NONSTANDARD METHODS FOR SOLVING SCHRODINGER'S EQUATION

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ABSTRACT. We consider the nonstandard Euler method applied to solving the 1-dimensional Schrodinger equation with zero potential, for the free particle, with given initial condition g. We find a nonstandard f specialising to g, for which the corresponding nonstandard solution F specialises to the classical solution.

We make some considerations in connection with the Schrodinger equation.

Definition 0.1. *We let $\mathcal{S}(\mathcal{R})$ denote the Schwartz space. $C(\mathcal{R})$ and $C^\infty(\mathcal{R})$ have their conventional meanings. We let $T = \mathcal{R} \times \mathcal{R}_{\geq 0}$ and let $T^0 = \mathcal{R} \times \mathcal{R}_{>0}$ denote its interior. We let $C(T) = \{G$, continuous on $T, G_t \in C(\mathcal{R})$, for $t \in \mathcal{R}_{\geq 0}\}$, $\mathcal{S}(T) = \{G \in C(T) : G_t \in \mathcal{S}(\mathcal{R})$, for $t \in \mathcal{R}_{\geq 0}, G|T^0 \in C^\infty(T^0)\}$. If $h \in \mathcal{S}(\mathcal{R})$, we define its Fourier transform by;*

$$\mathcal{F}(h)(y) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i y x} dx$$

for $y \in \mathcal{R}$.

If $g \in \mathcal{S}(T)$, we define its Fourier transform in space by;

$$\mathcal{F}(g)(y,t) = \int_{-\infty}^{\infty} g(x,t) e^{-2\pi i y x} dx$$

for $y \in \mathcal{R}$ and $t \in \mathcal{R}_{\geq 0}$.

We recall the facts that $\mathcal{F} : \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{S}(\mathcal{R})$ satisfies the following inversion theorem;

If $h \in \mathcal{S}(\mathcal{R})$, then;

$$h(x) = \int_{-\infty}^{\infty} \mathcal{F}(h)(y) e^{2\pi i x y} dy$$

for $x \in \mathcal{R}$, and, a similar result holds for $g \in \mathcal{S}(T)$.

Lemma 0.2. *If $g \in \mathcal{S}(\mathcal{R})$, there exists a unique $G \in \mathcal{S}(T)$, with $G_0 = g$, such that G satisfies the simplified Schrodinger equation;*

$$\frac{\partial G}{\partial t} = i\frac{\partial^2 G}{\partial x^2} \ (*)$$

on T^0.

Proof. Suppose, first, there exists such a solution G, then, applying $\mathcal{F}$ to $(*)$, we must have that;

$$\mathcal{F}(\frac{\partial G}{\partial t} - i\frac{\partial^2 G}{\partial x^2})(y,t) = 0 \ (t > 0, y \in \mathcal{R})$$

Differentiating under the integral sign, we have that;

$$\mathcal{F}(\frac{\partial G}{\partial t}) = \frac{\partial \mathcal{F}(G)}{\partial t}(y,t), \text{ for } t > 0, y \in \mathcal{R}$$

Integrating by parts and using the fact that $G_t \in \mathcal{S}(\mathcal{R})$, for $t > 0$, we have that;

$$\mathcal{F}\frac{\partial^2 G}{\partial x^2} = -4\pi^2 y^2 \mathcal{F}(G)(y,t), \text{ for } t > 0, y \in \mathcal{R}$$

We thus obtain the sequence of ordinary differential equations, indexed by $y \in \mathcal{R}$;

$$\frac{\partial \mathcal{F}(G)}{\partial t} + 4i\pi^2 y^2 \mathcal{F}(G)(y,t) = 0 \ (t > 0)$$

As $G \in C(T)$, $G_t \to G_0$ pointwise , as $t \to 0$, and, using the Dominated Convergence Theorem, $\mathcal{F}(G)(y,t) \to \mathcal{F}(G)(y,0)$, as $t \to 0$, for each $y \in \mathcal{R}$. By Picard's and Peano's Theorem, see [6], Chapter 4, this system of equations has a unique continuous solution, given by;

$$\mathcal{F}(G)(y,t) = e^{-4i\pi^2 y^2 t}\mathcal{F}(g)(y) \ (t \geq 0)$$

As $G_t \in \mathcal{S}(\mathcal{R})$, we have, by the inversion theorem, that, for $x \in \mathcal{R}$;

$$G_t(x) = \int_{-\infty}^{\infty} \mathcal{F}(G)(y,t)e^{2\pi iyx}dy$$

and, in particular, G_t is determined by its Fourier transform, for $t > 0$. It follows that G is a unique solution.

If $g \in \mathcal{S}(\mathcal{R})$, its Fourier series transform $\mathcal{F}(g) \in \mathcal{S}(\mathcal{R})$, hence;

$e^{-4i\pi^2 y^2 t}\mathcal{F}(g)(y)e^{2\pi iyx} \in \mathcal{S}(\mathcal{R})$

for $t > 0$, $x \in \mathcal{R}$. It follows that G defined by;

$G(x,t) = \int_{-\infty}^{\infty} e^{-4i\pi^2 y^2 t}\mathcal{F}(g)(y)e^{2\pi iyx}dx$

is a solution of the required form. □

As in [5], we make a nonstandard Fourier analysis of Schrodinger's equation, we introduce the following notation;

Definition 0.3. *If $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$, with η odd, we let;*

$\overline{\mathcal{V}_\eta} = {}^*\bigcup_{-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}} [\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$

so that $\overline{\mathcal{V}_\eta} = {}^*[-\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}})$. We let $\mathcal{D}_\eta$ denote the associated $*$-finite algebra, generated by the intervals $[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})$, for $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$, and μ_η the associated counting measure defined by $\mu_\eta([\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}})) = \frac{1}{\sqrt{\eta}}$. We let $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ denote the associated Loeb space. If $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, we let;

$\overline{\mathcal{T}_\nu} = {}^*\bigcup_{0 \leq i \leq \nu^2-1} [\frac{i}{\nu}, \frac{i+1}{\nu})$

so that $\overline{\mathcal{T}_\nu} = [0, \nu) \subset {}^*\mathcal{R}_{\geq 0}$. We let $\mathcal{C}_\nu$ denote the associated $*$-finite algebra, generated by the intervals $[\frac{i}{\nu}, \frac{i+1}{\nu})$, for $0 \leq i \leq \nu^2 - 1$, and λ_ν the associated counting measure defined by $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$. We let $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$ denote the associated Loeb space.

We let $(\mathcal{R} \cup \{+\infty, -\infty\}, \mathfrak{D}, \mu)$ denote the extended real line, with the completion $\mathfrak{D}$ of the extension of the Borel field, and μ the extension of Lebesgue measure, with $\mu(+\infty) = \mu(-\infty) = \infty$. We let $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$ denote the extended real half line, with the completion $\mathfrak{C}$ of the extended Borel field, and λ the extension of Lebesgue measure, with $\lambda(+\infty) = \infty$, see [6], Chapter 6.

We let $(\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, \mathcal{D}_\eta \times \mathcal{C}_\nu, \mu_\eta \times \lambda_\nu)$ be the associated product space and $(\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta \times \mathcal{C}_\eta), L(\mu_\eta \times \lambda_\nu))$ be the corresponding Loeb space. $(\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta) \times L(\mathcal{C}_\nu), L(\mu_\eta) \times L(\lambda_\nu))$ is the complete product of *the Loeb spaces* $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ *and* $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$*. Similarly,* $(\mathcal{R}\cup\{+\infty,-\infty\}\times(\mathcal{R}_{\geq 0}\cup\{+\infty\}, \mathfrak{D}\times\mathfrak{C}, \mu\times\lambda)$ *is the complete product of* $(\mathcal{R}\cup\{+\infty,-\infty\}, \mathfrak{D}, \mu)$ *and* $(\mathcal{R}_{\geq 0}\cup\{+\infty\}, \mathfrak{C}, \lambda)$*.*

We let $({}^*\mathcal{R}, {}^*\mathfrak{E})$ *denote the hyperreals, with the transfer of the Borel field* $\mathfrak{C}$ *on* $\mathcal{R}$*. A function* $f : (\overline{\mathcal{V}_\eta}, \mathcal{D}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{E})$ *is measurable, if* $f^{-1} : {}^*\mathfrak{E} \rightarrow \mathcal{D}_\eta$*. The same definition holds for* $\mathcal{T}_\nu$*. Similarly,* $f : (\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, \mathcal{D}_\eta\times\mathcal{C}_\nu) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{E})$ *is measurable, if* $f^{-1} : {}^*\mathfrak{E} \rightarrow \mathcal{D}_\eta\times\mathfrak{C}_\nu$*. Observe that this is equivalent to the definition given in* [3]*. We will abbreviate this notation to* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{R}$*,* $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{R}$ *or* $f : \overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{R}$ *is measurable,* $(*)$*. The same applies to* $({}^*\mathcal{C}, {}^*\mathfrak{E})$*, the hyper complex numbers, with the transfer of the Borel field* $\mathfrak{E}$*, generated by the complex topology. Observe that* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$*,* $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ $f : \overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *is measurable, in this sense, iff* $Re(f)$ *and* $Im(f)$ *are measurable in the sense of* $(*)$*.*

We let $\overline{\mathcal{S}_{\eta,\nu}} = \overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}$ *and;*

$$V(\overline{\mathcal{V}_\eta}) = \{f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C},\ f \textit{ measurable } d(\mu_\eta)\}$$

and, similarly, we define $V(\overline{\mathcal{T}_\nu})$*. Let;*

$$V(\overline{\mathcal{S}_{\eta,\nu}}) = \{f : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C},\ f \textit{ measurable } d(\mu_\eta\times\lambda_\nu)\}$$

Lemma 0.4. *The identity;*

$$i : (\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta\times\mathcal{C}_\nu), L(\mu_\eta\times\lambda_\nu))$$

$$\rightarrow (\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta)\times L(\mathcal{C}_\nu), L(\mu_\eta)\times L(\lambda_\nu))$$

and the standard part mapping;

$st : (\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta)\times L(\mathcal{C}_\nu), L(\mu_\eta)\times L(\lambda_\nu)) \rightarrow \mathcal{R}\cup\{+\infty,-\infty\}\times \mathcal{R}_{\geq 0}\cup\{+\infty\}$

are measurable and measure preserving.

Proof. The proof is similar work in Chapter 6 of [6], using Caratheodory's Extension Theorem and Theorem 22 of [1].

□

Definition 0.5. *Discrete Partial Derivatives*

Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *be measurable. We define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(\frac{i}{\sqrt{\eta}}) = \frac{\sqrt{\eta}}{2}(f(\frac{i+1}{\sqrt{\eta}}) - f(\frac{i-1}{\sqrt{\eta}}))$;

for $i \in {}^*\mathcal{Z}_{-\frac{(\eta-3)}{2} \leq i \leq \frac{(\eta-3)}{2}}$.

$f'(\frac{(\eta-1)}{2\sqrt{\eta}}) = \frac{\sqrt{\eta}}{2}(f(-\frac{(\eta-1)}{2\sqrt{\eta}}) - f(\frac{(\eta-3)}{2\sqrt{\eta}}))$

$f'(-\frac{(\eta-1)}{2\sqrt{\eta}}) = \frac{\sqrt{\eta}}{2}(f(-\frac{(\eta-3)}{2\sqrt{\eta}}) - f(\frac{(\eta-1)}{2\sqrt{\eta}}))$

Let $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *be measurable. We define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(\frac{i}{\nu}) = \nu(f(\frac{i+1}{\nu}) - f(\frac{i}{\nu}))$;

for $i \in {}^*\mathcal{N}_{0 \leq i \leq \nu^2-2}$.

$f'(\frac{\nu-1}{\nu}) = 0$;

If $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(\frac{j}{\sqrt{\eta}}) = f(\frac{j+1}{\sqrt{\eta}})$ *for* $-\frac{(\eta-1)}{2} \leq j \leq \frac{(\eta-3)}{2}$

$f^{lsh}(\frac{(\eta-1)}{2\sqrt{\eta}}) = f(-\frac{(\eta-1)}{2\sqrt{\eta}})$

$f^{rsh}(\frac{j}{\sqrt{\eta}}) = f(\frac{j-1}{\sqrt{\eta}})$ *for* $-\frac{(\eta-3)}{2} \leq j \leq \frac{(\eta-1)}{2}$

$f^{rsh}(-\frac{(\eta-1)}{2\sqrt{\eta}}) = f(\frac{(\eta-1)}{2\sqrt{\eta}})$

If $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(\frac{j}{\nu}) = f(\frac{j+1}{\nu})$ *for* $0 \leq j \leq \nu^2 - 2$

$f^{lsh}(\nu - \frac{1}{\nu}) = f(0)$

$f^{rsh}(\frac{j}{\nu}) = f(\frac{j-1}{\nu})$ *for* $1 \leq j \leq \nu^2 - 1$

$f^{rsh}(0) = f(\nu - \frac{1}{\nu})$

If $f:\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow {}^*\mathcal{C}$ is measurable. Then we define $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t}\}$ to be the unique measurable functions satisfying;

$\frac{\partial f}{\partial x}(\frac{i}{\sqrt{\eta}},t)=\frac{\sqrt{\eta}}{2}(f(\frac{i+1}{\sqrt{\eta}},t)-f(\frac{i-1}{\sqrt{\eta}},t))$;

for $i\in {}^*\mathcal{Z}_{-\frac{(\eta-3)}{2}\leq i\leq\frac{(\eta-3)}{2}}, t\in\overline{\mathcal{T}_\nu}$

$\frac{\partial f}{\partial x}(\frac{(\eta-1)}{2\sqrt{\eta}},t)=\frac{\sqrt{\eta}}{2}(f(-\frac{(\eta-1)}{2\sqrt{\eta}},t)-f(\frac{(\eta-3)}{2\sqrt{\eta}},t))$

$\frac{\partial f}{\partial x}(-\frac{(\eta-1)}{2\sqrt{\eta}},t)=\frac{\sqrt{\eta}}{2}(f(-\frac{(\eta-3)}{2\sqrt{\eta}},t)-f(\frac{(\eta-1)}{2\sqrt{\eta}},t))$

$\frac{\partial f}{\partial t}(x,\frac{j}{\nu})=\nu(f(x,\frac{j+1}{\nu})-f(x,\frac{j}{\nu}))$;

for $j\in {}^*\mathcal{N}_{0\leq j\leq\nu^2-2}, x\in\overline{\mathcal{V}_\eta}$

$\frac{\partial f}{\partial t}(x,\nu-\frac{1}{\nu})=0$

We define $\{f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t}\}$ *by;*

$f^{lsh_x}(x_0,t_0)=(f_{t_0})^{lsh}(x_0)$

$f^{lsh_t}(x_0,t_0)=(f_{x_0})^{lsh}(t_0)$

$f^{rsh_x}(x_0,t_0)=(f_{t_0})^{rsh}(x_0)$

$f^{rsh_t}(x_0,t_0)=(f_{x_0})^{rsh}(t_0)$

where, if $(x_0,t_0)\in\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}$;

$f_{t_0}(x_0)=f_{x_0}(t_0)=f(\frac{[\sqrt{\eta}x_0]}{\sqrt{\eta}},\frac{[\nu t_0]}{\nu})$

Lemma 0.6. *If f is measurable, then so are;*

$\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t},\frac{\partial^2 f}{\partial x^2},f_x,f_t,f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t},f^{lsh_x^2},f^{lsh_t^2},f^{rsh_x^2},f^{rsh_t^2}\}$

Proof. This follows immediately, by transfer, from the corresponding result for the discrete derivatives and shifts of discrete functions $f:\mathcal{H}_n\times\mathcal{T}_m\rightarrow\mathcal{C}$, where $n,m\in\mathcal{N}$, see [6], Chapter 6. □

Definition 0.7. *If $f:\overline{\mathcal{V}_\eta}\rightarrow {}^*\mathcal{C}$ is measurable, we define the internal integral $\int_{\overline{\mathcal{V}_\eta}} f d\mu_\eta$ as;*

$\frac{1}{\sqrt{\eta}}{}^*\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{(\eta-1)}{2}} f(\frac{j}{\sqrt{\eta}})$

Similar definitions hold for $\int_{\overline{\mathcal{T}_\nu}} g d\lambda_\nu$ and $\int_{\overline{\mathcal{S}_{\eta,\nu}}} hd(\mu_\eta\times\lambda_\nu)$, when $g:\overline{\mathcal{T}_\nu}\rightarrow {}^*\mathcal{C}$ and $h:\overline{\mathcal{S}_{\eta,\nu}}\rightarrow {}^*\mathcal{C}$ are measurable.

Lemma 0.8. *Let $g,h:\overline{\mathcal{V}_\eta}\rightarrow {}^*\mathcal{C}$ be measurable. Then;*

(i). $\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y)=0$

(ii). $(gh)'=g'h^{lsh}+g^{rsh}h'$

(iii). $\int_{\overline{\mathcal{V}_\eta}}(g'h)(y)d\mu_\eta(y)=-\int_{\overline{\mathcal{V}_\eta}} gh'd\mu_\eta(y)$

(iv). $\int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y)=\int_{\overline{\mathcal{V}_\eta}} g^{lsh}(y)d\mu_\eta(y)=\int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

(v). $(g')^{rsh}=(g^{rsh})'$, $(g')^{lsh}=(g^{lsh})'$

(vi). $\int_{\overline{\mathcal{V}_\eta}}(g''h)(y)d\mu_\eta(y)=\int_{\overline{\mathcal{V}_\eta}}(gh'')(y)d\mu_\eta(y)$

Proof. In the first part, for (i), we have, using Definition 0.5, that;

$\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y)$

$=\frac{1}{\sqrt{\eta}}[{}^*\sum_{-\frac{(\eta-3)}{2}\leq j\leq\frac{(\eta-3)}{2}}\frac{\sqrt{\eta}}{2}[g(\frac{j+1}{\sqrt{\eta}})-g(\frac{j-1}{\sqrt{\eta}})]$

$+\frac{\sqrt{\eta}}{2}[g(-\frac{(\eta-3)}{2\sqrt{\eta}})-g(\frac{(\eta-1)}{2\sqrt{\eta}})]+\frac{\sqrt{\eta}}{2}[g(-\frac{(\eta-1)}{2\sqrt{\eta}})-g(\frac{(\eta-3)}{2\sqrt{\eta}})]]=0$

For (ii), we calculate;

$(gh)'(\frac{j}{\sqrt{\eta}})=$

$=\frac{\sqrt{\eta}}{2}(gh(\frac{j+1}{\sqrt{\eta}})-gh(\frac{j-1}{\sqrt{\eta}}))$

$=\frac{\sqrt{\eta}}{2}(gh(\frac{j+1}{\sqrt{\eta}})-g(\frac{j-1}{\sqrt{\eta}})h(\frac{j+1}{\sqrt{\eta}})$

$+g(\frac{j-1}{\sqrt{\eta}})h(\frac{j+1}{\sqrt{\eta}})-gh(\frac{j-1}{\sqrt{\eta}}))$

$= g'(\frac{j}{\sqrt{\eta}})h(\frac{j+1}{\sqrt{\eta}}) + g(\frac{j-1}{\sqrt{\eta}})h'(\frac{j}{\sqrt{\eta}})$

$= (g'h^{lsh} + g^{rsh}h')(\frac{j}{\sqrt{\eta}})$

Combining $(i), (ii)$, we have;

$0 = \int_{\overline{\mathcal{V}_\eta}}(gh)'(x)d\mu_\eta(x)$

$= \int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh} + g^{rsh}h')(x)d\mu_\eta(x)$

and, rearranging, that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh})d\mu_\eta = -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}h')d\mu_\eta$

For (iv), we have that;

$\int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

$= \frac{1}{\sqrt{\eta}}({}^*\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{(\eta-1)}{2}} g^{rsh}(\frac{j}{\sqrt{\eta}}))$

$= \frac{1}{\sqrt{\eta}}({}^*\sum_{-\frac{(\eta-3)}{2}\leq j\leq\frac{(\eta-1)}{2}} g(\frac{j-1}{\sqrt{\eta}}) + g(\frac{(\eta-1)}{2\sqrt{\eta}}))$

$= \frac{1}{\sqrt{\eta}}({}^*\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{(\eta-1)}{2}} g(\frac{j}{\sqrt{\eta}}))$

$= \int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y)$

A similar calculation holds with g^{lsh}. For (v), we have for $-\frac{(\eta-5)}{2} \leq j \leq \frac{(\eta-3)}{2}$;

$(g')^{rsh}(\frac{j}{\sqrt{\eta}})$

$= g'(\frac{j-1}{\sqrt{\eta}})$

$= \frac{\sqrt{\eta}}{2}(g(\frac{j}{\sqrt{\eta}}) - g(\frac{j-2}{\sqrt{\eta}}))$

$(g^{rsh})'(\frac{j}{\sqrt{\eta}})$

$= \frac{\sqrt{\eta}}{2}(g^{rsh}(\frac{j+1}{\sqrt{\eta}}) - g^{rsh}(\frac{j-1}{\sqrt{\eta}}))$

$= \frac{\sqrt{\eta}}{2}(g(\frac{j}{\sqrt{\eta}}) - g(\frac{j-2}{\sqrt{\eta}}))$

Similar calculations hold for the remaining j to give that $(g')^{rsh} = (g^{rsh})'$, and the calculation $(g')^{lsh} = (g^{lsh})'$ is also similar.

It follows that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h)d\mu_\eta$

$= \int_{\overline{\mathcal{V}_\eta}}(g'(h^{rsh})^{lsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h^{rsh})')d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h')^{rsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(gh'))d\mu_\eta$

which gives (iii), using $(iv),(v)$. The calculation (vi) is immediate from (iii).

□

Lemma 0.9. *Similar results hold for* $\{lsh_x, rsh_x, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}$. *Namely, if* $g, h : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C}$ *are measurable. Then;*

(i). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} d(\mu_\eta \times \lambda_\nu) = 0$

(ii). $\frac{\partial (gh)}{\partial x} = \frac{\partial g}{\partial x} h^{lsh_x} + g^{rsh_x} \frac{\partial h}{\partial x}$

(iii). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} h d(\mu_\eta \times \lambda_\nu) = -\int_{\overline{\mathcal{S}_{\eta,\nu}}} g \frac{\partial h}{\partial x} d(\mu_\eta \times \lambda_\nu)$

(iv). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} g d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} g^{lsh_x} d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} g^{rsh_x} d(\mu_\eta \times \lambda_\nu)$

(v). $(\frac{\partial g}{\partial x})^{lsh_x} = \frac{\partial (g^{lsh_x})}{\partial x}$, *and, similarly, with* rsh_x *replacing* lsh_x.

(vi). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} (\frac{\partial^2 g}{\partial x^2} h) d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} (g \frac{\partial^2 h}{\partial x^2}) d(\mu_\eta \times \lambda_\nu)$ $(*)$

Proof. For (i), using (i) from the argument in the proof of Lemma 0.8, we have;

$\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} d(\mu_\eta \times \lambda_\nu)$

$= \int_{\overline{\mathcal{V}_\eta}} (\int_{\overline{\mathcal{T}_\nu}} (\frac{\partial g}{\partial x})_t d\mu_\eta) d\lambda_\nu(t)$

$= \int_{\overline{\mathcal{V}_\eta}}(\int_{\overline{\mathcal{T}_\nu}}(\frac{\partial g_t}{\partial x})d\mu_\eta)d\lambda_\nu(t)$

$= \int_{\overline{\mathcal{T}_\nu}} 0 d\lambda_\nu(t) = 0$

The proofs of $(ii), (iii), (iv)$ are similar to the proof of Lemma 0.8, relying on the result of (i). (v) follows easily from Definitions 0.5 and (vi) follows, repeating the result of (iii), and applying (v).

□

Lemma 0.10. *Given a measurable boundary conditions $f \in V(\overline{\mathcal{V}_\eta})$, there exists a unique measurable $F \in V(\overline{\mathcal{S}_{\eta,\nu}})$, satisfying the nonstandard Schrodinger equation;*

$\frac{\partial F}{\partial t} = i\frac{\partial^2 F}{\partial x^2}$

on $(\overline{\mathcal{T}_\nu} \setminus [\nu - \frac{1}{\nu}, \nu)) \times \overline{\mathcal{V}_\eta}$

with $F(0, x) = f(x)$, *for* $x \in \overline{\mathcal{V}_\eta}$, $(*)$.

Proof. The proof is a simple adaptation of Lemma 0.27 in [5].

□

Definition 0.11. *We let $exp_\eta(2\pi ixy), exp_\eta(-2\pi ixy) : \overline{\mathcal{R}_\eta}^2 \rightarrow {}^*\mathcal{C}$ be defined by;*

$exp_\eta(-2\pi ixy) = {}^*exp(-2\pi i\frac{[x\sqrt{\eta}]}{\sqrt{\eta}}\frac{[y\sqrt{\eta}]}{\sqrt{\eta}})$

$exp_\eta(2\pi ixy) = {}^*exp(2\pi i\frac{[x\sqrt{\eta}]}{\sqrt{\eta}}\frac{[y\sqrt{\eta}]}{\sqrt{\eta}})$

Given a measurable $f_\eta : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$, we define the nonstandard Fourier transform $\hat{f}_\eta : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ by;

$\hat{f}_\eta(y) = \int_{\overline{\mathcal{R}_\eta}} f_\eta(x)exp_\eta(-2\pi iyx)d\mu_\eta(x)$

Lemma 0.12. *Let hypotheses be as in Definition 0.11, then;*

$f_\eta(x) = \int_{\overline{\mathcal{R}_\eta}} \hat{f}_\eta(y)exp_\eta(2\pi ixy)d\mu_\eta(y)$ $(*)$

Proof. This can be shown using the method in [6], Chapter 5. Alternatively, we can calculate directly, with $x = \frac{k}{\sqrt{\eta}}$, for some $-\frac{(\eta-1)}{2} \leq k \leq$

$\frac{(\eta-1)}{2}$;

$\int_{\overline{\mathcal{R}_\eta}} \hat{f}_\eta(y) exp_\eta(2\pi ixy) d\mu_\eta(y)$

$= \frac{1}{\sqrt{\eta}} {}^*\sum_{-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}} \hat{f}_\eta(\frac{m}{\sqrt{\eta}}) {}^*exp(2\pi i \frac{k}{\sqrt{\eta}} \frac{m}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{\eta}} {}^*\sum_{-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}} \frac{1}{\sqrt{\eta}} {}^*\sum_{-\frac{(\eta-1)}{2} \leq j \leq \frac{(\eta-1)}{2}} f_\eta(\frac{j}{\sqrt{\eta}}) {}^*exp(-2\pi i \frac{j}{\sqrt{\eta}} \frac{m}{\sqrt{\eta}}) {}^*exp(2\pi i \frac{km}{\eta})$

$= {}^*\sum_{-\frac{(\eta-1)}{2} \leq j \leq \frac{(\eta-1)}{2}} f_\eta(\frac{j}{\sqrt{\eta}}) \frac{1}{\eta} {}^*\sum_{-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}} {}^*exp(2\pi im \frac{(k-j)}{\eta})$

For $k \neq j$, we are summing roots of unity, transferring the result for finite n, and we have that;

${}^*\sum_{-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}} {}^*exp(2\pi im(\frac{k-j}{\eta}))$

$= {}^*\sum_{0 \leq m \leq \eta-1} {}^*exp(2\pi im(\frac{k-j}{\eta})) = 0$

where we have used the fact that for $\frac{-(\eta-1)}{2} \leq m \leq -1$;

${}^*exp(2\pi im \frac{(k-j)}{\eta}) = {}^*exp(2\pi i(\eta+m) \frac{(k-j)}{\eta})$

and re-indexed. For $k = j$, we obtain;

${}^*\sum_{-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}} {}^*exp(2\pi im(\frac{k-j}{\eta})) = \eta$

which gives the result.

□

We relate the nonstandard Fourier transform and the standard Fourier transform;

Lemma 0.13. *Let $f \in S(\mathcal{R})$ and let $f_\eta \in V(\overline{\mathcal{R}_\eta})$ be defined by $f_\eta(x) = {}^*f(\frac{[\sqrt{\eta}x]}{\sqrt{\eta}})$, then, for $y \in \overline{\mathcal{R}_\eta}$, with y finite;*

${}^\circ\hat{f}_\eta(y) = \hat{f}({}^\circ y)$

Proof. This follows from (∗∗) in Theorem 6.24, Chapter 6, of [6].

