1^+ = g_{-1}^-$$

A similar result holds for f_η , replacing g by g_η and S by S_η .

Proof. We have that $S_1^+ = 1$ and $S_{-1}^- = -1$. Suppose that $f_1^+ + f_1^- \neq 0$ and $f_1^+ - f_1^- \neq 0$, (*), then we solve the equations;

$$1 = \lambda f_1^+ + \mu S_1^+ = \lambda f_1^+ + \mu = \lambda f_{-1}^- + \mu S_{-1}^- = \lambda f_{-1}^- - \mu$$

to obtain that;

$$\lambda = \frac{2}{f_1^+ + f_1^-} \neq 0$$

$$\mu = \frac{f_1^- - f_1^+}{f_1^+ + f_1^-} \neq 0$$

Setting $g = \lambda f + \mu S$ gives the result. For the general case, if $f_1^+ = f_{-1}^-$, we can take $g = f$. Otherwise, we can find a constant c , such that $(*)$ is satisfied for $f + c$. Applying the previous calculation, we can find g such that $g = \lambda(f + c) + \mu S$. Taking $\nu = \lambda c + \mu$ gives the result. The final claim is clear, taking the measurable versions. \square

Lemma 0.18. *For $\{f, g\} \subset V([-1, 1)_\eta)$, we have that;*

$$\int_{[-1, 1)_\eta} f g^D d\mu_\eta = - \int_{[-1, 1)_\eta} f^D g^{lsh} d\mu_\eta$$

Proof. We have $(fg)^D = f^D g^{lsh} + fg^D$, so that;

$$\begin{aligned} 0 &= \int_{[-1, \frac{\eta-1}{\eta}]} (fg)^D d\mu_\eta \\ &= \int_{[-1, \frac{\eta-1}{\eta}]} f^D g^{lsh} + fg^D d\mu_\eta \\ &= \int_{[-1, \frac{\eta-1}{\eta}]} f^D g^{lsh} d\mu_\eta + \int_{[-1, \frac{\eta-2}{\eta}]} fg^D d\mu_\eta \end{aligned}$$

Rearranging, we obtain that;

$$\int_{[-1, \frac{\eta-1}{\eta}]} fg^D d\mu_\eta = - \int_{[-1, \frac{\eta-1}{\eta}]} f^D g^{lsh} d\mu_\eta$$

\square

Lemma 0.19. *For $m \in \overline{\mathcal{Z}_\eta}$, we have that;*

$$\mathcal{F}_\eta(S_\eta)(m) = \frac{2(-1)^m}{\theta_\eta(m)}$$

$$\text{where } \theta_\eta(m) = \eta(\exp_\eta(\frac{-\pi m}{\eta}) - 1)$$

Moreover, for $m \in \overline{\mathcal{Z}_\eta}$ finite;

$${}^\circ\mathcal{F}_\eta(S_\eta)(m) = \mathcal{F}(S)(m)$$

Finally, for $x \in [-1, 1)_\eta$, with $^\circ x \in \{-1, 1\}$ there exists an infinite T , such that for $0 < k \leq T$, k infinite, $S_k(S_\eta)(x) \simeq 0$

Proof. Let $f(x) = x_\eta$, $h(x) = \exp_\eta(-\pi i x m)$, $h^D(x) = \theta_\eta(m) \exp_\eta(-\pi i x m)$, so if $g(x) = \frac{h(x)}{\theta_\eta(m)} = \frac{\exp_\eta(-\pi i x m)}{\theta_\eta(m)}$, then $g^D(x) = \exp_\eta(-\pi i x m)$ and $g^{lsh}(x) = \frac{\phi_\eta(m) \exp_\eta(-\pi i x m)}{\theta_\eta(m)}$, where $\phi_\eta(m) = \exp_\eta(-\frac{\pi i m}{\eta})$. We have that $f^D(x) = x_\eta^D = 1 + r \delta_{\frac{\eta-1}{\eta}}$, where $r = x_\eta^D(\frac{\eta-1}{\eta}) - 1 = \eta(x_\eta(-1) - x_\eta) - 1 = \eta(-1 - \frac{\eta-1}{\eta}) - 1 = -2\eta$, so that $x_\eta^D = 1 - 2\eta \delta_{\frac{\eta-1}{\eta}}$.

