

SCHRODINGER'S EQUATION AND RELATED CHARGE DENSITY

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ABSTRACT. We determine $|\Psi|^2$, in terms of standing waves, for a solution Ψ to Schrodinger's equation with $\Psi_0 \in C^\infty([-1, 1])$.

Definition 0.1. We let $C^\infty([-1, 1])$. $C(\mathcal{R})$ and $C^\infty(\mathcal{R})$ have their conventional meanings. We let $T = [-1, 1] \times \mathcal{R}_{\geq 0}$ and let $T^0 = (-1, 1) \times \mathcal{R}_{>0}$ denote its interior. We let $C(T) = \{G, \text{ continuous on } T, G_t \in C([-1, 1]), \text{ for } t \in \mathcal{R}_{\geq 0}\}$, $\mathcal{S}(T) = \{G \in C(T) : G_t \in C^{infy}([-1, 1]), \text{ for } t \in \mathcal{R}_{\geq 0}, G|T^0 \in C^\infty(T^0)\}$. If $h \in C^{infy}([-1, 1])$, and $m \in \mathcal{Z}$, we define its Fourier coefficient by;

$$\mathcal{F}(h)(m) = \int_{-\infty}^{\infty} h(x)e^{-\pi imx} dx$$

If $g \in \mathcal{S}(T)$, we define its Fourier coefficient in space by;

$$\mathcal{F}(g)(m, t) = \int_{-\infty}^{\infty} g(x, t)e^{-\pi imx} dx$$

for $m \in \mathcal{Z}$ and $t \in \mathcal{R}_{\geq 0}$.

We recall the facts that $\mathcal{F} : C^\infty([-1, 1]) \rightarrow V(\mathcal{Z})$ satisfies the following inversion theorem;

If $h \in C^\infty([-1, 1])$, then;

$$h(x) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \mathcal{F}(h)(m)e^{\pi ixm}$$

for $x \in [-1, 1]$, and, a similar result holds for $g \in \mathcal{S}(T)$.

If $h \in C^{infy}([-1, 1])$, and $m \in \mathcal{Z}_{\geq 1}$, we define its cosine Fourier coefficient by;

$$\mathcal{C}(h)(m) = \int_{-1}^1 h(x)\cos\pi mx dx$$

and its sine Fourier coefficient by;

$$\mathcal{S}(h)(m) = \int_{-1}^1 h(x) \sin \pi m x dx$$

and the constant coefficient by

$$\mathcal{C}(h)(0) = \frac{1}{2} \int_{-1}^1 h(x) dx$$

If $g \in \mathcal{S}(T)$, $m \in \mathcal{Z}_{\geq 1}$, and $t \in \mathcal{R}_{\geq 0}$, we define its cosine Fourier coefficient, sine Fourier coefficient and constant coefficient in space by;

$$\mathcal{C}(g)(m, t) = \int_{-1}^1 g(x, t) \cos \pi m x dx$$

$$\mathcal{S}(g)(m, t) = \int_{-1}^1 g(x, t) \sin \pi m x dx$$

$$\mathcal{C}(g)(0, t) = \frac{1}{2} \int_{-1}^1 g(x, t) dx$$

We recall the following fact from [?];

If $h \in C^\infty(S^1)$, then the series;

$$\sum_{m \geq 0} c_m \cos(\pi x m) + \sum_{m \geq 1} d_m \sin(\pi x m)$$

where $c_m = \int_{-1}^1 h(x) \cos(\pi x m) dx$, $m \geq 1$, $c_0 = \frac{1}{2} \int_{-1}^1 h(x) dx$ and $d_m = \int_{-1}^1 h(x) \sin(\pi x m) dx$, $m \geq 1$, converges uniformly to h on S^1 .

Lemma 0.2. *If $g \in C^\infty([-1, 1])$, there exists a unique $G \in \mathcal{S}(T)$, with $G_0 = g$, such that G satisfies the simplified Schrodinger equation;*

$$\frac{\partial G}{\partial t} = i \frac{\partial^2 G}{\partial x^2} \quad (*)$$

on T^0 .

Proof. Suppose, first, there exists such a solution G , then, applying \mathcal{F} to (*), we must have that;

$$\mathcal{F}\left(\frac{\partial G}{\partial t} - i \frac{\partial^2 G}{\partial x^2}\right)(m, t) = 0 \quad (t > 0, m \in \mathcal{Z})$$

Differentiating under the integral sign, we have that;

$$\mathcal{F}\left(\frac{\partial G}{\partial t}\right) = \frac{\partial \mathcal{F}(G)}{\partial t}(m, t), \text{ for } t > 0, m \in \mathcal{Z}$$

Integrating by parts and using the fact that $G_t \in C^\infty([-1, 1])$, for $t > 0$, we have that;

$$\mathcal{F}\frac{\partial^2 G}{\partial x^2} = -\pi^2 m^2 \mathcal{F}(G)(m, t), \text{ for } t > 0, m \in \mathcal{Z}$$