□

Definition 0.14. *Given a measurable* $f : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard vertical Fourier transform* $\hat{f} : \overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ *by;*

$$\hat{f}(t,y) = \int_{\overline{\mathcal{V}_\eta}} f(t,x) exp_\eta(-2\pi ixy) d\mu_\eta(x)$$

and, given a measurable $g : \overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard inverse vertical Fourier transform by;*

$$\check{g}(t,x) = \int_{\overline{\mathcal{V}_\eta}} g(t,y) exp_\eta(2\pi ixy) d\mu_\eta(y)$$

so that, by $(*)$ *in Lemma 0.12,* $f = \check{\hat{f}}$

Similar to Definition 6.20 of [6]*, Chapter 6, for* $f \in \overline{\mathcal{R}_\eta}$*, we let* $\phi_\eta : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ *be defined by;*

$$\phi_\eta(y) = \frac{\sqrt{\eta}}{2}\left(exp_\eta\left(\frac{-2\pi iy}{\sqrt{\eta}}\right) - exp_\eta\left(\frac{2\pi iy}{\sqrt{\eta}}\right)\right)$$

The following is the analogue of Lemma 5.14 in [6], Chapter 5, using the definition of the discrete derivative in Definition 0.5 and the discrete Fourier transform from Definition 0.11;

Lemma 0.15. *Let* $f : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ *be measurable; then, for* $y \in \mathcal{R}_\eta$*,*

$$\hat{f''}(y) = \phi_\eta^2(y)\hat{f}(y)$$

Proof. We have, using Lemma 0.8(iii), that;

$$(\hat{f'})(y) = \int_{\overline{\mathcal{R}_\eta}} f'(x) exp_\eta(-2\pi ixy) d\mu_\eta(x)$$

$$= -\int_{\overline{\mathcal{R}_\eta}} f(x)(exp_\eta)'(-2\pi ixy) d\mu_\eta(x)$$

A simple calculation shows that;

$$(exp_\eta)'(-2\pi ixy) = exp_\eta(-2\pi ixy)\phi_\eta(y)$$

Therefore;

$$(\hat{f'})(y) = -\phi_\eta(y)\hat{f}(y)$$

Then $\hat{f}''(y)$

$= -\phi_\eta(y)\hat{f}'(y)$

$= \phi_\eta^2(y)\hat{f}(y)$

as required. ∎

Definition 0.16. *Let $f \in V(\overline{\mathcal{R}_\eta})$. We denote by f^D the one step derivative defined by;*

$f^D(\frac{j}{\sqrt{\eta}}) = \sqrt{\eta}(f(\frac{j+1}{\sqrt{\eta}}) - f(\frac{j}{\sqrt{\eta}}))$

for $-\frac{(\eta-1)}{2} \leq j \leq \frac{(\eta-3)}{2}$*, and;*

$f^D(\frac{(\eta-1)}{2\sqrt{\eta}}) = \sqrt{\eta}(f(-\frac{(\eta-1)}{2\sqrt{\eta}}) - f(\frac{(\eta-1)}{2\sqrt{\eta}})$

Lemma 0.17. *Let $g \in S(\mathcal{R})$ and let g_η be defined on $\overline{\mathcal{R}_\eta}$ by;*

$g_\eta(\frac{j}{\sqrt{\eta}}) = {}^*g(\frac{j}{\sqrt{\eta}})$

for $-\frac{(\eta-1)}{2} \leq j \leq \frac{(\eta-1)}{2}$*, and;*

$g_\eta(x) = g_\eta(\frac{[\sqrt{\eta}x]}{\sqrt{\eta}})$

for $x \in \overline{\mathcal{R}_\eta}$. Then, there exists a constant $M \in \mathcal{R}$ with;

$max\{g_\eta, g_\eta^D, g_\eta^{D^2}\} \leq M$

Proof. This is very similar to Lemma 0.28 of [5]. Clearly $|g_\eta| \leq M_0$, where $M_0 \in \mathcal{R}$, as g is bounded. Using Taylor's Theorem, we have that;

$|\frac{1}{h}(g(x+h) - g(x)) - g'(x)|$

$= |\frac{1}{h}(g(x) + hg'(x) + \frac{h^2}{2}g''(c) - g(x) - hg'(x))|$

$= |\frac{h}{2}g''(c)| \leq Kh$

where $K \in \mathcal{R}$, $x \in \mathcal{R}$, $c \in [x, x+h]$. Here, we have used the fact that g'' is bounded. By transfer, it follows that, for $x \in \overline{\mathcal{R}_\eta} \setminus [\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}})$, $(g_\eta)^D(x) \simeq (g')_\eta(x)$. As $(g')_\eta$ is bounded by hypothesis, we obtain the result $|g_\eta^D \leq M_1$, where $M_1 \in \mathcal{R}_{>0}$, for $\overline{\mathcal{R}_\eta} \setminus [\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}})$. However, we have to consider the end interval separately. It is clearly sufficient to show that $g_\eta^D(\frac{(\eta-1)}{2\sqrt{\eta}})\simeq 0$. To see this, observe that as $g\in S(\mathcal{R})$, there exists a constant $C\in\mathcal{R}$, with;

$|g(x)|\leq\frac{C}{|x|^2}$, for $x\in\mathcal{R}$, $|x|>1$

Using this, we compute, by transfer;

$|g_\eta^D(\frac{(\eta-1)}{2\sqrt{\eta}})|\leq\sqrt{\eta}(|g_\eta(-\frac{(\eta-1)}{2\sqrt{\eta}})|+|g_\eta(\frac{(\eta-1)}{2\sqrt{\eta}})|)$

$\leq C\sqrt{\eta}(\frac{2}{\frac{(\eta-1)^2}{4\eta}})=\frac{C\eta^{\frac{3}{2}}}{2(\eta-1)^2}\simeq 0$

as required. For the final claim, we have that;

$g_\eta^{D^2}(x)=\eta(g_\eta(x+\frac{2}{\sqrt{\eta}})-2g_\eta(x+\frac{1}{\sqrt{\eta}})+g_\eta(x))$

for $x\in\overline{\mathcal{R}_\eta}\setminus[\frac{(\eta-5)}{2\sqrt{\eta}},\frac{(\eta+1)}{2\sqrt{\eta}})$

Using Taylor's Theorem, we have that;

$|\frac{1}{h^2}(g(x+2h)-2g(x+h)+g(x))-g''(x)|$

$=|\frac{1}{h^2}(g(x)+2hg'(x)+\frac{4h^2}{2}g''(x))+\frac{8h^3}{6}g'''(c)-2(g(x)+hg'(x)+\frac{h^2}{2}g''(x)+\frac{h^3}{6}g'''(d))+g(x)-g''(x)|$

$=|\frac{4h^3}{3h^2}g'''(c)-\frac{h^3}{3h^2}g'''(d)|$

$=|\frac{4h}{3}g'''(c)-\frac{h}{3}g'''(d)|\leq Lh$

where $c\in[x,x+2h]$, $d\in[x,x+h]$, $L\in\mathcal{R}$, $x\in\mathcal{R}$. Here, we have used the fact that g''' is bounded. By transfer, it follows that, for $x\in\overline{\mathcal{R}_\eta}\setminus[\frac{(\eta-3)}{2\sqrt{\eta}},\frac{(\eta+1)}{2\sqrt{\eta}})$, $(g_\eta)^{D^2}(x)\simeq(g'')_\eta(x)$. As $(g'')_\eta$ is bounded by hypothesis, we obtain the result $|g_\eta^{D^2}(x)|\leq M_2$, where $M_2\in\mathcal{R}_{>0}$, for $x\in\overline{\mathcal{R}_\eta}\setminus[\frac{(\eta-3)}{2\sqrt{\eta}},\frac{(\eta+1)}{2\sqrt{\eta}})$. Again, we have to consider the endpoints separately, the details are left to the reader. The result follows, taking $M=max(M_0,M_1,M_2)$.

□

Lemma 0.18. *We have the following properties of the operator D for $\{g,h\}\subset V(\overline{\mathcal{R}_\eta})$;*

(i). $\int_{\overline{\mathcal{R}_\eta}} g^D(x)d\mu_\eta(x) = 0$

(ii). $(gh)^D = g^D h^{lsh} + gh^D$.

(iii). $\int_{\overline{\mathcal{R}_\eta}} g^D h d\mu_\eta = -\int_{\overline{\mathcal{R}_\eta}} g(h^{rsh})^D d\mu_\eta$

(iv). $\int_{\overline{\mathcal{R}_\eta}} g(x)d\mu_\eta(x) = \int_{\overline{\mathcal{R}_\eta}} g^{lsh}(x)d\mu_\eta(x) = \int_{\overline{\mathcal{R}_\eta}} g^{rsh}(x)d\mu_\eta(x)$

(v). $(g^D)^{rsh} = (g^{rsh})^D$ *and similarly for* lsh.

(vi). $\int_{\overline{\mathcal{R}_\eta}} g^{D^2} h(x)d\mu_\eta(x) = \int_{\overline{\mathcal{R}_\eta}} g(h^{D^2})^{rsh^2}(x)d\mu_\eta(x)$

Proof. This can be shown in an analogous way to Lemma 0.8, the details are left to the reader.

□

Lemma 0.19. *If* $g \in V(\overline{\mathcal{R}_\eta})$, *then;*

$\hat{(}g^{D^2})(y) = \chi_\eta^2(y)V_\eta^2(y)\hat{g}(y)$ $(**)$

for $y \in \mathcal{R}_\eta$, *where;*

$\chi_\eta(y) = \sqrt{\eta}(exp_\eta(\frac{-2\pi iy}{\sqrt{\eta}}) - 1)$

$V_\eta(y) = exp_\eta(\frac{2\pi iy}{\sqrt{\eta}})$

for $y \in \mathcal{R}_\eta$.

Proof. The proof is similar to Lemma 0.15.

□

We now have the following estimate, analogous to Lemma 0.34 of [5];

Lemma 0.20. *Let* χ_η *be defined as in Lemma 0.19, then;*

$4|y| \leq |\chi_\eta(y)| \leq 8|y|$, *for* $y \in \overline{\mathcal{R}_\eta}$

Proof. In order to see this, we calculate, for $y \in \overline{\mathcal{R}_\eta}$;

$|\chi_\eta(y)| = |\sqrt{\eta}(exp_\eta(\frac{-2\pi iy}{\sqrt{\eta}}) - 1)|$

$= \sqrt{\eta}|cos_\eta(\frac{2\pi y}{\sqrt{\eta}}) - isin_\eta(\frac{2\pi y}{\sqrt{\eta}}) - 1|$

$= \sqrt{\eta}(2 - 2cos_\eta(\frac{2\pi y}{\sqrt{\eta}}))^{\frac{1}{2}}$

$= \sqrt{\eta}(4sin_\eta^2(\frac{\pi y}{\sqrt{\eta}}))^{\frac{1}{2}}$

$2\sqrt{\eta}|sin_\eta(\frac{\pi y}{\sqrt{\eta}})|$

We have that;

$\frac{2|x|}{\pi} \leq |sin_\eta(x)| \leq \frac{4|x|}{\pi}$, for $|x| \leq \frac{\pi}{2}$

Without loss of generality, we can assume that $-\frac{(\eta-1)}{2\sqrt{\eta}} \leq y \leq \frac{(\eta-1)}{2\sqrt{\eta}}$, and, with this hypothesis, letting $x = \frac{\pi y}{\sqrt{\eta}}$, we obtain;

$4|y| \leq |\chi_\eta(y)| \leq 8|y|$

as required.

□

Lemma 0.21. *Let $g \in S(\mathcal{R})$, then there exist constants $\{C_1, C_2\} \subset \mathcal{R}$ with;*

$|g_\eta^D(x)| \leq \frac{C_1}{|x|^2}$, $|g_\eta^{D^2}(x)| \leq \frac{C_2}{|x|^2}$

for $x \in \overline{\mathcal{R}_\eta}$, $|x| \geq 1$.

Proof. For the first part, observe that $x^2 g \in S(\mathcal{R})$, so, using the proof of Lemma 0.17, we have that $(x_\eta^2 g_\eta)^D \simeq ((x^2 g)')_\eta$, in particular, as $((x^2 g)')_\eta$ is bounded, there exists a constant $F \in \mathcal{R}$ with $|(x_\eta^2 g_\eta)^D| \leq F$. Using Lemma 0.18, we have that;

$(x_\eta^2 g_\eta)^D = (x_\eta^2)^D g_\eta^{lsh} + x_\eta^2 g_\eta^D$

Hence;

$|x_\eta^2 g_\eta^D| \leq F + |(x_\eta^2)^D g_\eta^{lsh}|$

A simple computation shows that $(x_\eta^2)^D = 2x_\eta + \frac{1}{\sqrt{\eta}}$, except at the endpoint $\frac{(\eta-1)}{2\sqrt{\eta}} \in \overline{\mathcal{R}_\eta}$, when $(x_\eta^2)^D(\frac{(\eta-1)}{2\sqrt{\eta}}) = 0$. It follows that;

$|x_\eta^2 g_\eta^D(x)| \leq F + |(2x_\eta + \frac{1}{\sqrt{\eta}})g_\eta^{lsh}(x)|$, $x \neq \frac{(\eta-1)}{2\sqrt{\eta}}$

$|x_\eta^2 g_\eta^D(x)| \leq F$, otherwise, $(*)$

As $(xg) \in S(\mathcal{R})$, using Lemma 0.17 again, we can find a constant $C \in \mathcal{R}$, with $|x_\eta g_\eta| \leq C$. It follows that $|x_\eta^{lsh} g_\eta^{lsh}| \leq C$. Then;

$|g_\eta^{lsh}(x)| \leq \frac{C}{|x_\eta^{lsh}|} = \frac{C}{|x_\eta + \frac{1}{\sqrt{\eta}}|} \leq \frac{C}{|x_\eta|}$

for $x \in \overline{\mathcal{R}_\eta}$, $|x| \geq 1$, $x \neq \frac{(\eta-1)}{2\sqrt{\eta}}$. We have that;

$|g_\eta^{lsh}(\frac{(\eta-1)}{2\sqrt{\eta}})| = |g_\eta(-\frac{(\eta-1)}{2\sqrt{\eta}})| = |^*g(-\frac{(\eta-1)}{2\sqrt{\eta}})| \leq \frac{D}{|\frac{(\eta-1)}{2\sqrt{\eta}}|}$

transferring the fact that $g \in S(\mathcal{R})$. Hence, taking $E = max(C, D)$, we have that $|g_\eta^{lsh}(x)| \leq \frac{E}{|x_\eta|}$, for $|x| \geq 1$. Substituting this bound into $(*)$ and using the fact that g_η^{lsh} is bounded, we obtain $G \in \mathcal{R}$, with $|x_\eta^2 g_\eta^D| \leq G$. Observing that $|x_\eta| \geq |x| - \frac{1}{\sqrt{\eta}} \geq \frac{|x|}{2}$, for $|x| \geq 1$, we obtain the first result taking $C_1 = 4G$.

For the second part, again observing that $x^2 g \in S(\mathcal{R})$. By the proof of Lemma 0.17, $(x_\eta^2 g_\eta)^{D^2} \simeq ((x^2 g)'')_\eta$, in particular, there exists a constant $V \in \mathcal{R}$ with $|(x_\eta^2 g_\eta)^{D^2}| \leq V$. We compute;

$(x_\eta^2 g_\eta)^{D^2} = (x_\eta^2)^{D^2} g_\eta^{lsh^2} + 2(x_\eta^2)^D (g_\eta^D)^{lsh} + (x_\eta^2) g_\eta^{D^2}$

Hence;

$|(x_\eta^2) g_\eta^{D^2}| \leq V + |(x_\eta^2)^{D^2} g_\eta^{lsh^2}| + |2(x_\eta^2)^D (g_\eta^D)^{lsh}|$

We have that;

$(x_\eta^2)^{D^2} = (2x_\eta + \frac{1}{\sqrt{\eta}})^D = 2$, for $x \in (\overline{\mathcal{R}_\eta} \setminus [\frac{(\eta-3)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}}))$

$(x_\eta^2)^{D^2}(\frac{(\eta-3)}{2\sqrt{\eta}}) = (x_\eta^2)^{D^2}(\frac{(\eta-1)}{2\sqrt{\eta}}) = 2 - \eta$

Hence;

$|(x_\eta^2) g_\eta^{D^2}(x)| \leq V + |2 g_\eta^{lsh^2}(x)| + |2(2x_\eta + \frac{1}{\sqrt{\eta}})(g_\eta^D)^{lsh}(x)|$ for $x \in (\overline{\mathcal{R}_\eta} \setminus [\frac{(\eta-3)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}}))$

$|(x_\eta^2)g_\eta^{D^2}(\frac{(\eta-3)}{2\sqrt{\eta}})| \leq V+|(2-\eta)g_\eta^{lsh^2}(\frac{(\eta-3)}{2\sqrt{\eta}})|+|2(2\frac{(\eta-3)}{2\sqrt{\eta}}+\frac{1}{\sqrt{\eta}})(g_\eta^D)^{lsh}(\frac{(\eta-3)}{2\sqrt{\eta}})|$

$|(x_\eta^2)g_\eta^{D^2}(\frac{(\eta-1)}{2\sqrt{\eta}})| \leq V+|(2-\eta)g_\eta^{lsh^2}(\frac{(\eta-1)}{2\sqrt{\eta}})|$

We have that $g_\eta^{sh^2}$ is bounded, and, using the bound in the first part of the lemma;

$|2(2x_\eta+\frac{1}{\sqrt{\eta}})(g_\eta^D)^{lsh}(x)|$

$= |2(2x_\eta+\frac{1}{\sqrt{\eta}})g_\eta^D(x+\frac{1}{\sqrt{\eta}})|$

$\leq \frac{|2C_1(2x_\eta+\frac{1}{\sqrt{\eta}})|}{|x+\frac{1}{\sqrt{\eta}}|}$

$\leq \frac{4C_1(2x_\eta+\frac{1}{\sqrt{\eta}})}{|x_\eta|} \leq B$ for $x \in (\overline{\mathcal{R}_\eta} \setminus [\frac{(\eta-3)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}}))$, $|x| \geq 1$

where $B \in \mathcal{R}$. This establishes a bound C in the first line. We have, transferring the property of the Schwartz class to g_η, that;

$|(2-\eta)g_\eta^{lsh^2}(\frac{(\eta-3)}{2\sqrt{\eta}})|$

$= |(2-\eta)g_\eta(-\frac{(\eta-1)}{2\sqrt{\eta}})|$

$\leq \frac{4D\eta(\eta-2)}{(\eta-1)^2} \leq E$

where $\{D, E\} \subset \mathcal{R}$. We have, using the result of the first part of the lemma, that;

$|2(\frac{2(\eta-3)}{2\sqrt{\eta}}+\frac{1}{\sqrt{\eta}})(g_\eta^D)^{lsh}(\frac{(\eta-3)}{2\sqrt{\eta}})|$

$= |\frac{2(\eta-2)}{\sqrt{\eta}}(g_\eta^D)(\frac{(\eta-1)}{2\sqrt{\eta}})|$

$\leq \frac{4C_1(\eta-2)}{(\eta-1)} \leq F$

where $F \in \mathcal{R}$. This establishes a bound in the second line. The bound in the third line is similar and easier. So we obtain a bound $H \in \mathcal{R}$ for $(x_\eta^2 g_\eta^{D^2})$, for $|x| \geq 1$, and using a similar argument as at the end of the first part, we can find a constant $C_1 \in \mathcal{R}$ giving the final result. □

Lemma 0.22. *Let $g \in S(\mathcal{R})$, then;*

$|\hat{g_\eta}(y) \leq \frac{F}{|y|^2}$ *for* $y \in \overline{\mathcal{R}_\eta}$, $|y| > 1$

Moreover, $\hat{g_\eta}$ *is* S*-integrable.*

Proof. By Lemmas 0.17 and 0.21, we have that $g_\eta^{D^2}$ is S-integrable on $\overline{\mathcal{R}_\eta}$. In particular $\hat{}(g_\eta^{D^2})$ is bounded. Using the results of Lemmas 0.19 and 0.20, and rearranging, we obtain the first result. This could also have been deduced from Lemmas 6.21 and 6.22 of Chapter 6 in [6]. By Lemma 0.13, and the fact that $\hat{g}$ is bounded, we have that $|\hat{g_\eta}(y)| \leq G$, for $|y| \leq 1$, where $G \in \mathcal{R}$, the last result follows. □

Lemma 0.23. *Let* $g \in S(\mathcal{R})$ *be as in Lemma 0.2. Then, if* $y_0 \in \mathcal{R}_{\eta,>0}$ *is infinite, with* $y_0 \leq \frac{(\eta-1)}{2\sqrt{\eta}}$*, there exists* $g_{y_0,\eta} \in V(\overline{\mathcal{R}_\eta})$*, with* $g_{y_0,\eta} \simeq g_\eta$*, such that;*

$\hat{g_{y_0,\eta}}(y) = \hat{g_\eta}(y)$*, for all* $y \in \overline{\mathcal{R}_\eta}$, $|\frac{[\sqrt{\eta}y]}{\sqrt{\eta}}| \leq y_0$

$\hat{g_{y_0,\eta}}(y) = 0$*, for all* $y \in \overline{\mathcal{R}_\eta}$, $|\frac{[\sqrt{\eta}y]}{\sqrt{\eta}}| > y_0$

In particular ${}^\circ g_{y_0,\eta} = st^* g_\infty$*, where* $st : \overline{\mathcal{V}_\eta} \rightarrow \mathcal{R} \cup \{+\infty, -\infty\}$ *is the standard part mapping, and* g_∞ *is the extension of* g *obtained by setting* $g(\infty) = g(-\infty) = 0$*.*

Proof. For ease of notation, we let $f = g_\eta$. Let f_{y_0} be defined by;

$$f_{y_0}(x) = \int_{-\frac{([y_0\sqrt{\eta}]+1)}{\sqrt{\eta}}}^{\frac{[y_0\sqrt{\eta}]}{\sqrt{\eta}}} \hat{f}(z) exp_\eta(2\pi ixz) d\mu_\eta(z) \ (*)$$

By Lemma 0.84, $\hat{f}$ is S-integrable on $\overline{\mathcal{R}_\eta}$. It follows that, the tail contribution;

$$\int_{(\overline{\mathcal{R}_\eta} \setminus [-\frac{([y_0\sqrt{\eta}]+1)}{\sqrt{\eta}}, \frac{[y_0\sqrt{\eta}]}{\sqrt{\eta}}])} \hat{f}(z) exp_\eta(2\pi ixz) d\mu_\eta(z) \simeq 0 \text{ for } x \in \overline{\mathcal{R}_\eta}$$

Using the inversion theorem for f and comparing with $(*)$, we obtain that $f_{y_0} \simeq f$. As $f(x) \simeq 0$, for $x \in \overline{\mathcal{R}_\eta}$, with x infinite, we obtain that ${}^\circ f_{y_0}(x) = 0$, for $x \in \overline{\mathcal{R}_\eta}$ with x infinite, $(\dagger)$. We have that;

$$\hat{f}_{y_0}(y) = \int_{-\frac{([y_0\sqrt{\eta}]+1)}{\sqrt{\eta}}}^{\frac{[y_0\sqrt{\eta}]}{\sqrt{\eta}}} \hat{f}(z) \left(\int_{\overline{\mathcal{R}_\eta}} exp_\eta(2\pi ix(z-y)) d\mu_\eta(x)\right) d\mu_\eta(z)$$

As in the proof of the inversion theorem, we have that;

$\int_{\overline{\mathcal{R}_\eta}} exp_\eta(2\pi ix(z-y))d\mu_\eta(x)=0$, for $\frac{[\sqrt{\eta}z]}{\sqrt{\eta}}\neq\frac{[\sqrt{\eta}y]}{\sqrt{\eta}}$

$\int_{\overline{\mathcal{R}_\eta}} exp_\eta(2\pi ix(z-y))d\mu_\eta(x)=\sqrt{\eta}$, for $\frac{[\sqrt{\eta}z]}{\sqrt{\eta}}=\frac{[\sqrt{\eta}y]}{\sqrt{\eta}}$

Hence, evaluating the outer integral, gives the result of the second part of the lemma. Finally, for finite $x\in\overline{\mathcal{R}}_\eta$, we can use the fact that $\hat{f}$ is S-integrable, exp_η is S-continuous, Lemmas 0.13 and 0.4, and the standard inversion Theorem, to obtain that;

$$^\circ f_{y_0}(x)={}^\circ\int_{-\frac{([y_0\sqrt{\eta}]+1)}{\sqrt{\eta}}}^{\frac{[y_0\sqrt{\eta}]}{\sqrt{\eta}}}\hat{f}(z)exp_\eta(2\pi ixz)d\mu_\eta(z)$$

$$=\int_{z\in\mathcal{R}_\eta,z\ finite}{}^\circ\hat{f}(z)exp_\eta(2\pi i^\circ xz)dL(\mu_\eta)(z)$$

$$=\int_{\mathcal{R}}\hat{g}(^\circ z)exp(2\pi i^\circ x^\circ z)d\mu(z)$$

$$=g(^\circ x)$$

Combining this result with (†) proves the final claim.

□

Lemma 0.24. *Let $g\in S(\mathcal{R})$ be as in Lemma 0.2. Let $y_0\in\overline{\mathcal{R}}_{\eta,>0}$ be infinite, with $y_0\leq\frac{(\eta-1)}{2\sqrt{\eta}}$, and f_{y_0} as in Lemma 0.23. Let F_{y_0} solve the nonstandard Schrodinger equation, as in Lemma 0.10, with initial condition f_{y_0}. Then;*

$$\hat{F_{y_0}}(y,t)=(1+\frac{i\phi_\eta^2(y)}{\nu})^{[\nu t]}\hat{f_{y_0}}(y)$$

for $y\in\mathcal{R}_\eta$, $t\in\overline{\mathcal{T}_\nu}$. In particular;

$$\hat{F_{y_0}}(y,t)=0,\ for\ y_0<|y|\leq\frac{(\eta-1)}{2\sqrt{\eta}}$$

Proof. The first claim follows by a similar argument to the first part of Lemma 0.35 in [5]. The second claim is then immediate from the construction of f_{y_0} in Lemma 0.23.

□

Lemma 0.25. *For $y\in\overline{\mathcal{R}_\eta}$, with $|y|\leq\eta^{\frac{1}{5}}$, there exists a constant $C\in\mathcal{R}$ with;*

$$|\phi_\eta^2(y)+4\pi^2y^2|\leq\frac{C}{\eta^{\frac{1}{5}}}\simeq 0$$

Proof. An easy calculation, using the analytic expansion of exp, yields the standard estimate;

$$|\frac{e^{-2\pi iyh}-e^{2\pi iyh}}{2h}+2\pi iy|\leq 8\pi^3|y|^3h^2(\frac{1}{1-4\pi^2|y|^2h^2})$$

for $h>0$, where $y\in\mathcal{R}$. Applying transfer, and letting $h=\frac{1}{\sqrt{\eta}}$, we have that;

$$|\phi_\eta(y)+2\pi iy|\leq\frac{8\pi^3|y|^3}{\eta}(\frac{1}{1-\frac{4\pi^2|y|^2}{\eta}})$$

$$\leq\frac{16|y|^3\pi^3}{\eta}$$

for $y\in\mathcal{R}_\eta$, and $|y|\leq\eta^{\frac{1}{5}}$, as $\frac{4\pi^2|y|^2}{\eta}\leq\frac{1}{2}$. Taking $|y|\leq\eta^{\frac{1}{5}}$, gives a constant $D\in\mathcal{R}$, with;

$$|\phi_\eta(y)+2\pi iy|\leq\frac{D}{\eta^{\frac{2}{5}}}\simeq 0$$

It follows that $\phi_\eta(y)^2=-4\pi^2y^2-4\pi iy\epsilon+\epsilon^2$, where $|\epsilon|\leq\frac{D}{\eta^{\frac{2}{5}}}$. Let $\delta=-4\pi iy\epsilon+\epsilon^2$, then $|\delta|\leq\frac{D_1}{\eta^{\frac{1}{5}}}+\frac{D_2}{\eta^{\frac{4}{5}}}\leq\frac{C}{\eta^{\frac{1}{5}}}$, for some $\{D_1,D_2,C\}\subset\mathcal{R}$. This gives the result.

□

Lemma 0.26. *For $y\in\mathcal{R}_\eta$, with $|y|\leq min\{\eta^{\frac{1}{5}},\nu^{\frac{1}{5}}\}$, we have that;*

$$|(1+\frac{i\phi_\eta^2(y)}{\nu})^\nu-e^{i\phi_\eta^2(y)}|\leq\frac{1}{\nu^{\frac{1}{2}}}\simeq 0$$

Proof. Proceeding with a standard estimate, we have that, for $x\in\mathcal{C}$, $n\in\mathcal{N}$;

$$(1+\frac{x}{n})^n=e^nlog(1+\frac{x}{n})$$

$$=e^{n(\frac{x}{n}-\frac{x^2}{2n^2}+\frac{x^3}{n^3}+\ldots)}$$

$$|(1+\frac{x}{n})^n-e^x|\leq|e^x||1-e^{h_n(x)}|$$

$$\leq|e^x|\frac{|h_n(x)|}{1-|h_n(x)|}$$

where $h_n(x)=\frac{-x}{2n}+\frac{x^2}{3n^2}+\ldots$. We have that;

$$|h_n(x)|\leq|\frac{x}{n}|+|\frac{x}{n}|^2+\ldots=\frac{|\frac{x}{n}|}{1-|\frac{x}{n}|}$$

Hence;

$|(1+\frac{x}{n})^n - e^x| \leq \frac{|e^x||\frac{x}{n}|}{|1-|\frac{x}{n}|||1-|h_n(x)|}$ $(*)$

An easy calculation show that if $|\frac{x}{n}| \leq \frac{1}{4}$, then , by $(*)$;

$|(1+\frac{x}{n})^n - e^x| \leq 4|e^x||\frac{x}{n}|$, $(**)$

Applying transfer to $(**)$, we have that if $|e^x||x| \leq \frac{\nu^{\frac{1}{2}}}{4}$, then;

$|(1+\frac{x}{\nu})^\nu - e^x| \leq \frac{1}{\nu^{\frac{1}{2}}} \simeq 0$

In particular, if $|e^{i\phi_\eta^2(y)}||\phi_\eta^2(y)| \leq \frac{\nu^{\frac{1}{2}}}{4}$, $(***)$, then;

$|(1+\frac{i\phi_\eta^2(y)}{\nu})^\nu - e^{i\phi_\eta^2(y)}| \leq \frac{1}{\nu^{\frac{1}{2}}} \simeq 0$

as required. By the assumptions of this Lemma and the previous result Lemma 0.25, we have that;

$|\phi_\eta^2(y) + 4\pi^2|y|^2| \leq \frac{C}{\eta^{\frac{1}{5}}}$.

It follows that;

$|e^{i\phi_\eta^2(y)}||\phi_\eta^2(y)|$

$\leq (4\pi^2|y|^2 + |\epsilon|)|e^{i\epsilon}|$, where $|\epsilon| \leq \frac{C}{\eta^{\frac{1}{5}}}$

$\leq \frac{\nu^{\frac{1}{2}}}{4}$, if $4\pi^2|y|^2 e^{\frac{C}{\eta^{\frac{1}{5}}}} \leq \frac{\nu^{\frac{1}{2}}}{8}$

Clearly then, we can achieve the estimate $(***)$, if $|y| \leq \nu^{\frac{1}{5}}$.

□

Lemma 0.27. *Let $y_0 \in \overline{\mathcal{R}_{\eta,>0}}$, with y_0 infinite and $y_0 \leq min\{\eta^{\frac{1}{5}}, \nu^{\frac{1}{5}}\}$. Let F_{y_0} be as in Lemma 0.24, then, for finite $t \in \overline{\mathcal{T}_\nu}$;*

$^\circ\hat{F_{y_0}}(y,t) = e^{-i4\pi^{2\circ}y^{2\circ}t^\circ}\hat{f_{y_0}}(y)$

for $y \in \overline{\mathcal{R}_\eta}$, with y finite.

$\hat{F_{y_0}}(y,t) = 0$, *for $y \in \overline{\mathcal{R}_\eta}$, $y_0 < |y| \leq \frac{\eta-1}{2\sqrt{\eta}}$*

$|\hat{F_{y_0}}(y,t)| \leq \frac{E}{|y|^2}$, *for some* $E \in \mathcal{R}$, $y \in \overline{\mathcal{R}_\eta}$, $1 \leq |y| \leq y_0$

Proof. The second claim was already observed in Lemma 0.24. The first claim follows from the fact also in Lemma 0.24, that;

$\hat{F_{y_0}}(y,t) = (1 + \frac{i\phi_\eta^2(y)}{\nu})^{[\nu t]} \hat{f_{m_0}}(y)$, $(*)$

for $y \in \overline{\mathcal{R}_\eta}$, $t \in \overline{\mathcal{T}_\nu}$. For $y \in \overline{\mathcal{R}_\eta}$, with y finite, we have that ${}^\circ\phi_\eta^2(y) = -4\pi^{2\circ}y^2$. Combining this with the facts that $lim_{n\rightarrow\infty}(1+\frac{x}{n})^n = e^x$ and ν is infinite, t is finite, we obtain the result. For the final claim, by Lemmas 0.25 and 0.26, if $y \in \overline{\mathcal{R}_\eta}$, $|y| \leq y_0$, we have that $i\phi_\eta^2(y) \simeq -i4\pi^2 y^2$ and $(1+\frac{i\phi_\eta^2(y)}{\nu})^\nu \simeq e^{i\phi_\eta^2(y)}$. Combining these, we obtain that, $e^{-4i\pi^2 y^2} \simeq (1+\frac{i\phi_\eta^2(y)}{\nu})^\nu$. Using $(*)$, and the fact that $|e^{-4i\pi^2 y^2}| = 1$, we obtain, for finite t, that $\hat{F_{y_0}}(y,t) \simeq e^{-4i\pi^2 y^2 t}\hat{f_{y_0}}(y)$. Now the result follows from Lemma 0.84 and the claim in Lemma 0.23.