Using the previous lemma, we obtain that;

$$\begin{aligned}
& \mathcal{F}_\eta(S_\eta)(m) \\
&= \int_{[-1, 1)_\eta} x_\eta \exp_\eta(-\pi i x m) d\mu_\eta \\
& \int_{[-1, 1)_\eta} f g^D d\mu_\eta \\
&= - \int_{[-1, 1)_\eta} f^D g^{lsh} d\mu_\eta \\
&= - \int_{[-1, 1)_\eta} (1 - 2\eta \delta_{\frac{\eta-1}{\eta}}) \left(\frac{\phi_\eta(m) \exp_\eta(-\pi i x m)}{\theta_\eta(m)} \right) d\mu_\eta \\
&= - \frac{\phi_\eta(m)}{\theta_\eta(m)} \int_{[-1, 1)_\eta} \exp_\eta(-\pi i x m) d\mu_\eta(x) - 2\eta \frac{\phi_\eta(m)}{\theta_\eta(m)} \int_{[-1, 1)_\eta} \delta_{\frac{\eta-1}{\eta}} \exp_\eta(-\pi i x m) d\mu_\eta \\
&= -2\eta \frac{1}{\eta} \frac{\phi_\eta(m)}{\theta_\eta(m)} \exp_\eta(-\pi i \frac{\eta-1}{\eta} m) \\
&= -2 \frac{\phi_\eta(m)}{\theta_\eta(m)} \phi_\eta(-m) \exp_\eta(-\pi i m) \\
&= \frac{2(-1)^m}{\theta_\eta(m)}
\end{aligned}$$

as required. For the final part, either use the fact that for m finite $x_\eta \exp_\eta(-\pi i x m)$ is S -continuous and S -integrable, or check directly, for finite m that;

$$\theta_\eta(m) = \exp_\eta(-\pi i x m)^D(0) \simeq (\exp(-\pi i x m))'_\eta(0) = -\pi i m$$

as $\exp(-\pi i x m)$ is analytic.

For x as in the statement of the lemma, we have already checked that, for finite m , $^\circ \mathcal{F}_\eta(S_\eta)(m) = \mathcal{F}(S)(m)$, in particular, for finite k ;

$$S_k(f)(x) \simeq S_k(S)(\circ x) = 0$$

$$|S_k(f)(x)| < \frac{1}{k}$$

so that, by overflow there exists an infinite T , such that for $k \leq T$, k infinite, $|S_k(S_\eta(x))| < \frac{1}{k} \simeq 0$

□

Lemma 0.20. *For h analytic on $(0,1)$, h finite, left and right differentiable at $\{1, -1\}$, with the limits h_x^+ and h_x^- and $\circ x \subset \{-1, 1\}$ there exists $Q \in {}^*\mathcal{N}$, with Q infinite, and $Q \leq \sqrt{\eta}$, such that for all $0 < k \leq Q$, k infinite;*

$$S_k(h) \simeq \frac{g_x^+ + g_x^-}{2}$$

Proof. Consider the case when $\circ x = 1$. If $h_x^+ = h_x^-$, the result is proved in Lemma 0.14. Otherwise, using Lemma 0.17, we can find λ, μ, ν such that $g = \lambda h + \mu S_\eta + \nu$, with $g_x^+ = g_x^-$. By Lemma 0.14, for k infinite, with $0 < k \leq Q = \min(T, P)$, we obtain;

$$\begin{aligned} & S_k(\lambda h + \mu S_\eta + \nu)(x) \\ &= \lambda S_k(h)(x) + \mu S_k(S_\eta)(x) + \nu \\ &= \lambda S_k(h)(x) + \nu \\ &= S_k(g)(x) \simeq g_x^+. \end{aligned}$$

Rearranging, with $\lambda \neq 0$, $S_k(h)(x+) = \frac{g_x^{(+)-\nu}}{\lambda}$. In the generic case, when $\nu = 0$, we obtain that;

$$\begin{aligned} S_k(h)(x) &= \frac{g_x^{(+)}}{\lambda} \\ &= \frac{g_x(f_x^+ + f_x^-)}{2} \\ &= \frac{(\lambda f_x^+ + \mu)(f_x^+ + f_x^-)}{2} \\ &= \frac{2f_x^+ + (f_x^- - f_x^+) f_x^+ + f_x^-}{f_x^+ + f_x^-} \\ &= \frac{f_x^+ + f_x^-}{2} \end{aligned}$$

The case, when $\nu \neq 0$ is similar, and left to the reader.

□

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