We thus obtain the sequence of ordinary differential equations, indexed by $m \in \mathcal{Z}$;

$$\frac{\partial \mathcal{F}(G)}{\partial t} + i\pi^2 m^2 \mathcal{F}(G)(m, t) = 0 \quad (t > 0)$$

As $G \in C(T)$, $G_t \rightarrow G_0$ pointwise, as $t \rightarrow 0$, and, using the Dominated Convergence Theorem, $\mathcal{F}(G)(m, t) \rightarrow \mathcal{F}(G)(m, 0)$, as $t \rightarrow 0$, for each $m \in \mathcal{Z}$. By Picard's and Peano's Theorem, see [?], Chapter 4, this system of equations has a unique continuous solution, given by;

$$\mathcal{F}(G)(m, t) = e^{-i\pi^2 m^2 t} \mathcal{F}(g)(m) \quad (t \geq 0)$$

As $G_t \in C^\infty([-1, 1])$, we have, by the inversion theorem, that, for $x \in [-1, 1]$;

$$G_t(x) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \mathcal{F}(G)(m, t) e^{\pi i x m}$$

and, in particular, G_t is determined by its Fourier transform, for $t > 0$. It follows that G is a unique solution.

If $g \in C^\infty([-1, 1])$, its Fourier series transform $\mathcal{F}(g) \in C^\infty([-1, 1])$, hence;

$$e^{-i\pi^2 m^2 t} \mathcal{F}(g)(m) e^{2\pi i m x} \in C^{infy}([-1, 1])$$

for $t > 0$, $m \in \mathcal{Z}$. It follows that G defined by;

$$G(x, t) = \sum_{m=-\infty}^{\infty} e^{-i\pi^2 m^2 t} \mathcal{F}(g)(m) e^{2\pi i m x} dx$$

is a solution of the required form. □

Lemma 0.3. *If $G \in \mathcal{S}(T)$, with $G_0 = g$, and $g \in C^\infty([-1, 1])$ as above, then $Re(G)$ and $Im(G)$ satisfy the same differential equations;*

$$\frac{\partial \operatorname{Re}(G)}{\partial t} = -\frac{\partial^4 \operatorname{Re}(G)}{\partial x^4}$$

$$\frac{\partial \operatorname{Im}(G)}{\partial t} = -\frac{\partial^4 \operatorname{Im}(G)}{\partial x^4}$$

Moreover, explicit solutions are given by;

$$\begin{aligned} u(x, t) &= (\mathcal{C}(u))(0, 0) + (\mathcal{C}(u_s))(0, 0)t \\ &+ \sum_{m \geq 1} \pi m \left(\frac{(\mathcal{C}(u))(m, 0)}{\pi m} \cos(\pi^2 m^2 t) + \frac{(\mathcal{C}(u_s))(m, 0)}{\pi m} \sin(\pi^2 m^2 t) \cos(\pi x m) \right) \\ &+ \sum_{m \geq 1} \pi m \left(\frac{(\mathcal{S}(u))(m, 0)}{\pi m} \cos(\pi^2 m^2 t) + \frac{(\mathcal{S}(u_s))(m, 0)}{\pi m} \sin(\pi^2 m^2 t) \sin(\pi x m) \right) \end{aligned}$$

and $v(x, t)$ is similar.

In particular;

$$|G|^2 = u^2 + v^2$$

where $\{u, v\}$ are as above.

Proof. Let $u(x, t) = \operatorname{Re}(G)(x, t)$, $v(x, t) = \operatorname{Im}(G)(x, t)$, then, as;

$$\frac{\partial G}{\partial t} = i \frac{\partial^2 G}{\partial x^2} \quad (*)$$

and equating real and imaginary parts, we have that;

$$\frac{\partial u + iv}{\partial t} = i \frac{\partial^2 (u + iv)}{\partial x^2}$$

and;

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 (v)}{\partial x^2}, (i)$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 (u)}{\partial x^2}, (ii), (**)$$

It follows, applying $\frac{\partial}{\partial t}$ to (i), and $\frac{\partial^2}{\partial x^2}$ to (ii), then, $\frac{\partial}{\partial t}$ to (ii), and $\frac{\partial^2}{\partial x^2}$ to (i) that;

$$u_{tt} = -v_{xxt}$$

$$v_{xxt} = u_{xxxx}$$

$$v_{tt} = u_{xxt}$$

$$u_{xxt} = -v_{xxxx}$$

so, $u_{tt} = -u_{xxxx}$, (iii) and $v_{tt} = -v_{xxxx}$, (iv).