□

Theorem 0.28. *Let* $g \in S(\mathcal{R})$, *and* G *be as in Lemma 0.2. Let* $y_0 \in \overline{\mathcal{R}_\eta}$ *with* $y_0 \leq min\{\eta^{\frac{1}{5}}, \nu^{\frac{1}{5}}\}$. *Let* F_{y_0} *be as in Lemma 0.24. Then, for finite* t, *and finite* x, $(x,t) \in \overline{\mathcal{S}_{\eta,\nu}}$, *we have that* ${}^\circ F_{y_0}(x,t) = G({}^\circ x, {}^\circ t)$.

Proof. By the inversion theorem $(*)$ in Lemma 0.12, Lemma 0.27, and Lemma 0.84 we have that;

$${}^\circ F_{y_0}(x,t) = \int_{y\in\overline{\mathcal{R}_\eta}, y\ finite} e^{-i4\pi^{2\circ}y^{2\circ}t^\circ}\hat{f_{y_0}}(y) exp(2\pi i^\circ y^\circ x) dL(\mu_\eta)(y) \ (*)$$

We have, by Lemma 0.23 that, for $y \in \overline{\mathcal{R}_\eta}$, y finite, $\hat{f_{y_0}}(y) = \hat{g}_\eta(y)$. By Lemma 0.13, for y finite, ${}^\circ\hat{g}_\eta(y) = \hat{g}({}^\circ y)$. Comparing $(*)$ with the expression;

$$G({}^\circ x, {}^\circ t) = \int_{\mathcal{R}} e^{-i4\pi^2 y^{2\circ}t}\mathcal{F}(g)(y) e^{2\pi i y^\circ x} d\mu$$

obtained in Lemma 0.2, gives the result that ${}^\circ F_{y_0}(x,t) = G({}^\circ x, {}^\circ t)$ as required.

□

Definition 0.29. *If* $\{f,g\} \subset V(\overline{\mathcal{R}_\eta})$, *then we define the convolution* $(f*g) \in V(\overline{\mathcal{R}_\eta})$ *by;*

$(f*g)(x)=\int_{\overline{\mathcal{V}_\eta}} f(x-y)g(y)d\mu_\eta(y)$

with the convention that;

$f(-\frac{(\eta+1)}{2})=f(\frac{(\eta-1)}{2})$ *and* $f(\frac{(\eta+1)}{2})=f(-\frac{(\eta-1)}{2})$

Lemma 0.30. *We have, for* $\{f,g\}\subset V(\overline{\mathcal{R}}_\eta)$*, that;*

$(f\hat{*}g)(y)=\hat{f}\hat{g}(y)$

for $y\in\overline{\mathcal{R}_\eta}$*.*

Proof. This is a straightforward computation. We have that;

$$\hat{f}\hat{g}(y)=\frac{1}{\eta}({}^*\sum_{j=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} f(\frac{j}{\sqrt{\eta}})exp_\eta(\frac{-2\pi iyj}{\sqrt{\eta}}))({}^*\sum_{k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} g(\frac{k}{\sqrt{\eta}})exp_\eta(\frac{-2\pi iyk}{\sqrt{\eta}}))$$

$$=\frac{1}{\eta}({}^*\sum_{j,k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} f(\frac{j}{\sqrt{\eta}})g(\frac{k}{\sqrt{\eta}})exp_\eta(\frac{-2\pi iy(k+j)}{\sqrt{\eta}}))$$

$$=\frac{1}{\eta}({}^*\sum_{k=-\frac{(\eta-1)}{2},l=k-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2},k+\frac{(\eta-1)}{2}mod\eta} f(\frac{l-k}{\sqrt{\eta}})g(\frac{k}{\sqrt{\eta}})exp_\eta(\frac{-2\pi iyl}{\sqrt{\eta}}))\ (l=k+j)$$

$$=\frac{1}{\sqrt{\eta}}({}^*\sum_{l=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}}(\int_{\mathcal{V}_\eta} f(\frac{l}{\sqrt{\eta}}-y)g(y)d\mu_\eta(y))exp_\eta(\frac{-2\pi iyl}{\sqrt{\eta}}))$$

$$=\frac{1}{\sqrt{\eta}}({}^*\sum_{l=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}}(f*g)(\frac{l}{\sqrt{\eta}}))exp_\eta(\frac{-2\pi iyl}{\sqrt{\eta}})$$

$$=(f\hat{*}g)(y)$$

□

Lemma 0.31. *There exists a unique* $K\in V(\overline{\mathcal{R}_\eta})$ *with;*

$\hat{K}(y)=-\frac{4\pi^2 i}{\eta}([\sqrt{\eta}y]^2)\ (*)$

for $y\in\overline{\mathcal{R}_\eta}$*.*

Moreover;

$K(0)=\frac{-\pi^2 i}{\eta^{\frac{3}{2}}}\frac{(\eta-1)\eta(\eta+1)}{3}$

and, there exists rational functions $\{p(z),q(z)\}\subset{}^*\mathcal{R}(z)$*, such that;*

$K(x)=-\frac{8\pi^2 i}{\eta^{\frac{3}{2}}}Re(W(x))$

where;

$W(\frac{j}{\sqrt{\eta}}) = p(exp_\eta(\frac{\pi ij}{\eta}))$, *for* $0 < |j| \leq \frac{(\eta-1)}{2}$, j *odd.*

$W(\frac{j}{\sqrt{\eta}}) = q(exp_\eta(\frac{\pi ij}{\eta}))$, *for* $0 < |j| \leq \frac{(\eta-1)}{2}$, j *even.*

Proof. Clearly, if K is defined by;

$K(x) = -4\pi^2 i \int_{\overline{\mathcal{R}}_\eta} y_\eta^2 exp_\eta(2\pi iyx) d\mu_\eta(y)$

then K has the property $(*)$, and K is unique by the inversion theorem. We proceed to calculate K explicitly. We have that;

$K(x) = \frac{-4\pi^2 i}{\sqrt{\eta}} {}^*\sum_{k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} \frac{k^2}{\eta} exp_\eta(\frac{2\pi ixk}{\sqrt{\eta}})$

$= \frac{-4\pi^2 i}{\eta^{\frac{3}{2}}}({}^*\sum_{k=1}^{\frac{(\eta-1)}{2}} k^2 exp_\eta(\frac{2\pi ix}{\sqrt{\eta}})^k + {}^*\sum_{k=1}^{\frac{(\eta-1)}{2}} k^2 exp_\eta(\frac{-2\pi ix}{\sqrt{\eta}})^k)$

We then have;

$K(0) = \frac{-4\pi^2 i}{\eta^{\frac{3}{2}}} 2 {}^*\sum_{k=1}^{\frac{(\eta-1)}{2}} k^2$

$= \frac{-8\pi^2 i}{\eta^{\frac{3}{2}}} \frac{(\eta-1)\eta(\eta+1)}{24}$

$= \frac{-\pi^2 i}{\eta^{\frac{3}{2}}} \frac{(\eta-1)\eta(\eta+1)}{3}$

transferring Faulhaber's formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

We now calculate;

${}^*\sum_{k=1}^{\frac{\eta-1}{2}} k^2 z^k$

We have that;

${}^*\sum_{k=1}^{\frac{\eta-1}{2}} z^k = \frac{z^{\frac{\eta+1}{2}}-1}{z-1}$, $(z \neq 1)$

Differentiating, we obtain;

${}^*\sum_{k=1}^{\frac{\eta-1}{2}} kz^{k-1} = \frac{(\eta+1)z^{\frac{\eta-1}{2}}}{2(z-1)} - \frac{z^{\frac{\eta+1}{2}}-1}{(z-1)^2}$

Hence;

$^{*}\sum_{k=1}^{\frac{\eta-1}{2}} kz^{k}=\frac{(\eta+1)z^{\frac{\eta+1}{2}}}{2(z-1)}-\frac{z(z^{\frac{\eta+1}{2}}-1)}{(z-1)^{2}}$

$=\frac{\frac{\eta-1}{2}(z^{\frac{\eta+1}{2}})+1}{(z-1)}-\frac{(z^{\frac{\eta+1}{2}}-1)}{(z-1)^{2}}$

Differentiating again;

$^{*}\sum_{k=1}^{\frac{\eta-1}{2}} k^{2}z^{k-1}=\frac{(\eta-1)(\eta+1)z^{\frac{\eta-1}{2}}}{4(z-1)}-\frac{\frac{\eta-1}{2}(z^{\frac{\eta+1}{2}})+1}{(z-1)^{2}}$

$-\frac{(\eta+1)z^{\frac{\eta-1}{2}}}{2(z-1)^{2}}+\frac{2(z^{\frac{\eta+1}{2}}-1)}{(z-1)^{3}}$

It follows that;

$^{*}\sum_{k=1}^{\frac{\eta-1}{2}} k^{2}z^{k}=\frac{(\eta-1)(\eta+1)z^{\frac{\eta+1}{2}}}{4(z-1)}-\frac{\frac{\eta-1}{2}(z^{\frac{\eta+3}{2}})+z}{(z-1)^{2}}$

$-\frac{(\eta+1)z^{\frac{\eta+1}{2}}}{2(z-1)^{2}}+\frac{2(z^{\frac{\eta+3}{2}}-z)}{(z-1)^{3}}$

$=\frac{\frac{\eta-1}{2}\frac{\eta+1}{2}z^{\frac{\eta+3}{2}}-\frac{\eta-1}{2}\frac{\eta+1}{2}z^{\frac{\eta+1}{2}}-\frac{\eta-1}{2}z^{\frac{\eta+3}{2}}-z-\frac{\eta+1}{2}z^{\frac{\eta+1}{2}}}{(z-1)^{2}}+\frac{2(z^{\frac{\eta+3}{2}}-z)}{(z-1)^{3}}$

$=\frac{(\frac{\eta-1}{2})^{2}z^{\frac{\eta+3}{2}}-(\frac{\eta+1}{2})^{2}z^{\frac{\eta+1}{2}}-z}{(z-1)^{2}}+\frac{2(z^{\frac{\eta+3}{2}}-z)}{(z-1)^{3}}$

$=\frac{(\frac{\eta-1}{2})^{2}z^{\frac{\eta+5}{2}}-(\frac{\eta+1}{2})^{2}z^{\frac{\eta+3}{2}}-z^{2}-(\frac{\eta-1}{2})^{2}z^{\frac{\eta+3}{2}}+(\frac{\eta+1}{2})^{2}z^{\frac{\eta+1}{2}}+z+2z^{\frac{\eta+3}{2}}-2z}{(z-1)^{3}}$

$=\frac{(\frac{\eta-1}{2})^{2}z^{\frac{\eta+5}{2}}-(\frac{\eta^{2}-3}{2})z^{\frac{\eta+3}{2}}+(\frac{\eta+1}{2})^{2}z^{\frac{\eta+1}{2}}-z^{2}-z}{(z-1)^{3}}$

Now substituting $z=exp_{\eta}(\frac{2\pi ix}{\sqrt{\eta}})$, and using the fact that $exp_{\eta}(\frac{2\pi ij}{\eta})^{\frac{\eta}{2}}=(-1)^{j}$, we obtain, for $0<|j|\leq\frac{(\eta-1)}{2}$ and $[\sqrt{\eta}x]\neq 0$, $x\in\overline{\mathcal{R}_{\eta}}$, that;

$K(x)=-\frac{8\pi^{2}i}{\eta^{\frac{3}{2}}}Re(W(x))$

where;

$W(\frac{j}{\sqrt{\eta}})=\frac{(-1)^{j}(\frac{(\eta-1)}{2})^{2}exp_{\eta}(\frac{5\pi ij}{\eta})-exp_{\eta}(\frac{4\pi ij}{\eta})+((-1)^{j+1}(\frac{\eta^{2}-3}{2}))exp_{\eta}(\frac{3\pi ij}{\eta})-exp_{\eta}(\frac{2\pi ij}{\eta})+((-1)^{j}(\frac{\eta+1}{2})^{2})exp_{\eta}(\frac{\pi ij}{\eta})}{(exp_{\eta}(\frac{2\pi ij}{\eta})-1)^{3}}$

Now, set;

$$p(z) = \frac{-(\frac{(\eta-1)}{2})^2 z^5 - z^4 + \frac{(\eta^2-3)}{2} z^3 - z^2 - (\frac{(\eta+1)}{2})^2 z}{(z^2-1)^3}$$

$$q(z) = \frac{(\frac{(\eta-1)}{2})^2 z^5 - z^4 - \frac{(\eta^2-3)}{2} z^3 - z^2 + (\frac{(\eta+1)}{2})^2 z}{(z^2-1)^3}$$

□

Lemma 0.32. *Given a measurable boundary condition $f \in V(\overline{\mathcal{R}_\eta})$, there exists a unique measurable $F \in \overline{\mathcal{S}_{\eta,\nu}}$, satisfying the nonstandard convolution equation;*

$$\frac{\partial F}{\partial t} = (K * F)$$

on $(\overline{\mathcal{T}_\nu} \setminus [\nu - \frac{1}{\nu}, \nu)) \times \overline{\mathcal{R}_\eta}$

with $F(0,x) = f(x)$, *for* $x \in \overline{\mathcal{R}_\eta}$.

Moreover, the solution is given explicitly by;

$$F_{\frac{i}{\nu}} = L^{(i)} * f$$

where $L^{(i)}$ is the i'th convolution product of L, and $L \in V(\overline{\mathcal{R}_\eta})$ is defined in terms of K by;

$$L(0) = \sqrt{\eta} + \frac{K(0)}{\nu}$$

$L(\frac{r}{\sqrt{\eta}}) = \frac{K(\frac{r}{\sqrt{\eta}})}{\nu}$, *for* $1 \leq |r| \leq \frac{(\eta-1)}{2}$

Proof. We have that $F(0,x) = f(x)$, and, for $i \geq 0$;

$$F(\tfrac{i+1}{\nu}, \tfrac{j}{\eta}) = F(\tfrac{i}{\nu}, \tfrac{j}{\eta}) + \tfrac{1}{\sqrt{\eta}\nu} {}^* \textstyle\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} K(\tfrac{j-k}{\sqrt{\eta}}) F(\tfrac{i}{\nu}, \tfrac{k}{\sqrt{\eta}})$$

$$= \tfrac{1}{\sqrt{\eta}} {}^* \textstyle\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} L(\tfrac{j-k}{\sqrt{\eta}}) F(\tfrac{i}{\nu}, \tfrac{k}{\sqrt{\eta}})$$

where L is defined as in the hypotheses. In particular, the values of $F_{i+1\nu}$ are determined by the values of the previous step $F_{\frac{i}{\nu}}$. Moreover;

$$F_{\frac{i+1}{\nu}} = L * F_{\frac{i}{\nu}}$$

$$F_{\frac{i}{\nu}} = L^{(i)} * f$$

where $L^{(i)}$ is the i'th convolution product of L, using associativity of the convolution product.

□

Lemma 0.33. *Let $f \in V(\overline{\mathcal{R}_\eta})$, and let F solve the nonstandard convolution equation, as in Lemma 0.32, with initial condition f. Then;*

$\hat{F}(y,t) = (1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})^{[\nu t]}\hat{f}(y)$

for $y \in \mathcal{R}_\eta$, $t \in \overline{\mathcal{T}_\nu}$.

Moreover, if $\nu \geq \eta^5$, for finite $t \in \overline{\mathcal{T}_\nu}$

$(1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})^{[\nu t]} \simeq {}^*exp(-4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}it)$

Then, if $g \in S(\mathcal{R})$ and $G \in S(T)$ are as as in Lemma 0.2, if g_η and G_η solves the nonstandard convolution equation, we have that;

${}^\circ G_\eta(x,t) = G({}^\circ x, {}^\circ t)$

for $x \in \overline{\mathcal{V}_\eta}$, x finite, $t \in \mathcal{T}_\nu$, t finite.

Proof. Taking the nonstandard Fourier transform of both sides of the equation, and, using the property of convolutions, Lemma 0.30 we obtain;

$\frac{d}{dt}(\hat{F}(y,t)) = \hat{K}(y)\hat{F}(y,t)$, $y \in \overline{\mathcal{R}_\eta}$

Using the definition of the nonstandard derivative, and the property of K in Lemma 0.31, we obtain that;

$\nu(\hat{F}(y,t+\frac{1}{\nu}) - \hat{F}(y,t)) = -4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}i\hat{F}(y,t)$

for $y \in \overline{\mathcal{R}_\eta}$. Rearranging, we obtain that;

$\hat{F}(y,t+\frac{1}{\nu}) = (1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})\hat{F}(y,t)$

for $t \in \overline{\mathcal{T}_\nu}$. Iterating this expression, and using the initial condition $F_0 = f$, we obtain;

$\hat{F}(y,t) = (1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})^{[\nu t]}\hat{f}(y)$, (†)

as required. Now, using the claim $(**)$ in Lemma 0.26 and the fact that $|{}^*exp(-4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}i)|=1$, we have that if $|-4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}i|\leq\frac{\nu^{\frac{1}{2}}}{4}$, $(*)$, then;

$$(1-\frac{4\pi^2[\sqrt{\eta}y]^2i}{\eta\nu})^\nu\simeq{}^*exp(-4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}i),\ (**)$$

Clearly, we can achieve $(*)$, for $y\in\overline{\mathcal{R}_\eta}$, by taking;

$$\nu^{\frac{1}{2}}\geq 16\pi^2\frac{(\eta-1)^2}{4}=4\pi^2(\eta-1)^2$$

which is true if $\nu\geq\eta^5$. As both sides of $(**)$ are finite, using S-continuity of x^t, for finite $t\in\overline{\mathcal{T}_\nu}$, on a bounded domain, we have that;

$$(1-\frac{4\pi^2[\sqrt{\eta}y]^2i}{\eta\nu})^{[\nu t]}\simeq(1-\frac{4\pi^2[\sqrt{\eta}y]^2i}{\eta\nu})^{\nu t}\simeq{}^*exp(-4\pi^2\frac{[\sqrt{\eta}y]^2}{\eta}it)$$

which gives the second result. For the final claim, we have, by the inversion theorem $(*)$ in Lemma 0.12, that;

$$G_\eta(x,t)=\int_{\overline{\mathcal{R}_\eta}}\hat{G}_\eta(y,t)exp_\eta(2\pi iyx)d\mu_\eta(y)$$

We claim that there exists a constant $H\in\mathcal{R}$, with;

$$|\hat{G}_\eta(y,t)|\leq\frac{H}{y^2},\text{ for }y\in\overline{\mathcal{R}_\eta},\ |y|\geq 1\ (***)$$

To see this, we can use the computation $(\dagger)$, Lemma 0.84, and, with the assumption on ν, the second result of this lemma, which implies that $(1-\frac{4\pi^2[\sqrt{\eta}y]^2i}{\eta\nu})^{[\nu t]}$ is uniformly bounded. It follows that we can ignore the nonstandard Fourier transform values $\hat{G}_\eta(y,t)$, for infinite $y\in\overline{\mathcal{R}_\eta}$, when taking standard parts, and obtain that;

$${}^\circ G_\eta(x,t)=\int_{y\in\overline{\mathcal{R}_\eta},y\ finite}{}^\circ\hat{G}_\eta(y,t){}^\circ exp_\eta(2\pi iyx)dL(\mu_\eta(y)),\ (****)$$

By the second claim of the lemma, we have that;

$${}^\circ\hat{G}_\eta(y,t)={}^\circ({}^*exp)(-4\pi^2it\frac{[\sqrt{\eta}y]^2}{\eta}){}^\circ\hat{g}_\eta(y)$$

$$=e^{-4\pi^2i{}^\circ t{}^\circ y^2}{}^\circ\hat{g}_\eta(y),\ (\dagger\dagger)$$

for finite $y\in\overline{\mathcal{R}_\eta}$, finite $t\in\overline{\mathcal{T}_\nu}$, and by Lemma 0.13;

$^{\circ}\hat{g}_\eta(y) = \hat{g}(^{\circ}y)$ $(\dagger\dagger\dagger)$

for finite $y \in \overline{\mathcal{R}_\eta}$. Combining $(***\,*), (\dagger\dagger), (\dagger\dagger\dagger)$, and using S-continuity of $exp_\eta(2\pi iyx)$, for finite $\{x, y\} \subset \overline{\mathcal{R}_\eta}$, gives;

$^{\circ}G_\eta(x,t) = \int_{\mathcal{R}} e^{-4\pi^{2\circ}y^{2}i^{\circ}t}\hat{g}(^{\circ}y)e^{2\pi i^{\circ}y^{\circ}x}d\mu(y)$, $(*****)$

Comparing $(*****)$ with the conclusion of Lemma 0.2, we obtain the final result.

□

Lemma 0.34. *Let notation be as in the previous lemma, then there exists a constant $C \in \mathcal{R}_{\geq 0}$ such that;*

$|\hat{g_\eta}(y)| \leq \frac{C}{|y|^4}$

$|(\hat{g_\eta})^D(y)| \leq \frac{C}{|y|^3}$

$|(\hat{g_\eta})^{D^2}(y) \leq \frac{C}{|y|^2}$

for $y \in \overline{\mathcal{R}_\eta}$, $|y| > 1$.

Proof. The first claim uses a simple generalisation of Lemma 0.84. Following the proof there, we can find a constant $D \in \mathcal{R}$ with;

$|\hat{g_\eta}(y)| \leq \frac{D|\hat{}((g_\eta)^{D^4})|}{|y|^4}$ $|y| > 1$

It is therefore sufficient to prove that $\hat{}((g_\eta)^{D^4})$ is bounded. This is true, using Lemma 0.17, if we can prove that;

$g_\eta^{D^4}(x) \leq \frac{E}{|x|^2}$ for $|x| > 1$

This is a simple generalisation of Lemma 0.21. As in the proof there, we can use the fact that $(x_\eta^2 g_\eta)^{D^4}(x) \simeq ((x^2 g)^{(4)})(x)$, for $x \in \overline{\mathcal{R}_\eta}$, to show that $(x_\eta^2 g_\eta)^{D^4}$ is bounded. Then, using the Liebniz rule for derivatives, as in Lemma 0.18, we compute this term and use inductive bounds on the terms involving $\{g_\eta, g_\eta^D, g_\eta^{D^2}, g_\eta^{D^3}\}$, together with the computation of the one step derivatives of x_η^2. The details are left to the reader. For the second claim, we compute;

$(\hat{g_\eta})^D(y) = (\int_{\overline{\mathcal{R}_\eta}} g_\eta(x) exp_\eta(-2\pi iyx) d\mu_\eta(x))^D$

$= \int_{\overline{\mathcal{R}_\eta}} g_\eta(x)\sqrt{\eta}(exp_\eta(-2\pi i(y+\frac{1}{\sqrt{\eta}})x) - exp_\eta(-2\pi iyx))d\mu_\eta(x)$

$= \int_{\overline{\mathcal{R}_\eta}} g_\eta(x)\chi_\eta(x) exp_\eta(-2\pi iyx) d\mu_\eta(x)$

$= (g_\eta \hat{\chi}_\eta)(y)$

where $\chi_\eta(x) = \sqrt{\eta}(exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}})-1)$ is as in Lemma 0.19. Again, generalising the proof of Lemma 0.84, we can find a constant $D \in \mathcal{R}$, with;

$|(g_\eta \hat{\chi}_\eta)(y)| \leq \frac{D|\hat{}((g_\eta\chi_\eta)^{D^3})|}{|y|^3}$, $|y| > 1$

so it is sufficient to prove that $\hat{}((g_\eta\chi_\eta)^{D^3})$ is bounded, which follows from;

$(g_\eta\chi_\eta)^{D^3} \leq \frac{E}{|x|^2}$ $|x| > 1$ $(*)$

for some $E \in \mathcal{R}$. We compute;

$(g_\eta\chi_\eta)^{D^3}(x) = g_\eta^{D^3}\chi_\eta^{lsh^3}(x) + 3g_\eta^{D^2}(\chi_\eta^D)^{lsh^2}(x) + 3g_\eta^D(\chi_\eta^{D^2})^{lsh}(x) + g_\eta\chi_\eta^{D^3}(x)$

$\chi_\eta^D(x) = \eta(exp_\eta(-\frac{2\pi i(x+\frac{1}{\sqrt{\eta}})}{\sqrt{\eta}}) - exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}}))$

$= \eta exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}})(exp_\eta(-\frac{2\pi i}{\eta}) - 1)$

and, similarly;

$\chi_\eta^{D^2}(x) = \eta^{\frac{3}{2}} exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}})(exp_\eta(-\frac{2\pi i}{\eta}) - 1)^2$

$\chi_\eta^{D^3}(x) = \eta^2 exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}})(exp_\eta(-\frac{2\pi i}{\eta}) - 1)^3$

We have that $\eta(exp_\eta(-\frac{2\pi i}{\eta}) - 1) \simeq 1$, and $|exp_\eta(-\frac{2\pi ix}{\sqrt{\eta}})| = 1$, for $x \in \overline{\mathcal{R}_\eta}$. It follows that $|\chi_\eta^D| \leq D$, $|\chi_\eta^{D^2}| \leq \frac{D}{\sqrt{\eta}}$, $|\chi_\eta^{D^3}| \leq \frac{D}{\eta}$, for some $D \in \mathcal{R}$, in particular, $\{\chi_\eta^D, \chi_\eta^{D^2}, \chi_\eta^{D^3}\}$ are bounded, and, similarly, for $\{(\chi_\eta^D)^{lsh^2}, (\chi_\eta^{D^2})^{lsh}, (\chi_\eta^{D^3})\}$. By Lemma 0.21, it follows that;

$max\{|g_\eta^{D^2}(\chi_\eta^D)^{lsh^2}(x)|, |g_\eta^D(\chi_\eta^{D^2})^{lsh}(x)|, |g_\eta(\chi_\eta^{D^3})(x)|\} \leq \frac{F}{|x|^2}$, $|x| > 1$ $(**)$

for some $F\in\mathcal{R}$. By Lemma 0.20, we have $4|x^{lsh^3}|\leq|\chi_\eta^{lsh^3}(x)|\leq 8|x^{lsh^3}|$, (†). Generalising Lemma 0.21, by considering $x^3g\in S(\mathcal{R})$, we can show that;

$|g_\eta^{D^3}|\leq\frac{H}{|x|^3}$, $|x|>1$

for some $H\in\mathcal{R}$. Combining this result with (†) and (∗∗) easily proves (∗) as required. For the final claim, we compute;

$$(\hat{g_\eta})^{D^2}(y)=(\int_{\overline{\mathcal{R}_\eta}}g_\eta(x)exp_\eta(-2\pi iyx)d\mu_\eta(x))^{D^2}$$

$$=\int_{\overline{\mathcal{R}_\eta}}g_\eta(x)\eta(exp_\eta(-2\pi i(y+\tfrac{2}{\sqrt{\eta}})x)-2exp_\eta(-2\pi i(y+\tfrac{1}{\sqrt{\eta}})x)+exp_\eta(-2\pi iyx))d\mu_\eta(x)$$

$$=\int_{\overline{\mathcal{R}_\eta}}g_\eta(x)\chi_\eta^2(x)exp_\eta(-2\pi iyx)d\mu_\eta(x)$$

$$=(g_\eta\hat{\chi}_\eta^2)(y)$$

Proceeding as in the proof of the second claim, it is sufficient to show that;

$(g_\eta\chi_\eta^2)^{D^2}\leq\frac{E}{|x|^2}$ $|x|>1$ $(***)$

for some $E\in\mathcal{R}$. We have that;

$$(g_\eta\chi_\eta^2)^{D^2}=g_\eta^{D^2}\chi_\eta^{2,lsh}+2g_\eta^D(\chi_\eta^{D,lsh}\chi_\eta^{lsh^2}+\chi_\eta^{lsh}\chi_\eta^{D,lsh})$$

$$+g_\eta(\chi_\eta^{D^2}\chi_\eta^{lsh^2}+\chi_\eta^{D^2}\chi_\eta^{lsh}+\chi_\eta^D\chi_\eta^{D,lsh}+\chi_\eta^D\chi_\eta^D)$$

As above, we have $\{\chi_\eta^D,\chi_\eta^{D^2},\chi_\eta^{D,lsh}\}$ are bounded. Using Lemma 0.20, we have that;

$16|x|^2\leq|\chi_\eta^2(x)|\leq 64|x|^2$, $x\in\overline{\mathcal{R}_\eta}$

and a similar estimate holds for $\chi_\eta^{2,lsh}$. Similarly, we can generalise the estimate of 0.20 to χ_η^{lsh} and $\chi_\eta^{lsh^2}$. To prove $(***)$, it is, therefore, sufficient to find a constant $D\in\mathcal{R}$, with;

$|g_\eta(x)|\leq\frac{D}{|x|^3}$, $|g_\eta^D(x)|\leq\frac{D}{|x|^3}$, $|g_\eta^{D^2}(x)|\leq\frac{D}{|x|^4}$, $|x|>1$

As above, this follows by generalising Lemma 0.21, the details are left to the reader.

□

Theorem 0.35. *Let hypotheses be as in Lemma 0.33, then, for finite $t \in \overline{\mathcal{T}_\nu}$, with ${}^\circ t > 0$, there exists a constant $C_{{}^\circ t} \in \mathcal{R}$ such that;*

$$|G_\eta(x,t)| \leq \frac{C_{{}^\circ t}}{|x|^2},\ |x| > 1$$

In, particular, $G_{\eta,t}$ is S-integrable and, for infinite $x \in \overline{\mathcal{R}_\eta}$, ${}^\circ G_\eta(x,t) = 0$. Moreover;

$$^\circ G_{\eta,t} = st^*(G_{{}^\circ t,\infty})$$

where $G_{{}^\circ t,\infty}$ is the extension of $G_{{}^\circ t}$ obtained by setting $G_{{}^\circ t}(\infty) = G_{{}^\circ t}(-\infty) = 0$.

Proof. We let $H_{\eta,t} = \hat{G_{\eta,t}}$. Generalising Lemma 0.19, we have that, if $h \in V(\overline{\mathcal{R}_\eta})$, then;

$$\check{(}h^{D^2})(x) = \psi_\eta^2(x) W_\eta^2(x)\check{h}(x)\ (*)$$

for $x \in \mathcal{R}_\eta$, where;

$$\psi_\eta(x) = \sqrt{\eta}(exp_\eta(\frac{2\pi i x}{\sqrt{\eta}}) - 1)$$

$$W_\eta(x) = exp_\eta(\frac{-2\pi i x}{\sqrt{\eta}})$$

for $x \in \mathcal{R}_\eta$. Generalising 0.20, we have again that;

$$4|x| \leq |\psi_\eta(x)| \leq 8|x|, \text{ for } x \in \overline{\mathcal{R}_\eta}\ (**)$$

Combining $(*), (**)$, if $\check{(}H_{\eta,t}^{D^2})$ is bounded, $(***)$, it follows that there exists a constant $C \in \mathcal{R}$ with;

$$|\check{H_{\eta,t}}(x)| \leq \frac{C}{|x|^2}, \text{ for } |x| > 1$$

The first result then follows by the nonstandard inversion theorem, that $G_{\eta,t} = \check{H_{\eta,t}}$. To show $(***)$, following the method of Lemma 0.34, it is sufficient to prove that $H_{\eta,t}^{D^2}$ is bounded and there exists a constant $D \in \mathcal{R}$, with;

$$|H_{\eta,t}^{D^2}(y)| \leq \frac{D}{|y|^2} \text{ for } |y| > 1\ (****)$$

By Lemma 0.33, we have that;

$H_{\eta,t}(y)=\theta_{\eta,\nu,,-4\pi^2,t}(y)\hat{g_\eta}(y)$

for $y\in\overline{\mathcal{R}_\eta}$, where $\theta_{\eta,\nu,-4\pi^2,t}(y)=(1-\frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})^{[\nu t]}$. We then compute;

$H_{\eta,t}^{D^2}(y)=\theta^{D^2}_{\eta,\nu,-4\pi^2,t}(\hat{g_\eta})^{lsh^2}(y)+2\theta^{D}_{\eta,\nu,-4\pi^2,t}(\hat{g_\eta})^{D,lsh}(y)+\theta_{\eta,\nu,-4\pi^2,t}(\hat{g_\eta})^{D^2}(y)$

We claim there exists a constant $F\in\mathcal{R}$, with;

$|\theta^{D^2}_{\eta,\nu,-4\pi^2,t}(y)|\leq F|y|^2$

$|\theta^{D}_{\eta,\nu,-4\pi^2,t}(y)|\leq F|y|$

$|\theta_{\eta,\nu,-4\pi^2,t}(y)|\leq F$ $(\dagger)$

Combining $(\dagger)$ with the result of Lemma 0.34 proves $(****)$ as required. To see $(\dagger)$, we let $\theta_{n,c}:\mathcal{R}\rightarrow\mathcal{C}$, for $n\in\mathcal{N}$, $c\in\mathcal{R}$, be defined by;

$\theta_{n,c}(y)=(1+\frac{icy^2}{n})^n$

Then, if $\epsilon>0$, we have, using the binomial expansion, swapping summation, and differentiating, that;

$\frac{\theta_{n,c}(y+\epsilon)-\theta_{n,c}(y)}{\epsilon}=\frac{(1+\frac{ic(y+\epsilon)^2}{n})^n-(1+\frac{icy^2}{n})^n}{\epsilon}$

$=\frac{1}{\epsilon}(\sum_{m=0}^n\frac{C^n_m(ic)^m(y+\epsilon)^{2m}}{n^m}-\sum_{m=0}^n\frac{C^n_m(ic)^m y^{2m}}{n^m})$

$=\frac{1}{\epsilon}(\sum_{m=0}^n\frac{C^n_m(ic)^m}{n^m}[(y+\epsilon)^{2m}-y^{2m}])$

$=\frac{1}{\epsilon}(\sum_{m=0}^n\frac{C^n_m(ic)^m}{n^m}[\sum_{r=1}^{2m}C^{2m}_r y^{2m-r}\epsilon^r])$

$=(\sum_{m=0}^n\frac{C^n_m(ic)^m}{n^m}[\sum_{r=1}^{2m}C^{2m}_r y^{2m-r}\epsilon^{r-1}])$

$=(\sum_{m=0}^n\frac{C^n_m(ic)^m}{n^m}[\sum_{r=0}^{2m-1}C^{2m}_{r+1}y^{2m-r-1}\epsilon^r])$

$=(\sum_{r=0}^{2m-1}[\sum_{m=0}^n\frac{C^n_m(ic)^m C^{2m}_{r+1}y^{2m-r-1}}{n^m}]\epsilon^r)$

$=(\sum_{r=0,r\ even}^{2n-1}[\sum_{m=\frac{r+2}{2}}^n\frac{C^n_m(ic)^m C^{2m}_{r+1}y^{2m-r-1}}{n^m}]\epsilon^r)+(\sum_{r=0,r\ odd}^{2n-1}[\sum_{m=\frac{r+1}{2}}^n\frac{C^n_m(ic)^m C^{2m}_{r+1}y^{2m-r-1}}{n^m}]\epsilon^r)$

$= (\sum_{r=0,r\ even}^{2n-1}[\sum_{m=\frac{r+2}{2}}^{n}\frac{C_m^n(ic)^m y^{2m}}{n^m(r+1)!}]^{(r+1)}\epsilon^r)+(\sum_{r=0,r\ odd}^{2n-1}[\sum_{m=\frac{r+1}{2}}^{n}\frac{C_m^n(ic)^m y^{2m}}{n^m(r+1)!}]^{(r+1)}\epsilon^r)$

$= (\sum_{r=0,r\ even}^{2n-1}[\sum_{m=0}^{n}\frac{C_m^n(ic)^m y^{2m}}{n^m}]^{(r+1)}\frac{\epsilon^r}{(r+1)!})+(\sum_{r=0,r\ odd}^{2n-1}[\sum_{m=0}^{n}\frac{C_m^n(ic)^m y^{2m}}{n^m}]^{(r+1)}\frac{\epsilon^r}{(r+1)!})$

$= (\sum_{r=0}^{2n-1}[(1+\frac{icy^2}{n})^n]^{(r+1)}\frac{\epsilon^r}{(r+1)!})$ (††)

Differentiating without collecting terms, and repeatedly using the product rule, we have that;

$[(1+\frac{icy^2}{n})^n]^{(r+1)} = \sum_{k=1}^{2^{r+1}}\alpha_k y^{i(k)}(1+\frac{iy^2}{n})^{j(k)}$

where $0 \leq i(k) \leq r+1$ and $n-(r+1) \leq j(k) \leq n-1$. For $|y| > 1$, we have that $|y|^{i(k)} \leq |y|^{r+1}$, and, as $|1+\frac{icy^2}{n}| > 1$, that $|(1+\frac{icy^2}{n})^{j(k)}| < |(1+\frac{icy^2}{n})^n|$. It follows that, for $|c| \geq 1$;

$|[(1+\frac{icy^2}{n})^n]^{(r+1)}| \leq (2|c|)^{r+1}max_{1\leq k\leq 2^{r+1}}(|\alpha_k|)|y|^{r+1}|(1+\frac{icy^2}{n})^n|$ (†††)

Considering routes through the binary tree formed by differentiations, we can, for r even, assume that;

$|\alpha_k| \leq max_{0\leq s\leq\frac{r-2}{2}}(2|c|)^{\frac{r+2}{2}+s}(\frac{r+2}{2}+s)\ldots(2s+2)$ (††††)

for $1 \leq k \leq 2^{r+1}$. We then have that;

$|\frac{(2|c|)^{r+1}\alpha_k}{(r+1)!}| \leq max_{0\leq s\leq\frac{r-2}{2}}\frac{(2|c|)^{\frac{3r}{2}+2+s}}{(r+1)\ldots(\frac{r}{2}+2+s)(2s+1)\ldots1}$

$\leq max_{0\leq s\leq\frac{r-2}{2}}\frac{(8|c|^3)^{\frac{r}{2}+s+1}}{(r+1)\ldots(\frac{r}{2}+2+s)(2s+1)\ldots1} \leq (8|c|^3)^{[8|c|^3]}$

where we have use the fact that there are $\frac{r}{2}+s+1$ terms in the denominator. A similar calculation follows for r odd and is left to the reader. It follows that;

$|\theta_{n,c}^{D_\epsilon}(y)| \leq (8|c|^3)^{[8|c|^3]}\sum_{r=0}^{2n-1}|y|^{r+1}|(1+\frac{icy^2}{n})^n|\epsilon^r$

$= (8|c|^3)^{[8|c|^3]}|y||(1+\frac{icy^2}{n})^n|\sum_{r=0}^{2n-1}(|y|\epsilon)^r$, $|y| > 1$

Now, for $|y|\epsilon < \frac{1}{2}$, we have that $\sum_{r=0}^{2n-1}(|y|\epsilon)^r < 2$. By (∗) of Lemma 0.26, we have, if $n > 4|c||y|^2$, then;

$|(1+\frac{icy^2}{n})^n - e^{icy^2}| \leq 4|e^{iy^2}||\frac{cy^2}{n}| \leq |e^{iy^2}| = 1$

$|(1+\frac{icy^2}{n})^n|\leq 2$

It follows that;

$|\theta_{n,c}^{D_\epsilon}(y)|\leq C|y|$, $|y|>1$ (!)

where $C=4.(8|c|^3)^{[8|c|^3]}$, for $|y|\epsilon<\frac{1}{2}$, $n>4|c||y|^2$. If $|c|<1$, we can drop the constant c from the above calculation and obtain $C=4.8^8$, with $n>4|y|^2$. Transferring this result, we obtain, that for $y\in(\overline{\mathcal{R}_\eta}\setminus(\frac{\eta-1}{2\sqrt{\eta}},\frac{\eta+1}{2\sqrt{\eta}}))$, $\frac{|y|}{\sqrt{\eta}}<\frac{1}{2}$, and, for $\nu\geq 2\eta$, it follows $\nu>max(4|y|^2,4|c||y|^2)$, therefore;

$|\theta_{\eta,\nu,c}^{D}(y)|\leq C|y|$ $|y|>1$ $(*****)$

where $\theta_{\eta,\nu,c}(y)=(1+\frac{ic[\sqrt{\eta}y]^2}{\eta\nu})^\nu$.

We now compute, using (††);

$\theta_{n,c}^{D_\epsilon^2}(y)=\frac{\theta_{n,c}^{D}(y+\epsilon)-\theta_{n,c}^{D}(y)}{\epsilon}$

$=\frac{1}{\epsilon}[(\sum_{r=0}^{2n-1}[(1+\frac{ic(y+\epsilon)^2}{n})^n]^{(r+1)}\frac{\epsilon^r}{(r+1)!})-(\sum_{r=0}^{2n-1}[(1+\frac{icy^2}{n})^n]^{(r+1)}\frac{\epsilon^r}{(r+1)!})]$

$=[(\sum_{r=0}^{2n-1}[\frac{1}{\epsilon}((1+\frac{ic(y+\epsilon)^2}{n})^n-(1+\frac{icy^2}{n})^n)]^{(r+1)}\frac{\epsilon^r}{(r+1)!})]$

$=[(\sum_{r=0}^{2n-1}[\sum_{s=0}^{2n-1}[(1+\frac{icy^2}{n})^n]^{(s+1)}\frac{\epsilon^s}{(s+1)!}]^{(r+1)}\frac{\epsilon^r}{(r+1)!})]$

$=\sum_{r,s=0}^{2n-1}[(1+\frac{icy^2}{n})^n]^{(r+s+2)}\frac{\epsilon^{r+s}}{(r+1)!(s+1)!}$

$=\sum_{t=0}^{4n-2}\sum_{r+s=t,0\leq r,s\leq 2n-1}[(1+\frac{icy^2}{n})^n]^{(t+2)}\frac{\epsilon^t}{(r+1)!(s+1)!}$

$=\sum_{t=0}^{4n-2}\sum_{r=max(0,t-(2n-1))}^{min(t,2n-1)}[(1+\frac{icy^2}{n})^n]^{(t+2)}\frac{\epsilon^t}{(r+1)!(t-r+1)!}$ (†††††)

We observe trivially that for t even and $0\leq k\leq\frac{t}{2}$;

$\frac{1}{(\frac{t}{2}+k+1)!(\frac{t}{2}-k+1)!}\geq\frac{1}{(\frac{t}{2}+k+2)!(\frac{t}{2}-k)!}$

It follows that, for $0\leq t\leq 2n-1$, t even;

$max(\frac{1}{(r+1)!(t-r+1)!}:0\leq r\leq t)=\frac{1}{(\frac{t+2}{2}!)^2}$

and the same result holds for $2n-1 < t \leq 4n-2$, t even, maximizing over the range $t-(2n-1) \leq r \leq 2n-1$. By similar reasoning, for $0 \leq t \leq 2n-1$, t odd;

$$max(\tfrac{1}{(r+1)!(t-r+1)!} : 0 \leq r \leq t) = \tfrac{1}{(\frac{t+3}{2}!)(\frac{t+1}{2}!)}$$

and the same result holds for $2n-1 < t \leq 4n-2$, t odd. It follows, counting terms, that;

$$|\theta_{n,c}^{D_\epsilon^2}(y)| \leq \sum_{0\leq t\leq 2n-1, t\ even} |[(1+\tfrac{icy^2}{n})^n]^{(t+2)}| \tfrac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2}$$

$$+\sum_{0\leq t\leq 2n-1, t\ odd} |[(1+\tfrac{icy^2}{n})^n]^{(t+2)}| \tfrac{(t+1)\epsilon^t}{(\frac{t+3}{2}!)(\frac{t+1}{2}!)}$$

$$+\sum_{2n-1<t\leq 4n-2, t\ even} |[(1+\tfrac{icy^2}{n})^n]^{(t+2)}| \tfrac{(4n-1-t)\epsilon^t}{(\frac{t+2}{2}!)^2}$$

$$+\sum_{2n-1<t\leq 4n-2, t\ odd} |[(1+\tfrac{icy^2}{n})^n]^{(t+2)}| \tfrac{(4n-1-t)\epsilon^t}{(\frac{t+3}{2}!)(\frac{t+1}{2}!)}, \ (\dagger\dagger\dagger\dagger\dagger\dagger)$$

As in $(\dagger\dagger\dagger)$, we have, assuming $|y| > 1$, that;

$$|[(1+\tfrac{icy^2}{n})^n]^{(t+2)}| \leq 2^{t+2} max_{1\leq k\leq 2^{t+1}} |\beta_k||y|^{t+2}(1+\tfrac{icy^2}{n})^n$$

where, similarly to $(\dagger\dagger\dagger\dagger)$, for t even;

$$|\beta_k| \leq max_{0\leq s\leq \frac{t+2}{2}} (2|c|)^{\frac{t+2}{2}+s}(\tfrac{t+2}{2}+s)\ldots(2s+1)$$

This time, we have to compute the bound explicitly. For $0 \leq t \leq 4n-2$, t even, and $0 \leq s \leq \frac{t+2}{2}$, we let;

$$\beta_{t,s} = (2|c|)^{\frac{t+2}{2}+s}(\tfrac{t+2}{2}+s)\ldots(2s+1)$$

Then, for $0 \leq s \leq \frac{t}{2}$;

$$\tfrac{\beta_{t,s}}{\beta_{t,s+1}} = \tfrac{(2s+2)(2s+1)}{2|c|(\frac{t+2}{2}+s+1)}$$

We define $\gamma_t : \mathcal{R}_{\geq 0} \rightarrow \mathcal{R}$ by;

$$\gamma_t(v) = \tfrac{(2v+2)(2v+1)}{2|c|(\frac{t+2}{2}+v+1)}$$

Solving the equation $\gamma_t'(v) = 0$, a simple calculation produces the quadratic equation;

$2v^2+4v(1+\frac{t+2}{2})+3\frac{t+2}{2}=0$

which has no positive roots. It follows that γ_t is strictly monotone in the range $[0,\frac{t}{2}]$. Moreover, $\gamma_t(0)=\frac{2}{t+3}$ and $\gamma_t(\frac{t}{2})=\frac{t+1}{2}$, so for $t\geq 1$, γ_t is strictly increasing in the range $[0,\frac{t}{2}]$, and, there exists a unique $v_0\in[0,\frac{t}{2}]$ for which $\gamma_t(v_0)=1$. We can solve the equation $\gamma_t(v)=1$ explicitly, to obtain the quadratic equation;

$4v^2+4v-(t+2)=0$

with solution $v_0=\frac{\sqrt{t+3}-1}{2}\in[0,\frac{t}{2}]$. Let $s_0=[v_0]$, then it follows that the sequence $\{\beta_{t,s}:0\leq s\leq\frac{t+2}{2}\}$ is strictly increasing in the range $[0,s_0+1]$ and strictly decreasing in the range $[s_0+1,\frac{t+1}{2}]$. It follows that it attains its maximum at $t_0=s_0+1=[\frac{\sqrt{t+3}+1}{2}]$. We now estimate, for $0\leq t\leq 2n-1$, t even, $|c|>1$;

$|[(1+\frac{icy^2}{n})^n]^{(t+2)}|\frac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2}$

$\leq(2|c|)^{t+2}max_{1\leq k\leq 2^{t+1}}|\beta_k||y|^{t+2}(1+\frac{icy^2}{n})^n\frac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2}$

$\leq|y|^{t+2}\epsilon^t(1+\frac{icy^2}{n})^n(\frac{(2|c|)^{t+2}(t+1)(2|c|)^{\frac{t+2}{2}+t_0}(\frac{t+2}{2}+t_0)\ldots(2t_0+1)}{(\frac{t+2}{2}!)^2})$

$\leq|y|^{t+2}\epsilon^t(1+\frac{icy^2}{n})^n(\frac{(t+1)(2|c|)^{\frac{3t+\sqrt{t+3}+7}{2}}(\frac{t+2}{2}+t_0)!}{(\frac{t+2}{2}!)^2})$

$\leq|y|^{t+2}\epsilon^t(1+\frac{icy^2}{n})^n(\frac{(t+1)(4|c|^2)^t(\frac{t+2}{2}+t_0)!}{(\frac{t+2}{2}!)^2})$, $(t\geq 15)$

By Stirling's approximation, for all positive integers m;

$\sqrt{2\pi}m^{m+\frac{1}{2}}e^{-m}\leq m!\leq em^{m+\frac{1}{2}}e^{-m}$

Applying the approximation in this case, we obtain that;

$|[(1+\frac{icy^2}{n})^n]^{(t+2)}|\frac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2}$

$\leq|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(t+1)(4|c|^2)^te(\frac{t+2}{2}+t_0)^{\frac{t+3}{2}+t_0}e^{-(\frac{t+2}{2}+t_0)}}{2\pi(\frac{t+2}{2})^{t+3}e^{-(t+2)}})$

$\leq\frac{e^2}{2\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(t+1)(4|c|^2e)^t(\frac{t+2}{2}+t_0)^{\frac{t+3}{2}+t_0}}{(\frac{t+2}{2})^{t+3}})$

$\leq \frac{e^2}{2\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(t+1)(4|c|^2e)^t(2t)^{\frac{t+3}{2}+t_0}}{(\frac{t}{2})^t})$, $(t \geq 15)$

$\leq \frac{e^2}{2\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(t+1)(16|c|^2e)^t(t)^{\frac{t+3}{2}+t_0}}{t^t})$, $(t \geq 19)$

$\leq \frac{e^2}{\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(16|c|^2e)^t(t)^{\frac{t+5}{2}+t_0}}{t^t})$

$\leq \frac{e^2}{\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{(16|c|^2e)^t(t)^{\frac{2t}{3}}}{t^t})$, $(t \geq 45)$

$= \frac{e^2}{\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|(\frac{((16|c|^2e)^3)^{\frac{t}{3}}}{t^{\frac{t}{3}}})$

$\leq \frac{e^2}{\pi}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|$, $(t \geq (16|c|^2e)^3)$

Now, noting that, when $t = 0$;

$|[(1+\frac{icy^2}{n})^n]^{(t+2)}|\frac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2}$

$= |[(1+\frac{icy^2}{n})^n]^{(2)}|$

$\leq 6|c|^2|[(1+\frac{icy^2}{n})^n]||y|^2$

If we let;

$C_1 = max_{1\leq t<(16|c|^2e)^3}(\frac{e^2}{\pi}, 6|c|^2, (\frac{(2|c|)^{t+2}(t+1)(2|c|)^{\frac{t+2}{2}+t_0}(\frac{t+2}{2}+t_0)...(2t_0+1)}{(\frac{t+2}{2}!)^2}))$

Then, it follows, for $n \geq \frac{(16|c|^2e)^3+1}{2}$, and $0 \leq t \leq 2n-1$, that;

$|[(1+\frac{icy^2}{n})^n]^{(t+2)}|\frac{(t+1)\epsilon^t}{(\frac{t+2}{2}!)^2} \leq C_1|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|$

Similarly, we can obtain bounds $\{C_2, C_3, C_4\} \subset \mathcal{R}$ for the remaining terms in (††††††), the details are left to the reader. If $|c| < 1$, we can again drop c from the above calculation, obtaining a corresponding bound for $n \geq \frac{(16e)^3+1}{2}$. It follows that we can find $C \in \mathcal{R}$, with;

$|\theta_{n,c}^{D_\epsilon^2}| \leq C\sum_{0\leq t\leq 4n-2}|y|^{t+2}\epsilon^t|(1+\frac{icy^2}{n})^n|$

for sufficiently large $n \in \mathcal{N}$. We then repeat the argument of the previous case, to obtain, if $n > 4|c||y|^2$, $D = 4C$, $|y|\epsilon < \frac{1}{2}$, that;

$|\theta_{n,c}^{D_\epsilon^2}(y)| \leq D|y|^2$ $|y| > 1$, $(*****{*})$

Again, if $|c|<1$, it is sufficient to take $n>4|y|^2$. As before, we can transfer this result, to obtain, for $y\in(\overline{\mathcal{R}_\eta}\setminus(\frac{\eta-3}{2\sqrt{\eta}},\frac{\eta+1}{2\sqrt{\eta}}))$, with η infinite, and $\nu\geq 2\eta$ so $\nu\geq max(4|y|^2,4|c||y|^2)$;

$$|\theta^{D^2}_{\eta,\nu,c}(y)|\leq D|y|^2\ (|y|>1)\ (*******)$$

By quantifying over the constant $c\in\mathcal{R}$, and noting that the bounds C and D of $(******)$ and (!) can be written as explicit functions of c, we can assume that, for $\nu\geq 2\eta$, $(*******)$ and $(*****)$ hold for any $c\in{}^*\mathcal{R}$ with c finite, where the bounds $C(c)$ and $D(c)$ in ${}^*\mathcal{R}$ are also finite. Now, for $t\in{}^*\mathcal{R}$, with t finite, ${}^\circ t>0$, and $\{c,\eta,\nu\}$ as above, let;

$$\theta_{\eta,\nu,c,t}(y)=(1+\frac{ic[\sqrt{\eta}y]^2}{\eta\nu})^{[\nu t]}$$

Setting $d=\frac{c[\nu t]}{\nu}$, we have $ct\leq d\leq ct+\frac{1}{\nu}$, so d is finite. Moreover;

$$\theta_{\eta,\nu,c,t}(y)=\theta_{\eta,[\nu t],d}(y),\text{ for }y\in\overline{\mathcal{R}_\eta}$$

As t and d are finite and ${}^\circ t>0$, we have that $\nu\geq 2\eta$ implies $[\nu t]>|d|\eta\geq 4|d||y|^2$, for $y\in\overline{\mathcal{R}_\eta}$, if $|d|\geq 1$. Similarly, we have that $\nu\geq 2\eta$ implies $[\nu t]>\eta\geq 4|y|^2$, for $y\in\overline{\mathcal{R}_\eta}$, if $|d|<1$. Hence, the results $(*****)$ and $(*******)$ holds for $\theta_{\eta,\nu,c,t}$ and, without loss of generality, the bound $F\in\mathcal{R}$, depending on c and t. The interval $(\frac{\eta-3}{2\sqrt{\eta}},\frac{\eta+1}{2\sqrt{\eta}})$ has to be treated separately due to the definition of the operator D. Observe, from the above proof, that, with $\nu\geq 2\eta$, we have that there exists a constant $F_c\in\mathcal{R}$, with;

$$|\theta_{\eta,\nu,c}(y)|\leq F_c,\ y\in\overline{\mathcal{R}_\eta}$$

Using the above proof, with the same assumptions on t, we can find a constant $F_{t,c}\in\mathcal{R}$, with;

$$|\theta_{\eta,\nu,c,t}(y)|=|\theta_{\eta,[\nu t],d}(y)|\leq F_{t,c},\ y\in\overline{\mathcal{R}_\eta}$$

where $d=\frac{c[\nu t]}{\nu}$. By definition of D, we have that;

$$|\theta^D_{\eta,\nu,c,t}(y)|\leq 2\sqrt{\eta}F_{t,c}$$

$$|\theta^{D^2}_{\eta,\nu,c,t}(y)|\leq 4\eta F_{t,c}$$

Now, observing that;

$2\sqrt{\eta}F_{t,c} < 6F_{t,c}|\frac{\eta-3}{2\sqrt{\eta}}|$

$4\eta F_{t,c} < 20F_{t,c}|\frac{\eta-3}{2\sqrt{\eta}}|^2$

we can obtain the results $(*****)$ and $(*******)$ for $\theta_{\eta,\nu,c,t}$, without the caveat on the endpoints. Taking $c = -4\pi^2$ gives the result (†) above, and the first claim of the theorem. The second claim follows from the first claim and the last claim of Lemma 0.33, which shows that $G_{\eta,t}$ is bounded for $|x| < 1$. The third claim is obvious from the first claim. The final claim follows easily by combining the first claim with the final result of Lemma 0.33.

□

As in [5], we switch to a statistical analysis of Schrodinger's equation. We require the following lemma.

Lemma 0.36. *There exists a unique* $K \in C(\mathcal{R}_{>0} \times \mathcal{R}^2)$*, such that for any* $t > 0$ K_t *is bounded, and if* $g \in S(\mathcal{R})$*, and* G *is given by Lemma 0.2, then;*

$G_t(x) = \int_{\mathcal{R}} K(t,x,y)g(y)dy$

Moreover, K *is given explicitly by;*

$K(t,x,y) = R(t,x-y)$

where;

$R(t,x) = \frac{1}{\sqrt{4\pi i t}}e^{\frac{ix^2}{4t}}$

and $\sqrt{}$ *denotes the principal branch of the square root.*

Proof. By the proof of Lemma 0.2, we have that;

$G_t(x) = \int_{\mathcal{R}} e^{2\pi i y x}(e^{-4i\pi^2 y^2 t}\mathcal{F}(g)(y))dy$

$= \int_{\mathcal{R}} e^{2\pi i y x} lim_{\epsilon \rightarrow 0}(e^{(-it-\epsilon)4\pi^2 y^2}\mathcal{F}(g)(y))dy$

$= lim_{\epsilon \rightarrow 0} \int_{\mathcal{R}} e^{2\pi i y x}(e^{(-it-\epsilon)4\pi^2 y^2 t}\mathcal{F}(g)(y))dy$, (DCT)

$= lim_{\epsilon\rightarrow 0}(\mathcal{F}^{-1}(e^{(-it-\epsilon)4\pi^2y^2t}\mathcal{F}(g)))(x)$

$= lim_{\epsilon\rightarrow 0}(\mathcal{F}^{-1}(e^{(-it-\epsilon)4\pi^2y^2t})*g)(x)$, (Convolution/Inversion Theorem)

$= lim_{\epsilon\rightarrow 0}(\frac{1}{\sqrt{4\pi(it+\epsilon)}}e^{\frac{-x^2}{4(it+\epsilon)}})*g(x)$

$= lim_{\epsilon\rightarrow 0}\int_{\mathcal{R}}\frac{1}{\sqrt{4\pi(it+\epsilon)}}e^{\frac{-y^2}{4(it+\epsilon)}}g(x-y)dy$

$= \int_{\mathcal{R}}\frac{1}{\sqrt{4\pi it}}e^{\frac{-y^2}{4it}}g(x-y)dy$, (DCT)

$= \int_{\mathcal{R}}\frac{1}{\sqrt{4\pi it}}e^{\frac{iy^2}{4t}}g(x-y)dy$

$= \int_{\mathcal{R}}R(t,x-y)g(y)dy$ (commutativity of convolution)

$= \int_{\mathcal{R}}K(t,x,y)g(y)dy$

where $R(t,x)$ and $K(t,x,y)$ are defined as above. Note the use of the Dominated Convergence Theorem is justified in both cases, as g and $\mathcal{F}(g)$ belong to the Schwartz class. For $\epsilon>0$ and $t>0$, $e^{(-it-\epsilon)4\pi^2y^2t}$ also belongs to the Schwartz class, so we can apply the operator $\mathcal{F}^{-1}$. It is a standard result, using the identity theorem from complex analysis, that this produces the function $\frac{1}{\sqrt{4\pi(it+\epsilon)}}e^{\frac{-x^2}{4(it+\epsilon)}}$. More explicitly, consider the function;

$f(z,x)=\int_{\mathcal{R}}e^{-zy^2}e^{2\pi ixy}dy$

For fixed $x\in\mathcal{R}$ and $z\in\mathcal{C}$ with $Re(z)>0$, f defines an analytic function. For $z>0$, we have that;

$f(z,x)=\sqrt{\frac{\pi}{z}}e^{\frac{-\pi^2x^2}{z}}$ $(*)$

Taking the principal branch of the square root, the right hand side of $(*)$ also produces an analyic function for $Re(z)>0$. Hence, using the identity theorem, we obtain the equality $(*)$ for all $z\in\mathcal{C}$ with $Re(z)>0$. Substituting $4\pi^2(it+\epsilon)$ for z produces the result. Noting that $K\in C(\mathcal{R}_{>0}\times\mathcal{R}^2)$, and, for fixed $t>0$, K_t is bounded, we have shown existence. To prove uniqueness, let $L(t,x,y)$ also satisfy the conditions of the lemma. Then for fixed $t>0$, for every $g\in S(\mathcal{R})$, and

$x \in \mathcal{R}$, we have that;

$\int_{\mathcal{R}}(K-L)(t,x,y)g(y)dy = 0$

Fixing x, as $(K-L)_{t,x}$ is continuous, it follows easily from the properties of $S(\mathcal{R})$ that $(K-L)_{t,x} = 0$ as required.

□

We now show an analogous nonstandard result.