This gives the first part. Now, apply the operators $\{\mathcal{C}, \mathcal{S}\}$ to (iii) and obtain;

$$\mathcal{C}(u_{ss} + u_{xxxx})(m, t) = 0 \quad (t > 0, m \in \mathcal{Z}_{\geq 0})$$

$$\mathcal{S}(u_{ss} + u_{xxxx})(m, t) = 0 \quad (t > 0, m \in \mathcal{Z}_{> 0})$$

Differentiating under the integral sign again, integrating by parts and using the fact that $u^t \in C^\infty([-1, 1])$, for $t > 0$, we have that, for $m \in \mathcal{Z}_{\geq 1}$;

$$(\mathcal{C}(u))_{ss}(m, t) = -\pi^4 m^4 (\mathcal{C}(u))(m, t)$$

$$(\mathcal{C}(u))_{ss}(0, t) = 0$$

for $m \in \mathcal{Z}_{\geq 1}$;

$$(\mathcal{S}(u))_{ss}(m, t) = -\pi^4 m^4 (\mathcal{S}(u))(m, t), \quad (\dagger)$$

with explicit solution, $m \geq 1$, given by;

$$(\mathcal{C}(u))(m, t) = A_m \cos(\pi^2 m^2 t) + B_m \sin(\pi^2 m^2 t)$$

$$(\mathcal{S}(u))(m, t) = C_m \cos(\pi^2 m^2 t) + D_m \sin(\pi^2 m^2 t)$$

$$(\mathcal{C}(u))(0, t) = A_0 + B_0 t$$

and, taking derivatives;

$$A_m = (\mathcal{C}(u))(m, 0), \quad B_m \pi^2 m^2 = (\mathcal{C}(u_s))(m, 0)$$

$$C_m = (\mathcal{S}(u))(m, 0), \quad D_m \pi^2 m^2 = (\mathcal{S}(u_s))(m, 0)$$

$$A_0 = (\mathcal{C}(u))(0, 0), \quad B_0 = (\mathcal{C}(u_s))(0, 0)$$

so that;

$$(\mathcal{C}(u))(m, t) = (\mathcal{C}(u))(m, 0)\cos(\pi^2 m^2 t) + \frac{(\mathcal{C}(u_s))(m, 0)}{\pi^2 m^2} \sin(\pi^2 m^2 t)$$

$$(\mathcal{S}(u))(m, t) = (\mathcal{S}(u))(m, 0)\cos(\pi^2 m^2 t) + \frac{(\mathcal{S}(u_s))(m, 0)}{\pi^2 m^2} \sin(\pi^2 m^2 t)$$

$$(\mathcal{C}(u))(0, t) = (\mathcal{C}(u))(0, 0) + (\mathcal{C}(u_s))(0, 0)t, (\dagger\dagger\dagger)$$

It follows, applying the inversion theorem and $(\dagger\dagger\dagger)$, that;

$$u(x, t) = (\mathcal{C}(u))(0, t) + \sum_{m \geq 1} (\mathcal{C}(u))(m, t)\cos(\pi x m) + \sum_{m \geq 1} (\mathcal{S}(u))(m, t)\sin(\pi x m)$$

$$= (\mathcal{C}(u))(0, 0) + (\mathcal{C}(u_s))(0, 0)t$$

$$+ \sum_{m \geq 1} (\mathcal{C}(u))(m, 0)\cos(\pi^2 m^2 t) + \frac{(\mathcal{C}(u_s))(m, 0)}{\pi^2 m^2} \sin(\pi^2 m^2 t)\cos(\pi x m)$$

$$+ \sum_{m \geq 1} (\mathcal{S}(u))(m, 0)\cos(\pi^2 m^2 t) + \frac{(\mathcal{S}(u_s))(m, 0)}{\pi^2 m^2} \sin(\pi^2 m^2 t)\sin(\pi x m)$$

$$u(x, t) = (\mathcal{C}(u))(0, t) + \sum_{m \geq 1} (\mathcal{C}(u))(m, t)\cos(\pi x m) + \sum_{m \geq 1} (\mathcal{S}(u))(m, t)\sin(\pi x m)$$

$$= (\mathcal{C}(u))(0, 0) + (\mathcal{C}(u_s))(0, 0)t$$

$$+ \sum_{m \geq 1} \pi m \left(\frac{(\mathcal{C}(u))(m, 0)}{\pi m} \cos(\pi^2 m^2 t) + \frac{(\mathcal{C}(u_s))(m, 0)}{\pi m} \sin(\pi^2 m^2 t)\cos(\pi x m) \right)$$

$$+ \sum_{m \geq 1} \pi m \left(\frac{(\mathcal{S}(u))(m, 0)}{\pi m} \cos(\pi^2 m^2 t) + \frac{(\mathcal{S}(u_s))(m, 0)}{\pi m} \sin(\pi^2 m^2 t)\sin(\pi x m) \right)$$

□

REFERENCES

- [1] A Note on the Convergence of Fourier Series, available at <http://www.curvalinea.net>, Tristram de Piro, (2019).

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