Lemma 0.37. *There exists a unique* $W \in V(\overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta}^2)$, *such that for any initial condition* $f \in V(\overline{\mathcal{R}_\eta})$, *if* F *is given by Lemma 0.32, then;*

$F(t,x) = \int_{\overline{\mathcal{R}_\eta}} W(t,x,y)f(y)d\mu_\eta(y)$ $(*)$

Moreover, $W(t,x,y) = < F_{\delta_y,t}, \delta_x > = F_{\delta_y}(t,x) = (L^{([t\nu])} * \delta_y)(x)$

where $L^{(i)}$, *for* $i \in {}^*\mathcal{N}$, *is given by Lemma 0.32, and, for fixed* $y \in \overline{\mathcal{R}_\eta}$, $\delta_y(x) = \sqrt{\eta}$ *for* $x \in \overline{\mathcal{R}_\eta}$, *with* $[x\sqrt{\eta}] = [y\sqrt{\eta}]$, *and* $\delta_y(x) = 0$ *otherwise.*

Proof. By Lemma 0.32, we have that;

$F(t,x) = (L^{([t\nu])} * f)(x)$

Therefore, setting $W(t,x,y) = L^{([t\nu])}(x-y)$, with the usual wrap around convention, proves existence. To prove uniqueness, suppose that $V(t,x,y)$ also satisfies the property $(*)$. Then, for any initial condition, $f \in V(\overline{\mathcal{R}_\eta})$, and fixed $t \in \overline{\mathcal{T}_\nu}$, $x \in \overline{\mathcal{R}_\eta}$ we have, by the uniqueness of F in Lemma 0.32, that;

$\int_{\overline{\mathcal{R}_\eta}}(W-V)(t,x,y)f(y)d\mu_\eta(y) = 0$

Now letting $f = \delta_{y_0}$, for fixed $y_0 \in \overline{\mathcal{R}_\eta}$, we have, by the definition of the internal integral, that $(W-V)_{t,x}(y_0) = 0$. As y_0 was arbitrary, we obtain the first result. For the second part, we compute;

$< F_{\delta_y,t}, \delta_x >$

$= \int_{\overline{\mathcal{R}_\eta}} F_{\delta_y,t}(z)\delta_x(z)d\mu_\eta(z)$

$= F_{\delta_y,t}(x)$

$= \int_{\overline{\mathcal{R}_\eta}} W(t,x,z)\delta_y(z)d\mu_\eta(z)$

$= W(t,x,y)$

The equality $F_{\delta_y,t}(x) = (L^{([t\nu])} * \delta_y)(x)$ follows from Lemma 0.32. □

Lemma 0.38. *Suppose that W is S-continuous, and bounded, then, for finite $(t,x,y) \in \overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta}^2$, with ${}^\circ t > 0$, we have that;*

${}^\circ W(t,x,y) = K({}^\circ t, {}^\circ x, {}^\circ y)$

and for finite $(t,y) \in \overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta}$, and infinite $x \in \overline{\mathcal{R}_\eta}$, with ${}^\circ t > 0$, we have that;

${}^\circ W(t,x,y) = 0$

In particular, it follows that $L^{[t\nu]}(z) \simeq 0$, for infinite $z \in \overline{\mathcal{R}_\eta}$, and finite t with ${}^\circ t > 0$.

Proof. Fix $g \in S(\mathcal{R})$ with corresponding $g_\eta \in V(\overline{\mathcal{R}_\eta})$. By the last claim in Theorem 0.35, we have that ${}^\circ G_{\eta,t} = st^*(G_{{}^\circ t,\infty})$, if t is finite and ${}^\circ t > 0$, $(*)$. For such t, and finite $x \in \overline{\mathcal{R}_\eta}$, we have that $W_{t,x}$ is S-continuous and bounded. As g_η is S-continuous, bounded and S-integrable, it follows that $W_{t,x}g_\eta$ is also S-continuous, bounded and S-integrable. Therefore, by $(*)$ and $(*)$ of Lemma 0.37, we have that;

$G_{{}^\circ t}({}^\circ x) = \int_{\overline{\mathcal{R}_\eta}} {}^\circ W(t,x,y){}^\circ g_\eta(y)dL(\mu_\eta)(y)$

$= \int_{y \in \overline{\mathcal{R}_\eta}, y\ finite} {}^\circ W(t,x,y){}^\circ g_\eta(y)dL(\mu_\eta)(y)$

$= \int_{y \in \overline{\mathcal{R}_\eta}, y\ finite} {}^\circ W({}^\circ t, {}^\circ x, y)st^*(g_\infty)(y)dL(\mu_\eta)(y)$

$= \int_{\mathcal{R}} {}^\circ W({}^\circ t, {}^\circ x, {}^\circ y)g({}^\circ y)d({}^\circ y)$

using Lemma 0.4, the fact that W is S-continuous and ${}^\circ g_\eta = st^*(g_\infty)$. As $g \in S(\mathcal{R})$ was arbitrary and ${}^\circ W({}^\circ t, {}^\circ x, {}^\circ y)$ is continuous and bounded, for finite (t,x,y), it follows, by the uniqueness claim in Lemma 0.36, that ${}^\circ W({}^\circ t, {}^\circ x, {}^\circ y) = K({}^\circ t, {}^\circ x, {}^\circ y)$ as required. For the second claim, when x is infinite, applying $(*)$ again and similar reasoning to the above, we obtain;

$\int_{\mathcal{R}} {}^{\circ}W({}^{\circ}t, {}^{\circ}x, {}^{\circ}y)g({}^{\circ}y)d({}^{\circ}y) = 0$

Again, as $g \in S(\mathcal{R})$ was arbitrary, ${}^{\circ}W({}^{\circ}t, {}^{\circ}x, {}^{\circ}y)$ is continuous and bounded, we conclude easily that ${}^{\circ}W({}^{\circ}t, {}^{\circ}x, {}^{\circ}y) = 0$, for (t, y) finite x infinite and ${}^{\circ}t > 0$. The last claim is an easy consequence of the fact that $W(t, x, y) = L^{([t\nu])}(x - y)$, noting that for x infinite and finite y $x - y$ is infinite even with the wrap around convention.

□

In general, W need not be bounded or S-continuous. However, we can now show the following;

Lemma 0.39. *When $t = \frac{m}{2\pi}$ and $m \in \mathcal{Z}_{>0}$, $\nu \geq \eta^5$, then $W(t, x, y)$ is bounded.*

Proof. By the proof of Lemma 0.37, we have that $W(t, x, y) = F_{\delta_y,t}(x)$. By Lemma 0.33, we have that;

$\hat{F_{\delta_y}}(z, t) = (1 - \frac{4\pi^2[\sqrt{\eta}z]^2 i}{\eta\nu})^{[\nu t]}\hat{\delta_y}(z)$

We have that;

$\hat{\delta_y}(z) = \int_{\overline{\mathcal{R}_\eta}} \delta_y(x) exp_\eta(-2\pi izx)d\mu_\eta(x)$

$= exp_\eta(-2\pi izy)$

Therefore, it follows;

$\hat{F_{\delta_y}}(z, t) = (1 - \frac{4\pi^2[\sqrt{\eta}z]^2 i}{\eta\nu})^{[\nu t]} exp_\eta(-2\pi izy)$

Using $(**)$ of Lemma 0.26, $(**)$ in Lemma 0.33, we have that, if $\nu \geq \eta^5$, then;

$|(1 - \frac{4\pi^2[\sqrt{\eta}z]^2 i}{\eta\nu})^\nu - {}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}i)| \leq \frac{1}{\nu^{\frac{1}{2}}}$, (†)

Transferring the result of footnote 6 in [7], we have that if $\{w', w\} \subset {}^*\mathcal{C}$, $t \in {}^*\mathcal{R}_{>0}$, with $|w' - w| < min\{\frac{|w|}{2}, \frac{|w|}{4t}\}$, $|w| \leq 1$, (this assumption isn't strictly necessary), $w \neq 0$, then $|w'^t - w^t| < \frac{4et|w' - w|}{|w|}$. Noting that $|{}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}i)| = 1$, and $\frac{1}{\nu^{\frac{1}{2}}} < min(\frac{1}{2}, \frac{1}{4t})$, for t finite, $t > 0$, we obtain, using (†), that;

$|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{\nu t}-{}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}it)|\leq\frac{4et}{\nu^{\frac{1}{2}}}\simeq 0$ (††)

We have that;

$|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{[\nu t]}-(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{\nu t}|$

$=|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{[\nu t]}||1-(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{s}|$

where $0\leq s<1$. By the proof of Lemma 0.35, with just the assumption that $\nu\geq 2\eta$, and ${}^\circ t>0$, t finite, there exists a constant $C_t\in\mathcal{R}$, with;

$|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{[\nu t]}|\leq C_t$

Using the result of footnote 6 in [7] again, and the fact that $|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})\simeq 1$, we have that;

$|1-(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{s}|$

$<4es|1-(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})|\leq\frac{16es\pi^2(\eta-1)^2}{\eta\nu}\leq\frac{\eta^2}{\nu}$

It follows that;

$|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{[\nu t]}-(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{\nu t}|\leq\frac{C_t\eta^2}{\nu}$

and, using the triangle inequality, and (††);

$|(1-\frac{4\pi^2[\sqrt{\eta}z]^2i}{\eta\nu})^{[\nu t]}-{}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}it)|\leq\frac{C_t\eta^2}{\nu}+\frac{4et}{\nu^{\frac{1}{2}}}<\frac{1}{\eta^{\frac{3}{2}}}$

Hence;

$|\hat{F_{\delta_y}}(z,t)-{}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}it)exp_\eta(-2\pi izy)|\leq\frac{1}{\eta^{\frac{3}{2}}}$

Applying the inversion theorem, we have that;

$|F_{\delta_y}(x,t)-\int_{\overline{\mathcal{R}_\eta}}{}^*exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}it)exp_\eta(-2\pi izy)exp_\eta(2\pi izx)d\mu_\eta(z)|$

$\leq\int_{\overline{\mathcal{R}_\eta}}\frac{d\mu_\eta(z)}{\eta^{\frac{3}{2}}}=\frac{\eta^{\frac{1}{2}}}{\eta^{\frac{3}{2}}}=\frac{1}{\eta}\simeq 0,\ (*)$

We have that;

$|\int_{\overline{\mathcal{R}_\eta}}{}^{*}exp(-4\pi^2\frac{[\sqrt{\eta}z]^2}{\eta}it)exp_\eta(-2\pi izy)exp_\eta(2\pi izx)d\mu_\eta(z)|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq k\leq\frac{(\eta-1)}{2}}{}^{*}exp(-4\pi^2\frac{k^2}{\eta}it)^{*}exp(\frac{2\pi ikl}{\eta})|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(-4\pi^2\frac{(w-\frac{(\eta-1)}{2})^2}{\eta}it)^{*}exp(\frac{2\pi i(w-\frac{(\eta-1)}{2})l}{\eta})|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(-4\pi^2\frac{w^2-w(\eta-1)}{\eta}it)^{*}exp(\frac{2\pi iwl}{\eta})|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}(wl-2\pi t(w^2-w(\eta-1))))|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}(wl-m(w^2-w(\eta-1))))|$

$=|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}(sw-mw^2))|$, $(**)$

where $s=l-m(\eta-1)\ (\mathrm{mod}\ \eta)$, $t=\frac{m}{2\pi}$, $m\in\mathcal{Z}_{>0}$, $[x\sqrt{\eta}]=a$, $[y\sqrt{\eta}]=b$, $a-b=l\ (\mathrm{mod}\ \eta)$, $w=k+\frac{(\eta-1)}{2}$.

Now considering the polynomial $p(w)\in F_\eta[w]$, $p(w)=sw-mw^2$, with $deg(p)=2$, we obtain, using the fact that η is prime, by transfer of the estimate for finite fields, see [9], that;

$|\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}(sw-mw^2))|\leq\frac{(\eta^{\frac{1}{2}}+1)}{\eta^{\frac{1}{2}}}\leq 2$

Combining this bound with $(*)$, we obtain that $W(t,x,y)$ is bounded, for the specific $t=\frac{m}{2\pi}$, $m\in\mathcal{Z}_{>0}$. □

Lemma 0.40. *Let* $t=\frac{m}{2\pi}$*, and let;*

$W_m(t,x,y)=\frac{1}{4m^2}{}^{*}\sum_{0\leq i,j\leq 2m-1}W(t,x+\frac{i}{\sqrt{\eta}},y+\frac{j}{\sqrt{\eta}})$

Then, for finite $\{x_1,x_2,y_1,y_2\}\subset\overline{\mathcal{R}_\eta}$*, with* $x_1\simeq x_2$ *and* $y_1\simeq y_2$*, we have that;*

$W_m(t,x_1,y_1)\simeq W_m(t,x_2,y_2)$

In particular, it follows that;

${}^{\circ}W_m(t,x,y)=K(t,{}^{\circ}x,{}^{\circ}y)$

for (x,y) finite.

Proof. Continuing the calculation in $(**)$ above, we have that;

$W(t,x,y)\simeq\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}(sw-mw^{2}))$

$=\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{2\pi i}{\eta}((l+m)w-mw^{2}))$

$=\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}(w^{2}-\frac{(l+m)}{m}w))$

$=\frac{1}{\sqrt{\eta}}{}^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m}{2m})^{2}-\frac{(l+m)^{2}}{4m^{2}}))$

$=\frac{1}{\sqrt{\eta}}{}^{*}exp(\frac{2\pi im(l+m)^{2}}{4m^{2}\eta})^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m}{2m})^{2})$

$=\frac{1}{\sqrt{\eta}}{}^{*}exp(\frac{\pi i(l+m)^{2}}{2m\eta})^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m}{2m})^{2})$

Replacing l by $l-2km$, where $\frac{k}{\sqrt{\eta}}\simeq 0$, and $k\in{}^{*}\mathcal{Z}$, we have that;

$W(t,x,y+\frac{2km}{\sqrt{\eta}})\simeq\frac{1}{\sqrt{\eta}}{}^{*}exp(\frac{\pi i(l+(1-2k)m)^{2}}{2m\eta})^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m-2km}{2m})^{2})$

$=\frac{1}{\sqrt{\eta}}{}^{*}exp(\frac{\pi i(l+(1-2k)m)^{2}}{2m\eta})^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m}{2m}+k)^{2})$

$=\frac{1}{\sqrt{\eta}}{}^{*}exp(\frac{\pi i(l+(1-2k)m)^{2}}{2m\eta})^{*}\sum_{0\leq w\leq\eta-1}{}^{*}exp(\frac{-2\pi im}{\eta}((w-\frac{l+m}{2m})^{2})$ $(re-indexing)$

The re-indexing step is justified, as $(w-q+\eta)^{2}=(w-q)^{2}+2\eta(w-q)+\eta^{2}$, and $n=2m(w-q)\in{}^{*}\mathcal{Z}$, when $2mq\in{}^{*}\mathcal{Z}$, so we obtain a phase shift of ${}^{*}exp(-2\pi in)=1$.

Now, we have that;

${}^{*}exp(\frac{\pi i(l+(1-2k)m)^{2}}{2m\eta})={}^{*}exp(\frac{\pi i(l+m-2km)^{2}}{2m\eta})$

$={}^{*}exp(\frac{\pi i(l+m)^{2}}{2m\eta})^{*}exp(\frac{-2\pi ik(l+m)}{\eta})^{*}exp(\frac{2\pi imk^{2}}{\eta})$

$\simeq{}^{*}exp(\frac{\pi i(l+m)^{2}}{2m\eta})$

as ${}^{*}exp(\frac{-2\pi ik(l+m)}{\eta})\simeq{}^{*}exp(\frac{2\pi imk^{2}}{\eta})\simeq 1$, because $m\in\mathcal{Z}_{>0}$, $\frac{k}{\sqrt{\eta}}\simeq 0$ and $l=[x\sqrt{\eta}]-[y\sqrt{\eta}]$ with (x,y) finite. It follows immediately that;

$W(t,x,y) \simeq W(t,x,y+\frac{2km}{\sqrt{\eta}})$

In the same way, we can show that, for (x,y) finite;

$W(t,x,y) \simeq W(t,x+\frac{2km}{\sqrt{\eta}},y)$

Now, if $y_1 \simeq y_2$, we can suppose that $y_2 = y_1 + \frac{s}{\sqrt{\eta}}$, with $\frac{s}{\sqrt{\eta}} \simeq 0$. We can find a permutation $b : \mathcal{Z}/2m\mathcal{Z} \rightarrow \mathcal{Z}/2m\mathcal{Z}$, such that $[y_1\sqrt{\eta}] + j \equiv [y_1\sqrt{\eta}] + s + b(j)$, (mod $2m$), for $0 \leq j \leq 2m-1$, in particular $s + b(j) = 2k(j)m$, with $\frac{k(j)}{\sqrt{\eta}} \simeq 0$. Applying the previous result, we obtain that;

$W(t,x,y_1+\frac{j}{\sqrt{\eta}}) \simeq W(t,x,y_1+\frac{s+b(j)}{\sqrt{\eta}})$

As $2m$ is finite, and b is a permutation, it follows that;

$W^1_m(t,x,y_1) = \frac{1}{2m}{}^* \sum_{0 \leq j \leq 2m-1} W(t,x,y_1+\frac{j}{\sqrt{\eta}})$

$\simeq \frac{1}{2m}{}^* \sum_{0 \leq j \leq 2m-1} W(t,x,y_1+\frac{s+j}{\sqrt{\eta}})$

$= W^1_m(t,x,y_2)$

and, similarly, if $x_1 \simeq x_2$;

$W^2_m(t,x_1,y) = \frac{1}{2m}{}^* \sum_{0 \leq i \leq 2m-1} W(t,x_1+\frac{i}{\sqrt{\eta}},y)$

$\simeq W^2_m(t,x_2,y)$

In the same way, we can show that, for $0 \leq i,j \leq 2m-1$ and (x_1,x_2,y_1,y_2) finite, with $x_1 \simeq x_2$ and $y_1 \simeq y_2$;

$W^1_m(t,x_1+\frac{i}{\sqrt{\eta}},y_1) \simeq W^1_m(t,x_1+\frac{i}{\sqrt{\eta}},y_2)$

$W^2_m(t,x_1,y_2+\frac{j}{\sqrt{\eta}}) \simeq W^2_m(t,x_2,y_2+\frac{j}{\sqrt{\eta}})$

It follows, again as $2m$ is finite, that;

$W_m(t,x_1,y_1) = \frac{1}{2m}{}^* \sum_{0 \leq i \leq 2m-1} W^1_m(t,x_1+\frac{i}{\sqrt{\eta}},y_1)$

$\simeq \frac{1}{2m}{}^* \sum_{0 \leq i \leq 2m-1} W^1_m(t,x_1+\frac{i}{\sqrt{\eta}},y_2)$

$= \frac{1}{2m}{}^{*}\sum_{0\leq j\leq 2m-1}W_{m}^{2}(t,x_{1},y_{2}+\frac{j}{\sqrt{\eta}})$

$\simeq \frac{1}{2m}{}^{*}\sum_{0\leq j\leq 2m-1}W_{m}^{2}(t,x_{2},y_{2}+\frac{j}{\sqrt{\eta}})$

$= W_{m}(t,x_{2},y_{2})$

This shows the first claim of the Lemma. If $n\in\mathcal{N}$, we define the restriction of the internal integral on $\overline{\mathcal{R}_{\eta}}$ to ${}^{*}[-n,n]$ in the obvious way. We then claim, for $f\in S(\mathcal{R})$, with measurable counterpart $f_{\eta}\in V(\overline{\mathcal{R}_{\eta}})$, that;

$\int_{{}^{*}[-n,n]}W_{m}^{1}(t,x,y)f_{\eta}(y)d\mu_{\eta}(y)\simeq\int_{{}^{*}[-n,n]}W(t,x,y)f_{\eta}(y)d\mu_{\eta}(y)$ $(*)$

To see $(*)$, we compute;

$\int_{{}^{*}[-n,n]}W_{m}^{1}(t,x,y)f_{\eta}(y)d\mu_{\eta}(y)$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-[n\sqrt{\eta}]\leq j\leq[n\sqrt{\eta}]}W_{m}^{1}(t,x,\frac{j}{\sqrt{\eta}})f_{\eta}(\frac{j}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-[n\sqrt{\eta}]\leq j\leq[n\sqrt{\eta}]}\frac{1}{2m}{}^{*}\sum_{0\leq k\leq 2m-1}W(t,x,\frac{j+k}{\sqrt{\eta}})f_{\eta}(\frac{j}{\sqrt{\eta}})$

$= \frac{1}{2m\sqrt{\eta}}{}^{*}\sum_{0\leq k\leq 2m-1}{}^{*}\sum_{-[n\sqrt{\eta}]\leq j\leq[n\sqrt{\eta}]}W(t,x,\frac{j+k}{\sqrt{\eta}})f_{\eta}(\frac{j}{\sqrt{\eta}})$

$= \frac{1}{2m\sqrt{\eta}}{}^{*}\sum_{0\leq k\leq 2m-1}{}^{*}\sum_{-[n\sqrt{\eta}]+k\leq j\leq[n\sqrt{\eta}]+k}W(t,x,\frac{j}{\sqrt{\eta}})f_{\eta}(\frac{j-k}{\sqrt{\eta}})$

Using the argument of Lemma 0.17, and the fact that $|f'|\leq C$, for some finite $C\in\mathcal{R}$, we have that;

$|f_{\eta}(\frac{j-k}{\sqrt{\eta}})-f_{\eta}(\frac{j}{\sqrt{\eta}})|\leq\frac{Ck}{\sqrt{\eta}}\leq\frac{C(2m-1)}{\sqrt{\eta}}$

By Lemma 0.39, we can find a finite $D\in\mathcal{R}$, with $|W(t,x,y)|\leq D$. It follows that;

$\int_{{}^{*}[-n,n]}W_{m}^{1}(t,x,y)f_{\eta}(y)d\mu_{\eta}(y)$

$= \frac{1}{2m\sqrt{\eta}}{}^{*}\sum_{0\leq k\leq 2m-1}{}^{*}\sum_{-[n\sqrt{\eta}]+k\leq j\leq[n\sqrt{\eta}]+k}W(t,x,\frac{j}{\sqrt{\eta}})f_{\eta}(\frac{j}{\sqrt{\eta}})+\epsilon$

where;

$|\epsilon|\leq\frac{CD}{2m\sqrt{\eta}}\frac{(2m-1)}{\sqrt{\eta}}2m(2[n\sqrt{\eta}]+1)=\frac{CD(2m-1)(2[n\sqrt{\eta}]+1)}{\eta}\simeq 0$

as $\{C, D, m, n\}$ are finite. As $W(t, x, y)$ and f_η are bounded, and m is finite, we have that;

$$\frac{1}{\sqrt{\eta}} {}^*\sum_{-[n\sqrt{\eta}]+k\leq j\leq [n\sqrt{\eta}]+k} W(t, x, \tfrac{j}{\sqrt{\eta}}) f_\eta(\tfrac{j}{\sqrt{\eta}})$$

$$\simeq \frac{1}{\sqrt{\eta}} {}^*\sum_{-[n\sqrt{\eta}]\leq j\leq [n\sqrt{\eta}]} W(t, x, \tfrac{j}{\sqrt{\eta}}) f_\eta(\tfrac{j}{\sqrt{\eta}})$$

$$= \int_{^*[-n,n]} W(t, x, y) f_\eta(y) d\mu_\eta(y)$$

It follows that;

$$\int_{^*[-n,n]} W^1_m(t, x, y) f_\eta(y) d\mu_\eta(y)$$

$$\simeq \frac{1}{2m} {}^*\sum_{0\leq k\leq 2m-1} \int_{^*[-n,n]} W(t, x, y) f_\eta(y) d\mu_\eta(y)$$

$$= \int_{^*[-n,n]} W(t, x, y) f_\eta(y) d\mu_\eta(y)$$

so $(*)$ is shown. Now, by overflow, we can find an infinite $\kappa \in {}^*\mathcal{N}$, with $|\kappa| \leq \frac{\eta-1}{2\sqrt{\eta}}$, such that;

$$\int_{^*[-\kappa,\kappa]} W^1_m(t, x, y) f_\eta(y) d\mu_\eta(y) \simeq \int_{^*[-\kappa,\kappa]} W(t, x, y) f_\eta(y) d\mu_\eta(y), \ (**)$$

Using the fact that $f \in S(\mathcal{R})$, we can find a finite constant $E \in \mathcal{R}$, with $|f_\eta| \leq \frac{E}{y_\eta^2}$. As $max(|W^1_m|, |W|) \leq D$, it follows that;

$$|\int_{\overline{^*\mathcal{R}_\eta}\setminus {}^*[-\kappa,\kappa]} W^1_m(t, x, y) f_\eta(y) d\mu_\eta(y)|$$

$$\leq \int_{\overline{^*\mathcal{R}_\eta}\setminus {}^*[-\kappa,\kappa]} \frac{DE}{y_\eta^2} d\mu_\eta(y) \simeq 0 \ (***)$$

and, similarly, for W. It follows, using $(**)$,$(***)$, that;

$$\int_{\overline{\mathcal{R}_\eta}} W^1_m(t, x, y) f_\eta(y) d\mu_\eta(y) \simeq \int_{\overline{\mathcal{R}_\eta}} W(t, x, y) f_\eta(y) d\mu_\eta(y) \ (****)$$

By the second part of Lemma 0.33 and the defining property of W, given in $(*)$ of Lemma 0.37, using the fact that $F \in S(\mathcal{R})$, we have that, for $x_1 \simeq x_2$, with x_1 finite, that;

$$\int_{\overline{\mathcal{R}_\eta}} W(t, x_1, y) f_\eta(y) d\mu_\eta(y) \simeq \int_{\overline{\mathcal{R}_\eta}} W(t, x_2, y) f_\eta(y) d\mu_\eta(y)$$

It follows, using this and $(****)$, that;

$\int_{\overline{\mathcal{R}_\eta}} W_m(t,x,y)f_\eta(y)d\mu_\eta(y)$

$= \frac{1}{2m} {}^* \sum_{0\leq i\leq 2m-1} \int_{\overline{\mathcal{R}_\eta}} W_m^1(t, x+\frac{i}{\sqrt{\eta}}, y)f_\eta(y)d\mu_\eta(y)$

$\simeq \frac{1}{2m} {}^* \sum_{0\leq i\leq 2m-1} \int_{\overline{\mathcal{R}_\eta}} W(t, x+\frac{i}{\sqrt{\eta}}, y)f_\eta(y)d\mu_\eta(y)$

$\simeq \frac{1}{2m} {}^* \sum_{0\leq i\leq 2m-1} \int_{\overline{\mathcal{R}_\eta}} W(t,x,y)f_\eta(y)d\mu_\eta(y)$

$= \int_{\overline{\mathcal{R}_\eta}} W(t,x,y)f_\eta(y)d\mu_\eta(y)$, $(*****)$

Now, using the first part, that W_t is S-continuous, the fact that W_t is bounded, and $(*****)$ we can repeat the argument of the first part of Lemma 0.38, to obtain the final claim of the lemma, that ${}^\circ W_m(t,x,y) = K(t, {}^\circ x, {}^\circ y)$, as required.

□

We now want to generalise this result, to include times of the form $t = \frac{m}{2\pi n}$, where $\{m,n\} \subset \mathcal{Z}_{>0}$. For this, we require the following definition;

Definition 0.41. *Let $n \in \mathcal{Z}_{>0}$ and let $f \in V(\overline{\mathcal{R}_\eta})$, we define the generalised n nonstandard Fourier transform by;*

$\hat{}^n f(y) = \int_{\overline{\mathcal{R}_\eta}} f(x)exp_\eta(-2\pi inyx)d\mu_\eta(x)$

Lemma 0.42. *The Inversion Theorem and the Convolution Theorem still hold for Definition 0.41, that is, if $\{f,g\} \subset V(\overline{\mathcal{R}_\eta})$, then;*

$f(x) = \int_{\overline{\mathcal{R}_\eta}} \hat{}^n f(y)exp_\eta(2\pi inxy)d\mu_\eta(y)$

$\hat{}^n(f*g) = \hat{}^n f \hat{}^n g$

Proof. The proof is almost identical to that of Lemma 0.12 and Lemma 0.30. The details are left to the reader. □

We relate the nonstandard Fourier transform to the generalised n-transform.

Lemma 0.43. *Let $f \in V(\overline{\mathcal{R}_\eta})$, then, for $y \in \overline{\mathcal{R}_\eta}$, with $[y\sqrt{\eta}] = y\sqrt{\eta}$;*

$\hat{}^{n} f(y) = \hat{} f([ny]_{\overline{\mathcal{R}_\eta}})$

where $[ny]_{\overline{\mathcal{R}_\eta}}$ *denotes the unique* $w \in \overline{\mathcal{R}_\eta}$ *for which there exists* $b \in \mathcal{Z}$ *with* $ny - w = b\sqrt{\eta}$.

Proof. This is a simple computation. We have that;

$\hat{} f([ny]_{\overline{\mathcal{R}_\eta}})$

$= \hat{} f(ny - b\sqrt{\eta})$, some $b \in \mathcal{Z}$

$= \int_{\overline{\mathcal{R}_\eta}} f(x) exp_\eta(-2\pi i x(ny - b\sqrt{\eta})) d\mu_\eta(x)$

$= \int_{\overline{\mathcal{R}_\eta}} f(x) exp_\eta(-2\pi i nxy) d\mu_\eta(x)$

$= \hat{}^{n} f(y)$

□

We have the following modification of Lemma 0.33.

Lemma 0.44. *Let* $f \in V(\overline{\mathcal{R}_\eta})$, *let* F *solve the nonstandard convolution equation, as in Lemma 0.32, with initial condition* f, *and let* $n \in \mathcal{Z}_{>0}$. *Then, there exists* $b_y \in \mathcal{Z}$ *with;*

$\hat{}^{n} F(y,t) = (1 - \frac{4\pi^2((ny)_\eta - b_y\sqrt{\eta})^2 i}{\nu})^{[\nu t]} \hat{}^{n} f(y)$

for $y \in \mathcal{R}_\eta$, *with* $[y\sqrt{\eta}] = y\sqrt{\eta}$, $t \in \overline{\mathcal{T}_\nu}$.

Proof. Taking the generalised n-transform of the equation, and using the convolution theorem in Lemma 0.42, we have;

$\frac{d}{dt}(\hat{}^{n} F(y,t)) = \hat{}^{n} K(y) \hat{}^{n} F(y,t)$

By Lemma 0.31 and Lemma 0.43, we have that;

$\hat{}^{n} K(y) = \hat{} K(ny - b_y\sqrt{\eta})$, for some uniquely determined $b_y \in \mathcal{Z}$

$= -\frac{4\pi^2 i}{\eta}([\sqrt{\eta}(ny - b_y\sqrt{\eta})]^2)$

$= -\frac{4\pi^2 i}{\eta}(([\sqrt{\eta} ny] - b_y \eta)^2)$

$= -4\pi^2 i(((ny)_\eta - b_y\sqrt{\eta})^2)$

Now following the remaining iteration steps in Lemma 0.33, we obtain the result.

□

We have the following generalisation of Lemma 0.39.

Lemma 0.45. *When* $t = \frac{m}{2n\pi}$ *and* $\{m,n\} \subset \mathcal{Z}_{>0}$, $\nu \geq \eta^5$, *then* $W(t,x,y)$ *is bounded.*

Proof. We follow the steps in Lemma 0.39. Using the fact that;

$\hat{}^{n}\delta_y(z) = \int_{\overline{\mathcal{R}_\eta}} \delta_y(x) exp_\eta(-2\pi inzx) d\mu_\eta(x)$

$= exp_\eta(-2\pi inzy)$

It follows, by Lemma 0.44, that;

$\hat{}^{n}F_{\delta_y}(z,t) = (1 - \frac{4\pi^2((nz)_\eta - b_z\sqrt{\eta})^2 i}{\nu})^{[\nu t]} exp_\eta(-2\pi inzy)$

The estimates up to $(*)$ of Lemma 0.39 are the same. Applying the generalised inversion theorem in Lemma 0.42, it follows that;

$|W(t,x,y)| \simeq |\int_{\overline{\mathcal{R}_\eta}} {}^{*}exp(-4\pi^2((nz)_\eta - b_z\sqrt{\eta})^2 it) exp_\eta(-2\pi inzy) exp_\eta(2\pi inzx) d\mu_\eta(z)|$

$= |\frac{1}{\sqrt{\eta}} {}^{*}\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} {}^{*}exp(-4\pi^2(\frac{nk}{\sqrt{\eta}} - b_k\sqrt{\eta})^2 it) {}^{*}exp(\frac{2\pi inkl}{\eta})|$

$= |\frac{1}{\sqrt{\eta}} {}^{*}\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} {}^{*}exp(-4\pi^2(\frac{n^2k^2 it}{\eta})) {}^{*}exp(8\pi^2(nkb_k it)) {}^{*}exp(-4\pi^2(b_k^2 \eta it)) {}^{*}exp(\frac{2\pi inkl}{\eta})|$

$= |\frac{1}{\sqrt{\eta}} {}^{*}\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} {}^{*}exp(-2\pi(\frac{nmk^2 i}{\eta})) {}^{*}exp(\frac{2\pi inkl}{\eta})|$ ($t = \frac{m}{2\pi n}$, η divisible by n)

$= |\frac{1}{\sqrt{\eta}} {}^{*}\sum_{0 \leq w \leq \eta-1} {}^{*}exp(-2\pi inm\frac{(w - \frac{(\eta-1)}{2})^2}{\eta}) {}^{*}exp(\frac{2\pi inl(w - \frac{(\eta-1)}{2})}{\eta})|$

$= |\frac{1}{\sqrt{\eta}} {}^{*}\sum_{0 \leq w \leq \eta-1} {}^{*}exp(\frac{2\pi i}{\eta}(nsw - nmw^2))|$

where $s = l + m(\eta - 1)$ (mod η), $t = \frac{m}{2\pi n}$, $\{m,n\} \subset \mathcal{Z}_{>0}$, $[x\sqrt{\eta}] = a$, $[y\sqrt{\eta}] = b$, $a - b = l$ (mod η), $w = k + \frac{(\eta-1)}{2}$.

We transfer the reciprocity law for quadratic Gauss sums, which says that if $\{a,b,c\}\subset\mathcal{Z}$, with $ac\neq 0$ and $ac+b$ even, then;

$$\sum_{n=0}^{|c|-1} e^{\frac{i\pi}{c}(an^2+bn)} = |\frac{c}{a}|^{\frac{1}{2}} e^{\pi i\frac{|ac|-b^2}{4ac}} \sum_{n=0}^{|a|-1} e^{\frac{-\pi i}{a}(cn^2+bn)}$$

This, of course, implies that;

$$|\sum_{n=0}^{|c|-1} e^{\frac{i\pi}{c}(an^2+bn)}| \leq |\frac{c}{a}|^{\frac{1}{2}}|a| = |c|^{\frac{1}{2}}|a|^{\frac{1}{2}}$$

Taking $a=-2mn$, $b=2ns$ and $c=\eta$, the conditions for reciprocity are satisfied, and we obtain that;

$$|W(t,x,y)| \simeq |\frac{1}{\sqrt{\eta}}^* \sum_{0\leq w\leq \eta-1}{}^* exp(\frac{2\pi i}{\eta}(nsw-nmw^2))|$$

$$\leq \frac{1}{\sqrt{\eta}}\sqrt{\eta}\sqrt{2mn} = \sqrt{2mn}$$

As $\{m,n\}\subset\mathcal{Z}_{>0}$, we obtain the result.

□

We have the following generalisation of Lemma 0.40;

Lemma 0.46. *Let $t=\frac{m}{2n\pi}$, and let;*

$$W_m(t,x,y) = \frac{1}{4m^2}^* \sum_{0\leq i,j\leq 2m-1} W(t, x+\frac{i}{\sqrt{\eta}}, y+\frac{j}{\sqrt{\eta}})$$

Then, for finite $\{x_1,x_2,y_1,y_2\}\subset\overline{\mathcal{R}_\eta}$, with $x_1\simeq x_2$ and $y_1\simeq y_2$, we have that;

$$W_m(t,x_1,y_1)\simeq W_m(t,x_2,y_2)$$

In particular, it follows that;

$$^\circ W_m(t,x,y) = K(t, {}^\circ x, {}^\circ y)$$

for (x,y) finite.

Proof. We follow the steps of Lemma 0.40. We have that;

$$W(t,x,y) \simeq \frac{c_{\eta,l,m,n}}{\sqrt{\eta}}^* \sum_{0\leq w\leq \eta-1}{}^* exp(\frac{2\pi i}{\eta}(nsw-nmw^2))$$

where $c_{\eta,l,m,n} = {}^*exp(\frac{2\pi in(\eta-1)}{\eta}(\frac{m(\eta-1)-2l}{4}))$. Therefore;

$W(t,x,y) \simeq \frac{c_{\eta,l,m,n}}{\sqrt{\eta}}{}^*\sum_{0\leq w\leq \eta-1}{}^*exp(\frac{2\pi i}{\eta}(n(l-m)w - nmw^2))$

$= \frac{c_{\eta,l,m,n}}{\sqrt{\eta}}{}^*exp(\frac{\pi in(l-m)^2}{2m\eta})^*\sum_{0\leq w\leq \eta-1}{}^*exp(\frac{-2\pi inm}{\eta}(w-\frac{l-m}{2m})^2)$

Now replacing l by $l-2km$, where $\frac{k}{\sqrt{\eta}} \simeq 0$ and $k \in {}^*\mathcal{Z}$, a similar calculation to that of Lemma 0.40, using the fact that $\{n,m\} \subset \mathcal{Z}_{>0}$, $\{x,y\} \subset \overline{\mathcal{R}_\eta}$ are finite, and $|c_{\eta,l,m,n}| = 1$, shows that;

$W(t,x,y) \simeq W(t,x,y+\frac{2km}{\sqrt{\eta}})$

The rest of the calculation is identical to lemma 0.40.

□

We now switch to a statistical analysis and require some more notation;

Definition 0.47. *We recall the notation of Definition 0.3. We define a probability measure P_η on the $*$-finite algebra $\mathcal{D}_\eta$ of $\overline{\mathcal{R}_\eta}$ by setting;*

$P_\eta([\frac{i}{\sqrt{\eta}},\frac{i+1}{\sqrt{\eta}})) = \frac{1}{\eta}$, *for* $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$

We let $(\overline{\mathcal{R}_\eta},\mathcal{D}_\eta,P_\eta)$ be the resulting measure space, in the sense of [3]. *We let $(\overline{\mathcal{R}_\eta}\times\overline{\mathcal{T}}_\nu,\mathcal{D}_\eta\times\mathcal{C}_\nu,P_\eta\times\lambda_\nu)$ denote the corresponding product space, where $(\overline{\mathcal{T}}_\nu,\mathcal{C}_\nu,\lambda_\nu)$ was introduced in Lemma 0.3 .*

Definition 0.48. *By analogy with Markov theory, see* [5], *Definition 0.6, we define a weight matrix $M_{\eta,\nu}$, associated by;*

$M_{\eta,\nu}(i,i-k) = \frac{1}{\sqrt{\eta}}L(\frac{k}{\sqrt{\eta}})$

where $-\frac{(\eta-1)}{2} \leq i,k \leq \frac{(\eta-1)}{2}$, and we take $i, i-k \bmod \eta$, and L is as defined in Lemma 0.32.

Definition 0.49. *Let $f : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable with respect to the $*\sigma$-algebra $\mathcal{D}_\eta$, in the sense of* [3]. *We define $F : \overline{\mathcal{R}_\eta}\times\overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ by;*

$F(\frac{i}{\sqrt{\eta}},\frac{j}{\nu}) = (\pi_f M^j_{\eta,\nu})(i)$, *for* $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$, $i \in {}^*\mathcal{Z}$, $j \in {}^*\mathcal{Z}_{\geq 0}$

$F(x,t) = F(\frac{[\sqrt{\eta}x]}{\sqrt{\eta}}, \frac{[\nu t]}{\nu}),\ (x,t) \in \overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu}$

where π_f is the nonstandard distribution vector corresponding to f, $M_{\eta,\nu}$ is the weight matrix of Definition 0.48, and $M^j_{\eta,\nu}$ denotes a nonstandard power.

Theorem 0.50. *Let F be as defined in Definition 0.49, then F is measurable with respect to $\mathcal{D}_\eta \times \mathcal{D}_\nu$, and, moreover F is the unique solution to the nonstandard convolution equation;*

$\frac{\partial F}{\partial t} = (K * f)$

with initial condition f, where K was defined in Lemma 0.31.

Proof. The first proposition follows by observing that the defining schema for F is internal and by hyperfinite induction, see Lemma 0.4 of [5] for the mechanics of this transfer process. For the second proposition, it follows from the proof of Lemma 0.32 and commutativity of the convolution product, that, if F satisfies the nonstandard convolution equation, then;

$F_{\frac{j+1}{\nu}} = (L * F_{\frac{j}{\nu}}) = (F_{\frac{j}{\nu}} * L)$

We observed in Lemma 0.32, that F is uniquely determined from the initial condition f. We have that $M_{\eta,\nu}$ is symmetric. This follows, by observing that K is even, from the definition in Lemma 0.31, and, therefore, L is even, by the definition in Lemma 0.32. It follows that;

$M_{\eta,\nu}(i,j) = \frac{1}{\sqrt{\eta}} L(\frac{(i-j)}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{\eta}} L(\frac{(j-i)}{\sqrt{\eta}})$

$= M_{\eta,\nu}(j,i)$

as required. We then obtain;

$F(\frac{i}{\sqrt{\eta}}, \frac{j+1}{\nu}) = \frac{1}{\sqrt{\eta}} {}^*\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} F(\frac{i-k}{\sqrt{\eta}}, \frac{j}{\nu}) L(\frac{k}{\sqrt{\eta}})$

$= {}^*\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} F(\frac{i-k}{\sqrt{\eta}}, \frac{j}{\nu}) M_{\eta,\nu}(i, i-k)$

$= {}^*\sum_{-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}} F(\frac{i-k}{\sqrt{\eta}}, \frac{j}{\nu}) M^t_{\eta,\nu}(i-k, i)$

$= {}^*\sum_{-\frac{(\eta-1)}{2}\leq k\leq\frac{(\eta-1)}{2}} F(\frac{i-k}{\sqrt{\eta}},\frac{j}{\nu})M_{\eta,\nu}(i-k,i)$

$=\pi_{F_j}M_{\eta,\nu}(i)$

Hence, $\pi_{F_{j+1}}=\pi_{F_j}M_{\eta,\mu}$, and, by iteration, $\pi_{F_j}=\pi_f M^j_{\eta,\mu}$, which agrees with the defining schema for F in Definition 0.49.

□

Definition 0.51. *Let $(\overline{\Omega_\eta},\mathcal{E}_\eta,\gamma_\eta)$ be a nonstandard $*$-finite measure space. We define a reverse filtration on $\overline{\Omega_\eta}$ to be an internal collection of $*\sigma$-algebras $\mathcal{E}_{\eta,i}$, indexed by $0\leq i\leq\kappa$, $\kappa\in{}^*\mathcal{N}\setminus\mathcal{N}$, such that;*

(i). $\mathcal{E}_{\eta,0}=\mathcal{E}_\eta$

(ii). $\mathcal{E}_{\eta,i}\subseteq\mathcal{E}_{\eta,j}$, *if* $0\leq j\leq i\leq\kappa$.

We say that $\overline{F}:\overline{\Omega_\eta}\times\overline{\mathcal{T}_{\nu,\kappa}}\rightarrow{}^*\mathcal{C}$ is adapted to the filtration if $\overline{F}$ is measurable with respect to $\mathcal{E}_\eta\times\mathcal{C}_\nu$ and $\overline{F}_{\frac{i}{\nu}}:\overline{\Omega_\eta}\rightarrow{}^*\mathcal{C}$ is measurable with respect to $\mathcal{E}_{\eta,i}$, for $0\leq i\leq\kappa$.

If $f:\overline{\Omega_\eta}\rightarrow{}^*\mathcal{C}$ is measurable with respect to $\mathcal{E}_{\eta,j}$ and $0\leq j\leq i\leq\kappa$, we define the conditional expectation $E_\eta(f|\mathcal{E}_{\eta,i})$ to be the unique $g:\overline{\Omega_\eta}\rightarrow{}^*\mathcal{C}$ such that g is measurable with respect to $\mathcal{E}_{\eta,i}$ and;

$\int_U g d\gamma_\eta=\int_U f d\gamma_\eta$

for all $U\in\mathcal{E}_{\eta,i}$. We say that $\overline{F}:\overline{\Omega_\eta}\times\overline{\mathcal{T}_{\nu,\kappa}}\rightarrow{}^*\mathcal{C}$ is a reverse martingale if;

(i). $\overline{F}$ is adapted to the reverse filtration on $\overline{\Omega_\eta}$

(ii). $E_\eta(\overline{F}_{\frac{j}{\nu}}|\mathcal{E}_{\eta,i})=\overline{F}_{\frac{i}{\nu}}$ *for* $0\leq j\leq i\leq\kappa$

Theorem 0.52. *Let F be as in Definition 0.49, and let F_κ be its restriction to $\overline{\mathcal{R}_\eta}\times\overline{\mathcal{T}_{\nu,\kappa}}$. Then there exists a reverse filtration on $\overline{\mathcal{R}_\eta}$ and $\overline{F}_\kappa$ such that $\overline{F}_\kappa$ is a reverse martingale, and $\overline{F}_{\kappa,\frac{\kappa}{\nu}}=F_{\frac{\kappa}{\nu}}$.*

Proof. We define the reverse filtration, by setting $\mathcal{E}_{\eta,i}$ to be internal unions of the intervals;

$[\frac{j}{\eta^{\kappa-i}\sqrt{\eta}},\frac{j+1}{\eta^{\kappa-i}\sqrt{\eta}})$ for $-\frac{(\eta-1)\eta^{\kappa-i}}{2}\leq j\leq\frac{(\eta+1)\eta^{\kappa-i}}{2}-1$, $0\leq i\leq\kappa$

Clearly, this is an internal collection. It follows that $\mathcal{E}_\eta = \mathcal{E}_{\eta,0}$ consists of internal unions of the intervals;

$[\frac{j}{\eta^\kappa\sqrt{\eta}}, \frac{j+1}{\eta^\kappa\sqrt{\eta}})$ for $-\frac{(\eta-1)\eta^\kappa}{2} \leq j \leq \frac{(\eta+1)\eta^\kappa}{2} - 1$

and we define the corresponding measure γ_η by setting $\gamma_\eta([\frac{j}{\eta^\kappa\sqrt{\eta}}, \frac{j+1}{\eta^\kappa\sqrt{\eta}})) = \frac{1}{\eta^\kappa\sqrt{\eta}}$. Observe that $\mathcal{E}_{\eta,\kappa} = \mathcal{D}_\eta$, the original $*\sigma$-algebra.

We define bijections;

$\Phi_i : {}^*\mathcal{Z} \cap [-\frac{(\eta-1)}{2}, \frac{(\eta-1)}{2}] \times \overline{\Omega}_{\kappa-i} \rightarrow {}^*\mathcal{Z} \cap [-\frac{(\eta-1)\eta^{\kappa-i}}{2}, \frac{(\eta+1)\eta^{\kappa-i}}{2} - 1]$

for $0 \leq i \leq \kappa$, where

$\overline{\Omega}_{\kappa-i} = \{(\omega_k) : \omega_k \in {}^*\mathcal{Z} \cap [-\frac{(\eta-1)}{2}, \frac{(\eta-1)}{2}], 1 \leq k \leq \kappa - i\}$

by;

$\Phi_i(j, \omega) = \eta^{\kappa-i} j + \eta^{\kappa-i*} \sum_{1\leq k\leq\kappa-i} \frac{\omega_k + \frac{(\eta-1)}{2}}{\eta^k}$

Define $\overline{F}_\kappa$ by;

$\overline{F}_\kappa(\frac{r}{\eta^{\kappa-i}\sqrt{\eta}}, \frac{i}{\nu}) = \eta^{\kappa-i} \prod_{1\leq k\leq\kappa-i} \Theta(\omega_k) F_{\frac{i}{\nu}}(\frac{j}{\sqrt{\eta}} + \frac{1}{\sqrt{\eta}}{}^* \sum_{1\leq k\leq\kappa-i} \omega_k)$

where $\Phi_i(j, \omega) = r$, for $-\frac{(\eta-1)\eta^{\kappa-i}}{2} \leq r \leq \frac{(\eta+1)\eta^{\kappa-i}}{2} - 1$, $0 \leq i \leq \kappa$, and $\Theta : {}^*\mathcal{Z} \cap [-\frac{(\eta-1)}{2}, \frac{(\eta-1)}{2}] \rightarrow {}^*\mathcal{C}$ is defined by $\Theta(s) = \frac{1}{\sqrt{\eta}} L(\frac{s}{\sqrt{\eta}}) = M_{\eta,\nu}(i, i+s)$ (for any i).

$\overline{F}_\kappa(x, t) = \overline{F}_\kappa(\frac{[\eta^{\kappa-[\nu t]}\sqrt{\eta}x]}{\eta^{\kappa-[\nu t]}\sqrt{\eta}}, \frac{[\nu t]}{\nu})$, $(x, t) \in \overline{\mathcal{R}}_\eta \times \overline{\mathcal{T}_{\nu,\kappa}}$

It is clear that $\overline{F}_\kappa$ is adapted to the reverse filtration on $\overline{\mathcal{R}}_\eta$. Moreover, it is straightforward to see that;

$\overline{F}_\kappa(\frac{r}{\sqrt{\eta}}, \frac{\kappa}{\nu}) = F_{\frac{\kappa}{\nu}}(\frac{r}{\sqrt{\eta}})$

as $\Phi_\kappa(r) = r$, so $\overline{F}_{\kappa,\frac{\kappa}{\nu}} = F_{\frac{\kappa}{\nu}}$. We claim that $\overline{F}_\kappa$ is a reverse martingale. We have verified condition (i) in Definition 0.51. To verify (ii), by the tower law for conditional expectation, it is sufficient to prove that $E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1}) = \overline{F}_{\kappa,\frac{i+1}{\nu}}$, for $0 \leq i \leq \kappa - 1$. We have that;

$E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1})(\frac{r}{\eta^{\kappa-i-1}\sqrt{\eta}})$

$= \eta^{\kappa-i-1}\sqrt{\eta}\int_{[\frac{r}{\eta^{\kappa-i-1}\sqrt{\eta}},\frac{r+1}{\eta^{\kappa-i-1}\sqrt{\eta}})} E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1})d\gamma_\eta$

$= \eta^{\kappa-i-1}\sqrt{\eta}\int_{[\frac{r}{\eta^{\kappa-i-1}\sqrt{\eta}},\frac{r+1}{\eta^{\kappa-i-1}\sqrt{\eta}})} \overline{F}_{\kappa,\frac{i}{\nu}}d\gamma_\eta$

$= \frac{\eta^{\kappa-i-1}\sqrt{\eta}}{\eta^{\kappa-i}\sqrt{\eta}}({}^*\sum_{m=0}^{\eta-1}\overline{F}_{\kappa,\frac{i}{\nu}}(\frac{\eta r+m}{\eta^{\kappa-i}\sqrt{\eta}}))$

$= \frac{1}{\eta}(\eta^{\kappa-i}\prod_{1\leq k\leq\kappa-i-1}\Theta(\omega_k)({}^*\sum_{-\frac{(\eta-1)}{2}\leq s\leq\frac{(\eta-1)}{2}}\Theta(s)F_{\frac{i}{\nu}}(x+\frac{s}{\sqrt{\eta}}))$

$= \eta^{\kappa-i-1}\prod_{1\leq k\leq\kappa-i-1}\Theta(\omega_k)F_{\frac{i+1}{\nu}}(x) = \overline{F}_{\kappa,\frac{i+1}{\nu}}(\frac{r}{\eta^{\kappa-i-1}\sqrt{\eta}})$

where $\Phi_{i+1}(j,\omega)=r$, $\omega=(\omega_k)_{1\leq k\leq\kappa-i-1}$ and $x=\frac{j}{\sqrt{\eta}}+\frac{1}{\sqrt{\eta}}({}^*\sum_{1\leq k\leq\kappa-i-1}\omega_k)$, as required.

□

Definition 0.53. *We let $W_{\kappa,1}:\Omega_\kappa\rightarrow{}^*\mathcal{R}$ be defined by;*

$W_{\kappa,1}(\omega)=\frac{1}{\sqrt{\kappa}}{}^*\sum_{1\leq i\leq\kappa}\omega_i$

Lemma 0.54. *There exist constants $\{c_j : 0\leq j\leq\frac{\eta-1}{2}\}\subset{}^*\mathcal{C}$, a function $\Psi:{}^*\mathcal{R}\rightarrow{}^*\mathcal{C}$, defined by;*

$\Psi(x)={}^*exp(\kappa c_0){}^*exp(\sqrt{\kappa}x)$

complex-valued random variables $\{\chi_k : 1\leq k\leq\kappa\}$, defined by;

$\chi_k={}^*\sum_{j=1}^{\frac{\eta-1}{2}}c_j\omega_k^{2j}$

$\chi=\frac{1}{\sqrt{\kappa}}{}^*\sum_{k=1}^{\kappa}\chi_k$

such that;

$\Psi(\chi)(\omega)=\eta^\kappa\prod_{1\leq k\leq\kappa}\Theta(\omega_k),\ (\omega\in\overline{\Omega}_\kappa)$

Proof. We first claim there exists an even $*$-polynomial g of degree $\eta-1$;

$g(x)={}^*\sum_{j=0}^{\frac{\eta-1}{2}}c_jx^{2j}$

with;

$g(0) = ln(\eta\Theta(0)) = a_0$

$g(i) = ln(\eta\Theta(i)) = a_i$, $1 \leq |i| \leq \frac{\eta-1}{2}$

To see this, as Θ is even, we have to solve the matrix equation $W\overline{c} = \overline{a}$, where $(\overline{a})_i = a_i$, for $0 \leq i \leq \frac{\eta-1}{2}$, $(\overline{c})_i = c_i$, for $0 \leq i \leq \frac{\eta-1}{2}$, and W is defined by $(W)_{i,0} = 1$, for $0 \leq i \leq \frac{\eta-1}{2}$, $(W)_{0,j} = 0$, for $1 \leq j \leq \frac{\eta-1}{2}$ and $(W)_{i,j} = i^{2j}$, for $1 \leq i, j \leq \frac{\eta-1}{2}$. It is sufficient to prove the matrix W is invertible, which follows from general facts about van der Monde matrices (fill in details). Now, let χ, χ_k, for $1 \leq k \leq \kappa$ and Ψ be as in the statement of the lemma. Then we compute;

$\Psi(\chi)(\omega) = {}^*exp(\kappa c_0)^*exp({}^*\sum_{1\leq k\leq\kappa}\chi_k(\omega))$

$= {}^*exp(\kappa c_0)^*\prod_{1\leq k\leq\kappa}{}^*exp(\chi_k(\omega))$

$= {}^*exp(\kappa c_0)^*\prod_{1\leq k\leq\kappa}{}^*exp(g(\omega_k) - c_0)$

$= {}^*\prod_{1\leq k\leq\kappa}{}^*exp(ln(\eta\Theta(\omega_k)))$

$= \eta^{\kappa*}\prod_{1\leq k\leq\kappa}\Theta(\omega_k)$

as required.

□

We now simplify the presentation of K given in Lemma 0.31.

Lemma 0.55. *Let K and W be given as in Lemma 0.31, and let Θ be given by Theorem 0.52. Then, for $0 < |j| \leq \frac{\eta-1}{2}$, j odd, we have that;*

$Re(W(\frac{j}{\sqrt{\eta}})) = -\frac{\eta}{4}\frac{{}^*cos(\frac{\pi j}{\eta})}{{}^*sin^2(\frac{\pi j}{\eta})}$

and for $0 < |j| \leq \frac{\eta-1}{2}$, j even, we have that;

$Re(W(\frac{j}{\sqrt{\eta}})) = \frac{\eta}{4}\frac{{}^*cos(\frac{\pi j}{\eta})}{{}^*sin^2(\frac{\pi j}{\eta})}$

In particular, we have that;

$\Theta(0) = 1 - \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2\nu}$

$\Theta(s)=\frac{2\pi^2 i}{\eta\nu}\frac{{}^*cos(\frac{\pi s}{\eta})}{{}^*sin^2(\frac{\pi s}{\eta})}$, $1\leq |s|\leq \frac{\eta-1}{2}$, s *odd*

$\Theta(s)=-\frac{2\pi^2 i}{\eta\nu}\frac{{}^*cos(\frac{\pi s}{\eta})}{{}^*sin^2(\frac{\pi s}{\eta})}$, $1\leq |s|\leq \frac{\eta-1}{2}$, s *even*

Moreover, for $0<|j|\leq \frac{\eta-1}{2}$;

$\frac{1}{\pi^2}\leq \frac{\eta^3-2\eta^2|j|}{4\pi^2 j^2}\leq |Re(W(\frac{j}{\sqrt{\eta}})|\leq \frac{\eta^3}{16j^2}\leq \frac{\eta^3}{16}$

Proof. Let $p(z)$ be as in Lemma 0.31. We have that;

$p(z)=-\frac{\eta^2}{4}\frac{(z^5-2z^3+z)}{(z^2-1)^3}+\frac{\eta}{2}\frac{(z^5-z)}{(z^2-1)^3}-\frac{1}{4}\frac{(z^5+4z^4+6z^3+4z^2+z)}{(z^2-1)^3}$

$=-\frac{\eta^2}{4}\frac{z(z^2-1)^2}{(z^2-1)^3}+\frac{\eta}{2}\frac{z(z^4-1)}{(z^2-1)^3}-\frac{1}{4}\frac{z(z+1)^4}{(z^2-1)^3}$

$=-\frac{\eta^2}{4}\frac{z}{z^2-1}+\frac{\eta}{2}\frac{z(z^2+1)}{(z^2-1)^2}-\frac{1}{4}\frac{z(z+1)}{(z-1)^3}$

We have that;

$\frac{z}{z^2-1}=\frac{1}{2}(\frac{1}{z+1}+\frac{1}{z-1})$

$Re(\frac{1}{z+1})=Re(\frac{\overline{z}+1}{|z+1|^2})$

$=Re(\frac{1+cos(x)-isin(x)}{(1+cos(x))^2+sin^2(x)})$ when $z=e^{ix}$

$=\frac{1+cos(x)}{2+2cos(x)}=\frac{1}{2}$

Similarly, we have that $Re(\frac{1}{z-1})=-\frac{1}{2}$, when $z=e^{ix}$. This proves that $Re(\frac{z}{z^2-1})=0$ when $z=e^{ix}$. We have that;

$\frac{z(z^2+1)}{(z^2-1)^2}=\frac{z}{(z^2-1)}+\frac{2z}{(z^2-1)^2}$

Hence, by the previous result;

$Re(\frac{z(z^2+1)}{(z^2-1)^2})=Re(\frac{2z}{(z^2-1)^2})$, when $z=e^{ix}$

Now, again using the previous result, we have that;

$Re(\frac{z}{(z^2-1)^2})=-Im(\frac{z}{(z^2-1)})Im(\frac{1}{z^2-1})$

We have;

$Im(\frac{1}{(z^2-1)}) = Im(\frac{cos(2x)-isin(2x)-1}{(cos(2x)-1)^2+(sin(2x))^2})$, $z = e^{ix}$

$= \frac{-sin(2x)}{2-2cos(2x)}$

$= \frac{-2sin(x)cos(x)}{4sin^2(x)}$

$= -\frac{cot(x)}{2}$

$Im(\frac{z}{(z^2-1)}) = Im(\frac{(cos(x)+isin(x))(cos(2x)-1-isin(2x))}{4sin^2(x)})$, $z = e^{ix}$

$= \frac{-cos(x)sin(2x)+sin(x)cos(2x)-sin(x)}{4sin^2(x)}$

$= \frac{-cos^2(x)sin(x)-sin^3(x)-sin(x)}{4sin^2(x)}$

$= \frac{-2sin(x)}{4sin^2(x)}$

$= \frac{-1}{2sin(x)}$

It follows that, for $z = e^{ix}$;

$Re(\frac{2z}{(z^2-1)^2}) = 2\frac{cot(x)}{2}\frac{-1}{2sin(x)}$

$= \frac{-cos(x)}{2sin^2(x)}$

Finally, we compute $Re(\frac{z(z+1)}{(z-1)^3})$, when $z = e^{ix}$. We have that;

$\frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)^2} + \frac{2z}{(z-1)^3}$

$= \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + 2(\frac{1}{(z-1)^2} + \frac{1}{(z-1)^3})$

$= \frac{1}{(z-1)} + \frac{3}{(z-1)^2} + \frac{2}{(z-1)^3}$

We computed $Re(\frac{1}{(z-1)}) = \frac{-1}{2}$. We have that;

$Re(\frac{3}{(z-1)^2}) = 3(Re(\frac{1}{(z-1)}))^2 - 3(Im(\frac{1}{(z-1)}))^2$

$= \frac{3}{4} - 3(Im(\frac{1}{(z-1)}))^2$

We have that;

$Im(\frac{1}{(z-1)}) = Im(\frac{cos(x)-1-isin(x)}{(cos(x)-1)^2+sin^2(x)})$, $(z = e^{ix})$

$= \frac{-sin(x)}{2-2cos(x)}$

$= -\frac{cot(\frac{x}{2})}{2}$

$Re(\frac{3}{(z-1)^2}) = \frac{3}{4}(1 - cot^2(\frac{x}{2}))$

Finally;

$Re(\frac{2}{(z-1)^3}) = 2(Re(\frac{1}{(z-1)}))^3 - 6Re(\frac{1}{(z-1)})(Im(\frac{1}{(z-1)}))^2$

$= 2(\frac{-1}{2})^3 - 6(\frac{-1}{2})(\frac{cot^2(\frac{x}{2})}{4})$ $(z = e^{ix})$

$= \frac{-1}{4} + \frac{3}{4}cot^2(\frac{x}{2})$

Combining these results, gives;

$Re(\frac{z(z+1)}{(z-1)^3}) = \frac{-1}{2} + \frac{3}{4}(1 - cot^2(\frac{x}{2})) - \frac{1}{4} + \frac{3}{4}cot^2(\frac{x}{2})$ $(z = e^{ix})$

$= 0$

This implies that;

$Re(p(e^{ix})) = -\frac{\eta}{4}(\frac{cos(x)}{sin^2(x)})$

and the result of the first part of the lemma follows by transfer. The second part follows by noting that $q(z)$ reverses the sign of the terms with coefficients involving η^2 and η. The third term becomes $\frac{1}{4}\frac{z(z-1)}{(z+1)^3}$. It is a similar exercise to verify that $Re(\frac{z(z-1)}{(z+1)^3}) = 0$, when $z = e^{ix}$. Again, the result follows by transfer.

For the claim involving Θ, we have, from Lemma 0.31 and the last result, that;

$K(0) = -\frac{\pi^2 i}{\eta^{\frac{3}{2}}}\frac{(\eta-1)\eta(\eta+1)}{3}$

$K(\frac{s}{\sqrt{\eta}}) = -\frac{8\pi^2 i}{\eta^{\frac{3}{2}}}Re(W(\frac{s}{\sqrt{\eta}}))$, $1 \leq |s| \leq \frac{\eta-1}{2}$

$= -\frac{8\pi^2 i}{\eta^{\frac{3}{2}}}\frac{-\eta}{4}\frac{{}^*cos(\frac{\pi s}{\eta})}{{}^*sin^2(\frac{\pi s}{\eta})}$, s odd

$= \frac{2\pi^2 i}{\eta^{\frac{1}{2}}} \frac{{}^*cos(\frac{\pi s}{\eta})}{{}^*sin^2(\frac{\pi s}{\eta})}$

and, similarly;

$K(\frac{s}{\sqrt{\eta}}) = -\frac{2\pi^2 i}{\eta^{\frac{1}{2}}} \frac{{}^*cos(\frac{\pi s}{\eta})}{{}^*sin^2(\frac{\pi s}{\eta})}$, s even

By the definition of L in 0.32 and the definition of Θ in Theorem 0.52, we have that;

$\Theta(s) = \frac{1}{\sqrt{\eta}} L(\frac{s}{\sqrt{\eta}})$, $|s| \leq \frac{\eta-1}{2}$

$L(0) = \sqrt{\eta} + \frac{K(0)}{\nu}$

$L(\frac{s}{\sqrt{\eta}}) = \frac{K(\frac{s}{\sqrt{\eta}})}{\nu}$, $1 \leq |s| \leq \frac{\eta-1}{2}$

We then compute, using the above calculation of K;

$\Theta(0) = 1 + \frac{K(0)}{\sqrt{\eta}\nu}$

$= 1 - \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2\nu}$

$\Theta(s) = \frac{K(\frac{s}{\sqrt{\eta}})}{\sqrt{\eta}\nu}$

$= \frac{2\pi^2 i}{\eta\nu} \frac{{}^*cos(\frac{\pi s}{\sqrt{\eta}})}{{}^*sin^2(\frac{\pi s}{\sqrt{\eta}})}$, s odd

$\Theta(s) = -\frac{2\pi^2 i}{\eta\nu} \frac{{}^*cos(\frac{\pi s}{\sqrt{\eta}})}{{}^*sin^2(\frac{\pi s}{\sqrt{\eta}})}$, s even

For the final result, we have the following inequalities, valid for $x \in (0, \frac{\pi}{2})$;

$1 - \frac{2x}{\pi} \leq cos(x) \leq 1$

$\frac{2x}{\pi} \leq sin(x) \leq x$

Combining these, we obtain that;

$\frac{1}{x^2} - \frac{2}{\pi x} \leq \frac{cos(x)}{sin^2(x)} \leq \frac{\pi^2}{4x^2}\ x \in (0, \frac{\pi}{2})$

Either by observing that $\frac{cos(x)}{sin^2(x)}$ is monotone decreasing on $(0, \frac{\pi}{2})$ and using these inequalities at the endpoints $\{\frac{1}{\sqrt{\eta}}, \frac{(\eta-1)}{2\sqrt{\eta}}\}$ of the nonstandard domain, or by substituting for j directly, we see that;

$|Re(W(\frac{j}{\sqrt{\eta}}))| \leq \frac{\eta}{4}\frac{\pi^2}{4(\frac{\pi j}{\eta})^2} = \frac{\eta^3}{16j^2} \leq \frac{\eta^3}{16}$ $(0 < j \leq \frac{\eta-1}{2})$

$|Re(W(\frac{j}{\sqrt{\eta}}))| \geq \frac{\eta}{4}(\frac{1}{(\frac{\pi j}{\eta})^2}) - \frac{2}{\pi(\frac{\pi j}{\eta})} = \frac{\eta^3-2\eta^2 j}{4\pi^2 j^2}$

$\geq \frac{\eta^2-2\eta(\frac{\eta-1}{2})}{\pi^2(\frac{\eta-1}{2})^2}$

$= \frac{\eta^2}{\pi^2(\eta-1)^2}$

$\geq \frac{1}{\pi^2}$ $(0 < j \leq \frac{\eta-1}{2})$

Now, we can extend the result to $0 < |j| \leq \frac{\eta-1}{2}$ by symmetry.

□

For technical reasons which will become apparent later, we alter slightly the definitions in Theorem 0.52.

Definition 0.56. *We define;*

$\overline{\Omega}^{new}_{\kappa-i} = \{(\omega_k) : \omega_k \in {}^*\mathcal{Z} \cap [1,\eta], 1 \leq k \leq \kappa - i\}$, $0 \leq i \leq \kappa$

We define maps;

$\Psi_i^{new} : \overline{\Omega}^{new}_{\kappa-i} \rightarrow \overline{\Omega}_{\kappa-i}$

by $\Psi_i(\omega)(k) = \omega_k - \eta \ (mod\ \eta)$*, for* $1 \leq k \leq \kappa - i$*,* $0 \leq i \leq \kappa$

We let Φ_i^{new} *be the composition* $\Phi_i \circ (Id_{\overline{\mathcal{R}_\eta}} \times \Psi_i^{new})$*. Then* Φ_i^{new} *is given explicitly by;*

$\Phi_i^{new}(j,\omega) = \eta^{\kappa-i}j + \eta^{\kappa-i*}\sum_{1\leq k\leq\kappa-i}\frac{(\omega_k - \frac{\eta+1}{2}(mod\eta))}{\eta^k}$

We transpose the weights by letting $\Theta^{new} : {}^*\mathcal{Z} \cap [1,\eta] \rightarrow {}^*\mathcal{C}$ *be defined by;*

$\Theta^{new}(s) = \Theta(s - \eta(mod\eta))$

Finally, we define $\overline{F}^{new}_\kappa$ *using the same definition as in Theorem 0.52, but replacing* Θ *by* Θ^{new}*,* Φ_i *by* Φ_i^{new}*, and using* $(\omega_k) \in \overline{\Omega}^{new}_{\kappa-i}$*.*

It is clear that $\overline{F}_{\kappa}^{new} = \overline{F}_{\kappa}$. *From now, we will drop the new superscript, emphasising that we are working with this representation in the path space* $\overline{\mathcal{R}_{\eta}} \times \overline{\Omega}_{\kappa}^{new}$.

Lemma 0.57. *Let* $1_{\overline{\mathcal{R}_{\eta}}}$ *be the indicator function on* $\overline{\mathcal{R}_{\eta}}$, *then an explicit solution to the convolution equation, with initial condition* $1_{\overline{\mathcal{R}_{\eta}}}$, *is given by;*

$$F_{1_{\overline{\mathcal{R}_{\eta}}}}(x,t) = 1$$

Moreover, we obtain the identities;

$${}^*\sum_{k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} \Theta(k) = 1$$

and;

$${}^*\sum_{0<|k|\leq\frac{(\eta-1)}{2}} (-1)^{k+1} \frac{{}^*cos(\frac{\pi k}{\eta})}{{}^*sin^2(\frac{\pi k}{\eta})} = \frac{(\eta-1)(\eta+1)}{6}.$$

Proof. By Lemma 0.33, we have that;

$$\hat{F_{1_{\overline{\mathcal{R}_{\eta}}}}}(y,t) = \left(1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu}\right)^{[\nu t]} \hat{1_{\overline{\mathcal{R}_{\eta}}}}(y)$$

By definition 0.11, we have that;

$$\hat{1_{\overline{\mathcal{R}_{\eta}}}}(y) = \int_{\overline{\mathcal{R}_{\eta}}} 1_{\overline{\mathcal{R}_{\eta}}}(x) exp_{\eta}(-2\pi ixy) d\mu_{\eta}(x)$$

Using the definition of the internal integral, we are summing roots of unity, when $[y\sqrt{\eta}] \neq 0$, so $\hat{1_{\overline{\mathcal{R}_{\eta}}}}(y) = 0$, and, when $y = 0$, we obtain that $\hat{1_{\overline{\mathcal{R}_{\eta}}}}(0) = \sqrt{\eta}$. Therefore, $\hat{1_{\overline{\mathcal{R}_{\eta}}}} = \delta_0$. Substituting 0 into $(1 - \frac{4\pi^2[\sqrt{\eta}y]^2 i}{\eta\nu})^{[\nu t]}$, we obtain that;

$$\hat{F_{1_{\overline{\mathcal{R}_{\eta}}}}}(y,t) = \delta_0$$

Now applying the inversion theorem, see Lemma 0.12, we obtain that $F_{1_{\overline{\mathcal{R}_{\eta}}}}(x,t) = 1$ as required.

Combining Theorem 0.50 and Definition 0.49, we have that;

$$F\left(\frac{i}{\sqrt{\eta}}, \frac{j}{\nu}\right) = (\pi_{1_{\overline{\mathcal{R}_{\eta}}}}) M_{\eta,\nu}^{j}$$

where $M_{\eta,\nu}$ is the weight matrix of Definition 0.48. A simple computation then shows that;

$(\pi 1_{\overline{\mathcal{R}_\eta}})M^j_{\eta,\nu} = C^j$

where $C = {}^*\sum_{k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} \Theta(k)$

with Θ defined as in Theorem 0.52. Applying the first result of the Lemma, we obtain that;

$C = {}^*\sum_{k=-\frac{(\eta-1)}{2}}^{\frac{(\eta-1)}{2}} \Theta(k) = 1$

Using Lemma 0.55, we have that;

$C = 1 - \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2\nu} + \frac{2\pi^2 i}{\eta\nu}({}^*\sum_{0<|k|\leq\frac{(\eta-1)}{2}}(-1)^{k+1}\frac{{}^*cos(\frac{\pi k}{\eta})}{{}^*sin^2(\frac{\pi k}{\eta})})$

A simple rearrangement proves the final result.

□

Definition 0.58. *We adopt the notation of Definition 0.56. We define the natural filtration on* $\overline{\Omega_\kappa}$. *If* $\omega' \in \overline{\Omega_i}$, *for some* $1 \leq i \leq \kappa$, *we let;*

$V_{\omega'} = \{\omega \in \Omega_\kappa : \omega|_{[1,i]} = \omega'\}$

We define the **-sigma algebras* $\mathcal{F}_i$ *to be generated from the internal collections* $\{V_{\omega'} : \omega' \in \overline{\Omega}_i\}$, *for* $1 \leq i \leq \kappa$, *and* $\mathcal{F}_0$ *to be generated by* $\{\emptyset, \overline{\Omega}_\kappa\}$. *Clearly, we have that* $\mathcal{F}_i \subseteq \mathcal{F}_j$, *for* $0 \leq i \leq j \leq \kappa$. *We define a probability measure* $P_{\overline{\Omega_\kappa}} : \mathcal{F}_\kappa \rightarrow {}^*[0,1]$ *by defining* $P_{\overline{\Omega_\kappa}}(V_\omega) = \frac{1}{\eta^\kappa}$, *where* $\omega \in \overline{\Omega_\kappa}$. *We define the random variables;*

$\overline{\omega_i} = \omega_i - E_\eta(\omega_i)$, $1 \leq i \leq \kappa$

$\overline{\omega_i^j} = \omega_i^j - E_\eta(\omega_i^j)$, $1 \leq i \leq \kappa$, $0 \leq j \leq \eta - 1$

where, by slight abuse of notation, we consider ω_i *as a random variable, taking the i'th coordinate of* $\omega \in \overline{\Omega_\kappa}$ *as its value, and, interpret* $\overline{\omega_i^0} = 1$. *We define the random variables;*

$\overline{\omega_i^{\overline{j}}} = \overline{\omega_1^{j_1}\omega_2^{j_2}} \ldots \overline{\omega_i^{j_i}}$, *where* $1 \leq i \leq \kappa$, $0 \leq j_i \leq \eta - 1$

Lemma 0.59. *Let* $X : \overline{\Omega}_\kappa \times \mathcal{T}_\kappa \rightarrow {}^*\mathcal{C}$ *be a martingale, then, for all* $0 \leq t \leq \kappa - 1$*;*

$$X_{t+1} = {}^*\sum_{j=1}^{\eta-1} \overline{\omega_{t+1}^{j}} f_j(\overline{\omega_1}, \ldots, \overline{\omega_t}) + g_t(\overline{\omega_1}, \ldots, \overline{\omega_t})$$

where $\{f_1, \ldots, f_{\eta-1}, g\} \subset {}^*\mathcal{C}[x_1, \ldots, x_t]$ *are* **-polynomials of degree at most* $\eta - 1$ *in each variable* x_k, $1 \leq k \leq t$, $g_t(\overline{\omega_1}, \ldots, \overline{\omega_t}) = X_t$, *and, we interpret* $\overline{\omega_k}^j = \overline{\omega_k^j}$, *for* $0 \leq j \leq \eta - 1$, $1 \leq k \leq t$.

Proof. We prove by induction that, for $1 \leq i \leq \kappa$, the monomials $\{\overline{\omega_i^{\bar{j}}} : 0 \leq j_i \leq \eta - 1\}$ form a basis for $V(\mathcal{F}_i)$, (†). As ${}^*dimV(\mathcal{F}_i) = \eta^i$, it is sufficient to prove they are $*$-linearly independent. The case $i = 1$, follows from the fact that $\{\omega_1^j : 0 \leq j \leq \eta - 1\}$ are $*$-linearly independent, which is a consequence of the invertibility of the van der Monde matrix M given by;

$$M_{1,j} = 1,\ 1 \leq j \leq \eta\ M_{i,1} = 1,\ 1 \leq i \leq \eta,\ M_{i,j} = i^{j-1},\ 2 \leq i, j \leq \eta$$

Now, if $\{\overline{\omega_1^j} : 0 \leq j \leq \eta - 1\}$ were $*$-linearly dependent, then clearly we would obtain $*$-linear dependence of $\{\omega_1^j : 0 \leq j \leq \eta - 1\}$ which is a contradiction. Similarly, one can show that for any $1 \leq i \leq \kappa$, $\{\overline{\omega_i^j} : 0 \leq j \leq \eta - 1\}$ are $*$-linearly independent, $(*)$. Now assume, inductively, that the monomials $\{\overline{\omega_s^{\bar{j}}} : 0 \leq j_i \leq \eta - 1\}$ are $*$-linearly independent, for some $1 \leq s < \kappa$, we show that the monomials $\{\overline{\omega_{s+1}^{\bar{j}}} : 0 \leq j_i \leq \eta - 1\}$ are $*$-linearly independent. Suppose not, then, without loss of generality, we can witness the dependence in the form;

$$\overline{\omega_s^{\bar{j}_1}} f_1(\overline{\omega_{s+1}}, \ldots, \overline{\omega_{s+1}^{\eta-1}})$$

$$+ \ldots + \overline{\omega_s^{\bar{j}_k}} f_k(\overline{\omega_{s+1}}, \ldots, \overline{\omega_{s+1}^{\eta-1}})$$

$$+ \ldots + \overline{\omega_s^{\bar{j}_r}} f_r(\overline{\omega_{s+1}}, \ldots, \overline{\omega_{s+1}^{\eta-1}}) = 0\ (**)$$

where $1 \leq r \leq \eta^s$ and $\{f_i : 1 \leq k \leq r\} \subset {}^*\mathcal{C}[x_1, \ldots, x_{\eta-1}]$ are non-zero linear polynomials. Now choose the first non-zero coefficient $\lambda_{1,i}$ in f_1, which matches with $\overline{\omega_{s+1}}^i$, for some $0 \leq i \leq \eta - 1$, and let $\{\lambda_{2,i}, \ldots, \lambda_{r,i}\}$ denote the corresponding coefficients in subsequent rows. By the inductive hypothesis, we have that $\{\overline{\omega_s^{\bar{j}_1}},\ldots,\overline{\omega_s^{\bar{j}_r}}\}$ are $*$-linearly independent, hence we can find $\omega\in\overline{\Omega}_\kappa$, with;

$$^*\sum_{k=1}^{r}\lambda_{k,i}\overline{\omega_s^{\bar{j}_k}}(\omega)\neq 0$$

Then, for all $\omega'\in\overline{\Omega}_\kappa$, with $\omega'|_{[1,s]}=\omega|_{[1,s]}$, using the fact that $\{\overline{\omega_s^{\bar{j}_k}}:1\leq k\leq r\}$ are $\mathcal{F}_s$ measurable, substituting in $(**)$, we obtain a fixed $*$-linear dependence of $\{\overline{\omega_{s+1}^w}:0\leq w\leq\eta-1\}$, evaluated at such ω'. As the $s+1$'th coordinate of ω' is free to vary between 1 and η, $\{\overline{\omega_{s+1}^w}:0\leq w\leq\eta-1\}$ are independent of the first s-coordinates, and $\mathcal{F}_{s+1}$ measurable, we clearly obtain $*$-linear dependence of $\{\overline{\omega_{s+1}^w}:0\leq w\leq\eta-1\}$, contradicting $(*)$. This shows $(\dagger)$. Now, using this result and the fact that X_{t+1} is $\mathcal{F}_{t+1}$ measurable, it is clear we can find the representation of X_{t+1} given in the lemma. It remains to show that $g_t(\overline{\omega_1},\ldots,\overline{\omega_t})=X_t$. Using the fact that $E_\eta(X_{t+1}|\mathcal{F}_t=X_t$, the facts that $g_t(\overline{\omega_1},\ldots,\overline{\omega_t})$ and $\{f_j((\overline{\omega_1},\ldots,\overline{\omega_t})):1\leq j\leq\eta-1\}$ are $\mathcal{F}_t$ measurable, taking out what is known, it is sufficient to prove that $E_\eta(\overline{\omega_{t+1}^j}|\mathcal{F}_t)=0$, for $1\leq j\leq\eta-1$. This follows from $*$-independence of $\{\overline{\omega_{t+1}^j}:1\leq j\leq\eta-1\}$ with respect to $\mathcal{F}_t$ and the fact that $E_\eta(\overline{\omega_{t+1}^j})=0$, for $1\leq j\leq\eta-1$.

□

Lemma 0.60. *Let $F_{1_{\mathcal{R}_\eta}}$ be as defined in Lemma 0.57, and let $\overline{F_{1_{\mathcal{R}_\eta},\kappa}}$ be the corresponding reverse martingale. Let;*

$$\Phi^*\overline{F_{1_{\mathcal{R}_\eta},\kappa}}:\overline{T}_{\kappa,\nu}^{rev}\times\overline{\Omega}_\kappa\times\overline{\mathcal{R}_\eta}\rightarrow{}^*\mathcal{C}$$

be its time-reversed pullback, defined by;

$$\Phi^*\overline{F_{1_{\mathcal{R}_\eta},\kappa}}(\tfrac{i}{\nu},\omega,\tfrac{j}{\sqrt{\eta}})=\overline{F_{1_{\mathcal{R}_\eta},\kappa}}(\Phi_{\kappa-i}(j,\omega)),\ -\tfrac{\eta-1}{2}\leq j\leq\tfrac{\eta-1}{2},0\leq i\leq\kappa$$

which, we also denote by $\overline{F_{1_{\mathcal{R}_\eta},\kappa}}$. Then;

$$(\overline{F_{1_{\mathcal{R}_\eta},\kappa}})_{t+1}-(\overline{F_{1_{\mathcal{R}_\eta},\kappa}})_t={}^*\sum_{j=1}^{\eta-1}\overline{\omega_{t+1}^j}r_j(\overline{F_{1_{\mathcal{R}_\eta},\kappa}})_t$$

where;

$$r_j=\eta^*\sum_{k=1}^{\eta-1}(\overline{A}-\overline{E})_{jk}^{-1}\Theta(k)-{}^*\sum_{k=1}^{\eta-1}(\overline{A}-\overline{E})_{jk}^{-1}$$

and $\overline{A}$, $\overline{E}$ *are the matrices defined by;*

$(\overline{A})_{jk} = j^k$, *for* $1 \leq j, k \leq \eta - 1$

$(\overline{E})_{jk} = E_\eta(\overline{\omega_1^k})$, *for* $1 \leq j, k \leq \eta - 1$

Proof. We abbreviate the martingale $\overline{F_{1_{\mathcal{R}_\eta},\kappa}}$ to X. Using the result of Lemma 0.59, we can determine the values of $\{f_1, \ldots f_{\eta-1}\}$ at $\omega|_{[1,t]}$, by the matrix equation;

$(A - E)\overline{f} = \overline{x}$

where;

$\overline{f} = (f_1, \ldots, f_{\eta-1})|_{\omega|_{[1,t]}}$

$\overline{x} = ((X_{t+1} - X_t)(\omega|_{[1,t]}, 1), \ldots (X_{t+1} - X_t)(\omega|_{[1,t]}, \eta))$

and A, E are the matrices defined by;

$(A)_{jk} = j^k$, $1 \leq j \leq \eta$, $1 \leq k \leq \eta - 1$

$(E)_{jk} = E_\eta((\overline{\omega_1^k}))$, $1 \leq j \leq \eta$, $1 \leq k \leq \eta - 1$

By $*$-linear independence of $\{\overline{\omega_1^1}, \ldots, \overline{\omega_1^{\eta-1}}\}$, we have that ${}^*rank(A - E) = \eta - 1$. We claim that $\overline{A} - \overline{E}$ is invertible, $(*)$. If not then, **Row Rank*$(\overline{A} - \overline{E}) \leq \eta - 2$. However;

${}^*\sum Rows(A - E) = \overline{0}$

using the identity ${}^*\sum_{j=1}^{\eta} j^k - \eta E_\eta(\overline{\omega_1^k}) = 0$

It follows that **Row Rank*$(A - E) = {}^*$*Row Rank*$(\overline{A} - \overline{E}) \leq \eta - 2$. This contradicts the fact that ${}^*rank(A - E) = \eta - 1$, hence $(*)$ is proved. It follows that;

$\overline{f} = (\overline{A} - \overline{E})^{-1}\overline{x}_{red}$

where $\overline{x}_{red}$ is the reduced vector of increments given by;

$\overline{x}_{red}=((X_{t+1}-X_t)(\omega|_{[1,t]},1),\ldots(X_{t+1}-X_t)(\omega|_{[1,t]},\eta-1))$

Now, by the definition of X_t, see Lemma 0.52, we have that;

$X_t(\omega)=\eta^t\Theta(\omega_1)\ldots\Theta(\omega_t)$

$(X_{t+1}-X_t)(\omega|_{[1,t]},i)=\eta^t\Theta(\omega_1)\ldots\Theta(\omega_t)(\eta\Theta(i)-1)$, $1\leq i\leq\eta-1$

By linearity, we then obtain that;

$\overline{f}=\eta^t\Theta(\omega_1)\ldots\Theta(\omega_t)(\overline{A}-\overline{E})^{-1}\overline{y}_{red}$

$=X_t(\omega|_{[1,t]})(\overline{A}-\overline{E})^{-1}\overline{y}_{red}$ $(*)$

where;

$\overline{y}_{red}=(\eta\Theta(1)-1,\ldots,\eta\Theta(\eta-1)-1)$

Now, by a simple rearrangement, and substituting the values of $\overline{f}$ into the equation for the increment from Lemma 0.59, we obtain the result, with;

$r_j=((\overline{A}-\overline{E})^{-1}\overline{y}_{red})_j$

□

Lemma 0.61. *We have that;*

$r_j=((\frac{\overline{A}^{-1}\overline{E}}{1-C}+Id)\overline{A}^{-1}\overline{y}_{red})_j$

where $C={}^*\sum_{1\leq i,j\leq\eta-1}\mu_i b_{ij}$

$\mu_i=E_\eta(\overline{\omega_1^i})$, $(\overline{A}^{-1})_{ij}=b_{ij}$, *and;*

$b_{ij}={}^*\sum_{k=i}^{\eta-1}\frac{(-1)^{k+j}C_j^k S_k^i}{k!}$

where $C_j^k=1$ *if* $j=0$, $C_j^k=0$, *if* $j>k$, *and* S_k^i *are signed Sterling numbers of the first kind.*

Proof. We first obtain an explicit expression for $\overline{f}$ from Lemma 0.60. Rearranging $(*)$ in the same Lemma, we obtain;

$\overline{Af}=\overline{Ef}+X_t(\omega|_{[1,t]})\overline{y}_{red}$ $(*)$

By standard results on van der Monde matrices, $\overline{A}$ is invertible, and we obtain;

$$\overline{f} = \overline{A}^{-1}\overline{Ef} + X_t(\omega|_{[1,t]})\overline{A}^{-1}\overline{y}_{red} \ (**)$$

We now compute $\overline{Ef}$. We multiply both sides of $(*)$ by $\overline{EA}^{-1}$, to obtain that;

$$\overline{Ef} = \overline{EA}^{-1}\overline{Ef} + X_t(\omega|_{[1,t]})\overline{EA}^{-1}\overline{y}_{red} \ (***)$$

It is a straightforward exercise to verify that;

$$\overline{EA}^{-1}\overline{E} = C\overline{E}$$

where C is given in the statement of the Lemma. We need to check that $C \neq 1$. Suppose not, then;

$$\overline{EA}^{-1}\overline{E} = \overline{E}$$

Multiplying both sides by $\overline{A}^{-1}$, we obtain that;

$$(\overline{A}^{-1}\overline{E})^2 = \overline{A}^{-1}\overline{E}$$

As $\overline{A}$ is invertible and $\overline{E} \neq 0$, there exists $\overline{v} \neq 0$, with;

$$(\overline{A}^{-1}\overline{E})\overline{v} = \overline{v}$$

Then, rearranging, we obtain that;

$$(\overline{A} - \overline{E})\overline{v} = \overline{0}$$

This contradicts the fact that $(\overline{A} - \overline{E})$ is invertible. Rearranging $(***)$, we obtain that;

$$(1 - C)\overline{Ef} = X_t(\omega|_{[1,t]})\overline{EA}^{-1}\overline{y}_{red} \ (****)$$

Substituting the value of $\overline{Ef}$ from $(****)$ into $(**)$, we obtain;

$$\overline{f} = \frac{X_t(\omega|_{[1,t]})}{1-C}\overline{A}^{-1}\overline{EA}^{-1}\overline{y}_{red} + X_t(\omega|_{[1,t]})\overline{A}^{-1}\overline{y}_{red} \ (*****)$$

The explicit calculation of b_{ij} is given in [4]. Rearranging $(*****)$, we obtain the required expression for the r_j.

$\square$

Lemma 0.62.

Proof. We compute the values of r_j, for $1 \leq j \leq \eta - 1$. We first calculate $\overline{EA}^{-1}$. We have that;

$$(\overline{EA}^{-1})_{ij} = \overline{\mu} \centerdot \overline{A}^{-1}(\overline{e_j}) \ 1 \leq i, j \leq \eta - 1$$

where $\overline{\mu} = (\mu_1, \ldots, \mu_{\eta-1})$ and $\{\overline{e_j} : 1 \leq j \leq \eta - 1\}$ are standard basis vectors. By the definition of $\overline{A}$ and $\{\mu_j : 1 \leq j \leq \eta - 1\}$, we have that;

$$\overline{A}^t(\overline{1}) = (\eta\mu_1 - \eta, \ldots, \eta\mu_{\eta-1} - \eta^{\eta-1})$$

Rearranging this, it follows that;

$$\overline{\mu} = \frac{\overline{A}^t(\overline{1})}{\eta} + \overline{\nu}$$

where $\overline{\nu} = (1, \eta, \ldots, \eta^{\eta-2})$. Then, we have;

$$< \overline{\mu}, \overline{A}^{-1}(\overline{e_j}) >$$

$$=< \frac{\overline{A}^t(\overline{1})}{\eta} + \overline{\nu}, \overline{A}^{-1}(\overline{e_j}) >$$

$$=< \frac{\overline{1}}{\eta}, \overline{AA}^{-1}(\overline{e_j}) > + < \overline{\nu}, \overline{A}^{-1}(\overline{e_j}) >$$

$$= \frac{1}{\eta} + < \overline{\nu}, \overline{A}^{-1}(\overline{e_j}) > \ (\dagger\dagger\dagger\dagger)$$

We introduce an extended matrix, defined by;

$$(\overline{A}_{ext})_{ij} = i^j, \text{ for } 1 \leq i, j \leq \eta$$

Then;

$$\overline{A}^t_{ext}(\overline{e_\eta}) = (\eta, \ldots, \eta^\eta)$$

$$\frac{\overline{A}^t_{ext}(\overline{e_\eta})}{\eta} = \overline{\nu}_{ext} \ (******)$$

where $\overline{\nu}_{ext} = (1, \eta, \ldots, \eta^{\eta-1})$. We have that $(\overline{A}^{-1})_{ij} = b_{ij}$, for $1 \leq i, j \leq \eta - 1$, and we let $(\overline{A}^{-1}_{ext}) = c_{ij}$, for $1 \leq i, j \leq \eta$. Then, using

$(*****);$

$$< \overline{\nu}, \overline{A}^{-1}(\overline{e_j}) >$$

$$= {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}b_{ij}$$

$$= {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(b_{ij}-c_{ij}) + {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}c_{ij}$$

$$= {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(b_{ij}-c_{ij}) + {}^*\sum_{i=1}^{\eta}\eta^{i-1}c_{ij} - \eta^{\eta-1}c_{\eta j}$$

$$= {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(b_{ij}-c_{ij}) - \eta^{\eta-1}c_{\eta j} + < \frac{\overline{A}^t_{ext}(\overline{e_\eta})}{\eta}, \overline{A}^{-1}_{ext}(\overline{e_j}) >$$

$$= {}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(b_{ij}-c_{ij}) - \eta^{\eta-1}c_{\eta j} \ (\dagger\dagger\dagger\dagger\dagger)$$

as $\overline{e_\eta}$ is orthogonal to $\overline{e_j}$, for $1 \leq j \leq \eta-1$. Now, as observed in the previous lemma;

$$b_{ij} = (-1)^{i+j*}\sum_{k=i}^{\eta-1}\frac{C_j^k[{}^k_i]}{k!}, \ 1 \leq i, j \leq \eta - 1$$

$$c_{ij} = (-1)^{i+j*}\sum_{k=i}^{\eta}\frac{C_j^k[{}^k_i]}{k!}, \ 1 \leq i, j \leq \eta$$

Therefore;

$$b_{ij} - c_{ij} = (-1)^{i+j+1}\frac{C_j^\eta[{}^\eta_i]}{\eta!}, \ 1 \leq i \leq j \leq \eta - 1$$

$$c_{\eta j} = (-1)^{\eta+j}\frac{C_j^\eta[{}^\eta_\eta]}{\eta!} = (-1)^{\eta+j}\frac{C_j^\eta}{\eta!}, \ 1 \leq j \leq \eta - 1$$

Hence;

$${}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(b_{ij}-c_{ij}) - \eta^{\eta-1}c_{\eta j}$$

$$= ({}^*\sum_{i=1}^{\eta-1}\frac{\eta^{i-1}(-1)^{i+j+1}C_j^\eta[{}^\eta_i]}{\eta!}) + \frac{\eta^{\eta-1}(-1)^{\eta+j+1}C_j^\eta}{\eta!}$$

$$= \frac{(-1)^{j+1}C_j^\eta}{\eta!}(({}^*\sum_{i=1}^{\eta-1}\eta^{i-1}(-1)^i[{}^\eta_i]) + \eta^{\eta-1}(-1)^\eta)$$

$$= \frac{(-1)^{j+1}C_j^\eta}{\eta!}({}^*\sum_{i=1}^{\eta}\eta^{i-1}(-1)^i[{}^\eta_i]) \ (\dagger\dagger\dagger\dagger\dagger\dagger)$$

We have that the falling factorial;

$$(x)_\eta = x(x-1)\ldots(x-\eta+1)$$

$= {}^*\sum_{i=1}^{\eta} S(\eta, i)x^i$

$= {}^*\sum_{i=1}^{\eta}(-1)^{\eta-i}[{}^{\eta}_{i}]x^i$

where the S denote unsigned Stirling numbers of the first kind. Substituting $x = \eta$, we obtain;

$(\eta)_\eta = \eta! = {}^*\sum_{i=1}^{\eta}\eta^i(-1)^{\eta-i}[{}^{\eta}_{i}]$

$= (-1)^\eta\eta^* \sum_{i=1}^{\eta}\eta^{i-1}(-1)^i[{}^{\eta}_{i}]$

Rearranging, this gives that;

${}^*\sum_{i=1}^{\eta}\eta^{i-1}(-1)^i[{}^{\eta}_{i}] = -(\eta - 1)!$ (η *odd*) (†††††††)

Combining (†††††), (††††††), (†††††††), we obtain that;

$< \overline{\nu}, \overline{A}^{-1}(\overline{e}_j) >= \frac{-(\eta-1)!C^{\eta}_{j}(-1)^{j+1}}{\eta!}$

$= \frac{(-1)^j C^{\eta}_{j}}{\eta}$

and, using (††††);

$(\overline{EA}^{-1})_{ij} = \overline{\mu} \centerdot \overline{A}^{-1}(\overline{e}_j)$

$= \frac{1}{\eta}(1 + (-1)^j C^{\eta}_{j})$ $(1 \leq i, j \leq \eta - 1)$

We now compute C from the previous lemma. We have that;

$C = {}^*\sum_{1\leq i,j\leq\eta-1}\mu_i b_{ij}$

where;

$(\overline{EA}^{-1})_{ij} = {}^*\sum_{k=1}^{\eta-1} b_{kj}$

By the previous result, we obtain immediately, that;

$C = {}^*\sum_{j=1}^{\eta-1}\frac{1}{\eta}(1 + (-1)^j C^{\eta}_{j})$

$= \frac{\eta-1}{\eta} + \frac{1}{\eta}{}^*\sum_{j=1}^{\eta-1}(-1)^j C^{\eta}_{j}$

Using symmetry of the binomial coefficients, we have that;

$^*\sum_{j=0}^{\eta}(-1)^j C_j^{\eta} = {}^*\sum_{j=1}^{\eta-1}(-1)^j C_j^{\eta} = 0$

Hence, $C = \frac{\eta-1}{\eta}$, in particular, $(1-C) = \frac{1}{\eta}$ and $\frac{1}{(1-C)} = \eta$. Now, we compute $\overline{A}^{-1}\overline{EA}^{-1}$. We have that;

$(\overline{A}^{-1}\overline{EA}^{-1})_{ij} = d_j$

where;

$d_j = \frac{1}{\eta}(1+(-1)^j C_j^{\eta})\overline{A}^{-1}(\overline{1})$

We have that;

$(\overline{A}^{-1}(\overline{1}))_i = {}^*\sum_{s=1}^{\eta-1} b_{is}$

$= {}^*\sum_{s=1}^{\eta-1}(-1)^{i+s*}\sum_{k=i}^{\eta-1}\frac{C_s^k[{k \atop i}]}{k!}$

$= {}^*\sum_{k=i}^{\eta-1}\frac{(-1)^i[{k \atop i}]}{k!}{}^*\sum_{s=1}^{k}(-1)^s C_s^k$

$= {}^*\sum_{k=i}^{\eta-1}\frac{(-1)^{i+1}[{k \atop i}]}{k!}$

using the fact that $^*\sum_{s=1}^{k} C_s^k = -1$. It follows that;

$(\overline{A}^{-1}\overline{EA}^{-1})_{ij} = \frac{1}{\eta}(1+(-1)^j C_j^{\eta})^*\sum_{k=i}^{\eta-1}\frac{(-1)^{i+1}[{k \atop i}]}{k!}$

and, by the previous result;

$(\frac{\overline{A}^{-1}\overline{EA}^{-1}}{(1-C)})_{ij} = (1+(-1)^j C_j^{\eta})^*\sum_{k=i}^{\eta-1}\frac{(-1)^{i+1}[{k \atop i}]}{k!}$

We can now compute $\frac{\overline{A}^{-1}\overline{EA}^{-1}}{(1-C)}(\overline{y}_{red})$. We have already observed that the weights $\{\Theta(j) : 1 \leq j \leq \eta-1\}$ are symmetric, so $\overline{y}_{red}$ is also symmetric. It follows, by symmetry of the binomial coefficients, that;

$^*\sum_{j=1}^{\eta-1}(-1)^j C_j^{\eta}(\overline{y}_{red})_j = 0$

Moreover, by definition of $\overline{y}_{red}$, Lemmas 0.55 and 0.57, we have;

$^*\sum_{j=1}^{\eta-1}(\overline{y}_{red})_j$

$= \eta^* \sum_{j=1}^{\eta-1} \Theta(j) - (\eta - 1)$

$= \eta \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2\nu} - (\eta - 1)$

$= \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta\nu} - (\eta - 1)$

Therefore;

$(\frac{\overline{A}^{-1}\overline{EA}^{-1}}{(1-C)}(\overline{y}_{red}))_i$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^{i+1}[{k \atop i}]}{k!} (\frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta\nu} - (\eta - 1))$

We now compute $(\overline{A})^{-1}(\overline{y}_{red})_i$. Observe that we can write $\overline{y}_{red} = \overline{z}_{red} - \overline{1}$, where;

$\overline{z}_{red} = (\eta\Theta(1), \ldots, \eta\Theta(\eta - 1))$

We have that;

$(\overline{A}^{-1}\overline{1})_i = {}^* \sum_{j=1}^{\eta-1} b_{ij}$

where;

${}^* \sum_{j=1}^{\eta-1} b_{ij} = {}^* \sum_{j=1}^{\eta-1} {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^{k+j} C_j^k S_k^i}{k!}$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^k S_k^i}{k!} {}^* \sum_{j=1}^{\eta-1} (-1)^j C_j^k$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^k S_k^i}{k!} {}^* \sum_{j=1}^{k} (-1)^j C_j^k$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^{k+1} S_k^i}{k!}$

We have that $S_k^i = (-1)^{k-i}[{k \atop i}]$ where $[{k \atop i}]$ are unsigned Sterling numbers of the first kind. Therefore;

$(\overline{A}^{-1}\overline{1})_i$

$= {}^* \sum_{j=1}^{\eta-1} b_{ij}$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^{k+1}(-1)^{k-i}[{k \atop i}]}{k!}$

$= (-1)^{i+1*}\sum_{k=i}^{\eta-1}\frac{[{k \atop i}]}{k!}$

In a similar way, we have that;

$(\overline{A})^{-1}(\overline{z}_{red})_i$

$= \eta^*\sum_{k=i}^{\eta-1}\frac{(-1)^k S_k^i}{k!}{}^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)$

$= \eta^*\sum_{k=i}^{\eta-1}\frac{(-1)^i[{k \atop i}]}{k!}{}^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)$

and;

$(\overline{A})^{-1}(\overline{y}_{red})_i$

$= {}^*\sum_{k=i}^{\eta-1}\frac{(-1)^i[{k \atop i}]}{k!}(\eta^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)+1)$

$(\frac{\overline{A}^{-1}\overline{E}\overline{A}^{-1}}{(1-C)}+\overline{A}^{-1})(\overline{y}_{red})_i$

$= {}^*\sum_{k=i}^{\eta-1}\frac{(-1)^i[{k \atop i}]}{k!}(\frac{-\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta\nu}+(\eta-1)+\eta^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)+1)$

$= {}^*\sum_{k=i}^{\eta-1}\frac{(-1)^i[{k \atop i}]}{k!}(\frac{-\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta\nu}+\eta+\eta^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j))$ $(+)$

We now compute ${}^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)$. First observe that the transposed weights are given by;

$\Theta(j) = \frac{2\pi^2 i}{\eta\nu}(-1)^{j+1}\frac{{}^*cos(\frac{\pi j}{\eta})}{{}^*sin^2(\frac{\pi j}{\eta})}$ $(1 \leq j \leq \frac{\eta-1}{2})$

$\Theta(j) = \frac{2\pi^2 i}{\eta\nu}(-1)^{j+1-\eta}\frac{{}^*cos(\frac{\pi(j-\eta)}{\eta})}{{}^*sin^2(\frac{\pi(j-\eta)}{\eta})}$

$= \frac{2\pi^2 i}{\eta\nu}(-1)^{j+1}\frac{{}^*cos(\frac{\pi j}{\eta})}{{}^*sin^2(\frac{\pi j}{\eta})}$ $(\frac{\eta+1}{2} \leq j \leq \eta-1)$

$\Theta(\eta) = 1 - \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2\nu}$

Transposing Lemma 0.31, we have that;

${}^*\sum_{j=1}^{k}(-1)^j C_j^k\Theta(j)$

$= \frac{1}{\sqrt{\eta}\nu}{}^*\sum_{j=1}^{k}(-1)^j C_j^k S(j)$

where $\hat{}S = SQ$ and;

$SQ(j) = \frac{-4\pi^2 j^2 i}{\eta}$, $(1 \leq j \leq \frac{\eta-1}{2})$

$SQ(j) = \frac{-4\pi^2 (j-\eta)^2 i}{\eta}$, $(\frac{\eta+1}{2} \leq j \leq \eta - 1)$

$SQ(\eta) = 0$

where the transposed nonstandard Fourier transform and its inverse are defined by;

$\hat{}S(s) = \frac{1}{\sqrt{\eta}}{}^* \sum_{1 \leq j \leq \eta} S(j)^* exp(\frac{-2\pi ijs}{\eta})$

$\check{}S(s) = \frac{1}{\sqrt{\eta}}{}^* \sum_{1 \leq j \leq \eta} S(j)^* exp(\frac{2\pi ijs}{\eta})$

We define M_k by;

$M_k(j) = (-1)^j C^k_j$, $(1 \leq j \leq k)$

$M_k(j) = 0$ otherwise, $(1 \leq j \leq \eta)$

Then;

$\hat{}M_k(s) = \frac{1}{\sqrt{\eta}}{}^* \sum_{1 \leq j \leq k} (-1)^j C^{k*}_j exp(\frac{-2\pi ijs}{\eta})$

$= \frac{1}{\sqrt{\eta}}{}^* \sum_{1 \leq j \leq k} (-1)^j C^k_j (\xi^{-s})^j$

$= \frac{1}{\sqrt{\eta}}((1 - \xi^{-s})^k - 1) = H_k(s)$ $(1 \leq s \leq \eta)$

where ξ is the $*$-primitive root of unity defined by ${}^*exp(\frac{2\pi i}{\eta})$. By the inversion theorem, we have that $M_k = \check{}H_k$ and $S = \check{}SQ$. We then have, using the inversion and convolution theorems, that;

$\frac{1}{\sqrt{\eta}}{}^* \sum_{j=1}^{k} (-1)^j C^k_j S(j)$

$= \frac{1}{\sqrt{\eta}}{}^* \sum_{j=1}^{k} M_k(j) S(j)$

$= \hat{}(M_k S)(\eta)$

$= \hat{}(H_k \check{}SQ)(\eta)$

$= \hat{\breve{(}}(H_k * SQ))(\eta)$

$= (H_k * SQ)(\eta)$

where the convolution is defined by;

$(H_k * SQ)(\eta) = \frac{1}{\sqrt{\eta}}{}^* \sum_{s=1}^{\eta} H_k(\eta - s)SQ(s)$, $(H_k(0) = H_k(\eta))$

$= \frac{1}{\sqrt{\eta}}{}^* \sum_{s=1}^{\frac{\eta-1}{2}} \frac{-4\pi^2 i}{\eta}((\eta - s) - \eta)^2 \frac{1}{\sqrt{\eta}}((1 - \xi^{-s})^k - 1)$

$+ \frac{1}{\sqrt{\eta}}{}^* \sum_{s=\frac{\eta+1}{2}}^{\eta-1} \frac{-4\pi^2 i}{\eta}(\eta - s)^2 \frac{1}{\sqrt{\eta}}((1 - \xi^{-s})^k - 1)$

$= \frac{-4\pi^2 i}{\eta^2}({}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2((1-\xi^{-s})^k - 1) + {}^* \sum_{s=\frac{\eta+1}{2}}{}^{\eta-1} (\eta - s)^2((1-\xi^{-s})^k - 1))$

$= \frac{-4\pi^2 i}{\eta^2}(-2^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2 + {}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2((1 - \xi^{-s})^k + (1 - \xi^s)^k))$

$= \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2} - \frac{8\pi^2 i}{\eta^2} Re({}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2((1 - \xi^s)^k)$

It follows that;

${}^* \sum_{j=1}^{k} (-1)^j C_j^k \Theta(j)$

$= \frac{\pi^2 i(\eta-1)\eta(\eta+1)}{3\eta^2 \nu} - \frac{8\pi^2 i}{\eta^2 \nu} Re({}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2((1 - \xi^s)^k)$

Substituting into (+), we obtain that;

$(\frac{\overline{A}^{-1}\overline{EA}^{-1}}{(1-C)} + \overline{A}^{-1})(\overline{y}_{red})_i$

$= {}^* \sum_{k=i}^{\eta-1} \frac{(-1)^i [{k \atop i}]}{k!}(\eta - \frac{8\pi^2 i}{\eta\nu} Re({}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^2((1 - \xi^s)^k)))$

$= \eta^* \sum_{k=i}^{\eta-1} \frac{(-1)^i [{k \atop i}]}{k!} - \frac{8\pi^2 i}{\eta\nu} Re({}^* \sum_{s=1}^{\frac{\eta-1}{2}} s^{2*} \sum_{k=i}^{\eta-1} \frac{(-1)^i [{k \atop i}](1-\xi^s)^k}{k!})$ (++)

....

□

Lemma 0.63.

Proof. We fix an infinitesimal $\delta \simeq 0$, with $\delta > 0$. We define a new process $X^{\delta,\nu} : \overline{\Omega}_\kappa \times \mathcal{T}_\kappa \to {}^*\mathcal{C}$ by;

$X^{\delta,\nu}(\omega, 0) = 1$

$X^{\delta,\nu}(\omega,\frac{i}{\nu})=\eta^i\prod_{1\leq j\leq i}\Delta(\omega_j)$, $1\leq i\leq\kappa$

where;

$\Delta(s)=\Theta(s)+\frac{\delta}{\nu}$, $1\leq s\leq\eta-1$

$\Delta(\eta)=\Theta(\eta)-\frac{(\eta-1)\delta}{\nu}$

Observe that $X^{\delta,\nu}$ defines a martingale, as, by the definition and Lemma 0.57;

${}^*\sum_{1\leq s\leq\eta}\Delta(s)={}^*\sum_{1\leq s\leq\eta}\Theta(s)=1$

As in the proof of Lemma 0.60, we have that;

$dX^{\delta,\nu}_t=X^{\delta,\nu}_{t+\frac{1}{\nu}}-X^{\delta,\nu}_t=X^{\delta,\nu}_t inc_{t+\frac{1}{\nu}}$, $t\in\mathcal{T}_\kappa\setminus\{\frac{\kappa}{\nu}\}$, $(\dagger)$

where $inc:\overline{\Omega}_\kappa\times\mathcal{T}_\kappa\setminus[0,\frac{1}{\nu})\rightarrow{}^*\mathcal{C}$ is defined by;

$inc(\omega,\frac{i}{\nu})=\eta\Delta(\omega_i)-1=Inc(\omega_i)$

where $Inc:{}^*\mathcal{Z}\cap[1,\eta]\rightarrow{}^*\mathcal{C}$ is defined in the obvious way. For $1\leq j\leq\eta-1$, we have that;

$Inc(j)=\eta\Delta(j)-1$

$=\eta(\Theta(j)+\frac{\delta}{\nu})-1$

$=\frac{i\eta\lambda_j}{\nu}+(\frac{\eta\delta}{\nu}-1)$, $\lambda_j\in{}^*\mathcal{R}$

Then, using the calculation of the weights in Lemma 0.62;

$|Inc(j)|<1$ iff $\frac{\eta^2\lambda_j^2}{\nu^2}+(\frac{\eta\delta}{\nu}-1)^2<1$

iff $\nu>\frac{\lambda_j^2\eta}{2\delta}+\frac{\delta\eta}{2}$

iff $\nu>\frac{2\pi^4}{\eta\delta}\frac{{}^*cos^2(\frac{\pi j}{\eta})}{{}^*sin^4(\frac{\pi j}{\eta})}+\frac{\delta\eta}{2}$, $(1\leq j\leq\eta-1)$ $(*)$

Similarly;

$Inc(\eta) = \eta\Delta(\eta) - 1$

$= \eta(\Theta(\eta) - \frac{(\eta-1)\delta}{\nu}) - 1$

$= \eta(1 - \frac{ic}{\nu}) - \frac{\eta(\eta-1)\delta}{\nu} - 1$

$= (\eta - 1)(1 - \frac{\eta\delta}{\nu}) - \frac{ic\eta}{\nu}$, $c \in {}^*\mathcal{R}$

and;

$|Inc(\eta)| > 1$ iff $(\eta - 1)^2(1 - \frac{\eta\delta}{\nu})^2 + \frac{c^2\eta^2}{\nu^2} > 1$

if $(\eta - 1)^2(1 - \frac{\eta\delta}{\nu})^2 > 1$

if $\nu > \frac{(\eta-1)\eta\delta}{(\eta-2)}$, $(**)$

Using the simple inequality;

$\frac{{}^*cos^2(\frac{\pi j}{\eta})}{{}^*sin^4(\frac{\pi j}{\eta})} \leq \frac{\eta^4}{\pi^4}$, $1 \leq j \leq \eta - 1$

we can satisfy both conditions $(*), (**)$, when $\nu > \frac{3\eta^3}{\delta}$. Using the fact that $X^{\delta,\nu}$ defines a martingale, Lemmas 0.60 and 0.61, we have that;

$dX_t^{\delta,\nu} = X_t^{\delta,\nu} {}^*\sum_{j=1}^{\eta-1} s_j \overline{\omega_{t+1}^j}$, $t \in \overline{T}_\kappa \setminus \{\frac{\kappa}{\nu}\}$, $(\dagger\dagger)$

where;

$s_j = ((\frac{\overline{A}^{-1}\overline{E}}{1-C} + Id)\overline{A}^{-1}(Inc))_j$

and Inc is the vector $(Inc(1), \ldots, Inc(\eta - 1))$. Comparing $(\dagger)$, with $(\dagger\dagger)$, and using the fact that $X^{\delta,\nu}$ is never zero, we must have that;

$inc_t = {}^*\sum_{j=1}^{\eta-1} s_j \overline{\omega_t^j}$, for $t \in \overline{T}_\kappa \setminus [0, \frac{1}{\nu})$

As $|Inc(j)| < 1$, for $1 \leq j \leq \eta - 1$, we have, for $\omega \in \overline{\Omega}_\kappa$, with $\omega_{t+\frac{1}{\nu}} \neq \eta$, that;

$d^*ln(X_t^{\delta,\nu})(\omega) = ({}^*ln(X_t^{\delta,\nu} + dX_t^{\delta,\nu}) - {}^*ln(X_t^{\delta,\nu}))(\omega)$

$= ({}^{*}ln(1+\frac{dX_{t}^{\delta,\nu}}{X_{t}^{\delta,\nu}}))(\omega)$

$= ({}^{*}ln(1+inc_{t+\frac{1}{\nu}}))(\omega)$

$= ({}^{*}\sum_{k\in{}^{*}\mathcal{Z}_{\geq 1}}\frac{(-1)^{k+1}inc_{t+\frac{1}{\nu}}^{k}}{k})(\omega)$

$= ({}^{*}\sum_{k\in{}^{*}\mathcal{Z}_{\geq 1}}\frac{(-1)^{k+1}({}^{*}\sum_{j=1}^{\eta-1}s_{j}\overline{\omega_{t+1}^{j}})^{k}}{k})(\omega)$ $(***)$

As $|Inc(\eta)|>1$, we have, for $\omega\in\overline{\Omega}_{\kappa}$, with $\omega_{t+\frac{1}{\nu}}=\eta$, that;

$d^{*}ln(X_{t}^{\delta,\nu})(\omega) = ({}^{*}ln(dX_{t}^{\delta,\nu}(1+\frac{X_{t}^{\delta,\nu}}{dX_{t}^{\delta,\nu}}))-{}^{*}ln(X_{t}^{\delta,\nu}))(\omega)$

$= ({}^{*}ln(dX_{t}^{\delta,\nu})+{}^{*}ln(1+\frac{1}{inc_{t+\frac{1}{\nu}}})-{}^{*}ln(X_{t}^{\delta,\nu}))(\omega)$

$= ({}^{*}ln(X_{t}^{\delta,\nu})+{}^{*}ln(inc_{t+\frac{1}{\nu}})+{}^{*}ln(1+\frac{1}{inc_{t+\frac{1}{\nu}}})-{}^{*}ln(X_{t}^{\delta,\nu}))(\omega)$

$= ({}^{*}ln(inc_{t+\frac{1}{\nu}})+{}^{*}ln(1+\frac{1}{inc_{t+\frac{1}{\nu}}}))(\omega)$

$= {}^{*}ln(1+inc_{t+\frac{1}{\nu}})(\omega)$ $(****)$

We denote by K^{δ}, as in Lemma 0.31, the corresponding convolution factor to the new set of weights defined by Δ. Transposing the weights, using Definition 0.56 and using the relation with the convolution factor, given in Lemma 0.55, we see that;

$K^{\delta}(\frac{j}{\sqrt{\eta}})=K(\frac{j}{\sqrt{\eta}})+\sqrt{\eta}\delta,\ 1\leq|j|\leq\frac{(\eta-1)}{2}$

$K^{\delta}(0)=K(0)-\sqrt{\eta}(\eta-1)\delta$

We consider the perturbed nonstandard convolution equation $\frac{\partial F}{\partial t}=K^{\delta}*F$, with initial condition $f\in V(\overline{\mathcal{R}_{\eta}})$, and nonstandard propagator $W^{\delta}(t,x,y)$. We have that;

$\hat{K^{\delta}}(y)=\hat{K}(y)+\hat{E}(y)$

where;

$E(0)=-\sqrt{\eta}(\eta-1)\delta,\ E(\frac{i}{\sqrt{\eta}})=\sqrt{\eta}\delta,\ 1\leq|i|\leq\frac{(\eta-1)}{2}$

Computing the nonstandard Fourier transform, we have that;

$\hat{E}(0)=0$

$\hat{E}(\frac{j}{\sqrt{\eta}})=-(\eta-1)\delta+\delta^{*}\sum_{-\frac{(\eta-1)}{2}\leq k\leq\frac{(\eta-1)}{2},k\neq 0}\xi^{jk}$

$=-(\eta-1)\delta-\delta=-\eta\delta$, $(\xi={}^{*}exp(-\frac{2\pi i}{\eta}))$

We now go through the Lemmas, Theorems and Definitions from 0.32 to 0.46, to see how the averaged propagators $\{W_{m}^{\delta}(t,x,y):m\in\mathcal{Z}_{>0}\}$ are related to the standard propagator for the original Schrodinger equation. Lemma 0.32 is the same. In Lemma 0.33, if

$|-4\pi^{2}i\frac{[\sqrt{\eta}y]^{2}}{\eta}-\eta\delta|\leq\frac{\nu^{\frac{1}{2}}}{4}$ (†††)

is satisfied on $\overline{\mathcal{R}_{\eta}}$, and $\delta\leq\frac{1}{\eta^{2}}$, (+), then;

$(1-4\pi^{2}i\frac{[\sqrt{\eta}y]^{2}}{\eta\nu}-\frac{\eta\delta}{\nu})^{\nu}\simeq{}^{*}exp(-\eta\delta)^{*}exp(-4\pi^{2}i\frac{[\sqrt{\eta}y]^{2}}{\eta})$

$\simeq{}^{*}exp(-4\pi^{2}i\frac{[\sqrt{\eta}y]^{2}}{\eta})$

A simple calculation shows that we can satisfy (†††), with the choice $\nu\geq 16\pi^{4}\eta^{2}$, (++), then the rest of the claim in Lemma 0.33 goes through, that is, if $g\in S(\mathcal{R})$ and $G\in S(T)$, are as in the Lemma, and, $\{g_{\eta},G_{\eta}^{\delta}\}$ solve the nonstandard perturbed convolution equation, then;

${}^{\circ}G_{\eta}^{\delta}(x,t)=G({}^{\circ}x,{}^{\circ}t)$, for $x\in\overline{\mathcal{R}_{\eta}}$, $t\in\mathcal{T}_{\nu}$, finite

Lemma 0.34 is the same and we can also obtain the claim of Theorem 0.35, for the perturbed equation, by noting the factorisation;

$(1+\frac{icy^{2}}{n}+\frac{d}{n})$

$=(1+\frac{d}{n})^{n}(1+\frac{iey^{2}}{n})^{n}$

where $\{c,d,e\}\subset\mathcal{R}$, $n\in\mathcal{N}$ and $e=\frac{cn}{n+d}$, together with the fact that, for given $c\in\mathcal{R}$, and $d\in\mathcal{R}_{<0}$, we have a uniform bound on e, independent of n. Lemma 0.36 is the same and Lemmas 0.37 and 0.38 now follow, replacing W by W^{δ}. In the proof of Lemma 0.39, we need a stronger bound, $\nu\geq\eta^{4}$ will suffice. Lemmas 0.40 to 0.46 are the same. We conclude that the averaged nonstandard perturbed propagators specialise to the value of the standard propagator, with the appropriate bounds on ν and δ.

We return to the statistical calculation. Observe that we can now obtain the conditions $(*), (**)$, for $\delta = \frac{1}{\eta^2}$, when $\nu > 3\eta^5$, this also guarantees the specialisation condition for the propagators. Now consider the processes, $X^{\delta,\nu,W^\delta,x,y} : \overline{\Omega}_\kappa \times \mathcal{T}_\kappa \rightarrow {}^*\mathcal{C}$, defined by;

$$X^{\delta,\nu,W^\delta,x,y}(\omega, 0) = W^\delta(\tfrac{\kappa}{\nu}, x, y)$$

$$X^{\delta,\nu}(\omega, \tfrac{i}{\nu}) = \eta^i \prod_{1\leq j\leq i} \Delta(\omega_j) W^\delta(\tfrac{\kappa-i}{\nu}, x + \tfrac{1}{\sqrt{\eta}}{}^* \textstyle\sum_{1\leq k\leq i}\omega_k, y),\ 1 \leq i \leq \kappa$$

By the proof in Theorem 0.52, and the fact, from Lemma 0.37, that $W^\delta(t, x, y) = F_{\delta_y}(t, x)$, we have that the processes $X^{\delta,\nu,W^\delta,x,y}$, for $\{x, y\} \subset \overline{\mathcal{R}_\eta}$ define martingales. It follows, abbreviating ${}^*\sum_{1\leq k\leq\kappa}\omega_k$ to $S(\omega)$, that;

$$W_{\frac{\kappa}{\nu}}(x, y) = E_\eta(X^{\delta,\nu,W^\delta,x,y}_{\frac{\kappa}{\nu}})$$

$$= \tfrac{\sqrt{\eta}}{\eta^\kappa}{}^* \textstyle\sum_{\omega\in\overline{\Omega}_\kappa, S(\omega)=[\sqrt{\eta}y]-[\sqrt{\eta}x], mod\eta}\eta^\kappa \prod_{1\leq j\leq\kappa} \Delta(\omega_j)$$

$$= \tfrac{\sqrt{\eta}}{\eta^\kappa}{}^* \textstyle\sum_{\omega\in\overline{\Omega}_\kappa, S(\omega)=[\sqrt{\eta}y]-[\sqrt{\eta}x], mod\eta}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega)$$

We proceed by calculating;

$${}^* \textstyle\sum_{\omega\in\overline{\Omega}_\kappa, S(\omega)=z, mod\eta}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega),\ z \in {}^*\mathcal{Z} \cap [1, \eta]$$

Let $d(\omega) = \{j : \omega_j = \eta, 1 \leq j \leq \kappa\}$ and let $e(\omega) = {}^*Card(d(\omega))$. Then, for $\omega \in \overline{\Omega}_\kappa$, with $e(\omega) = \lambda$, we have, using $(****)$ that;

$${}^*ln(X^{\delta,\nu}_{\frac{\kappa}{\nu}})(\omega) = {}^* \textstyle\sum_{0\leq t\leq\kappa-1} d^*ln(X^{\delta,\nu}_{\frac{t}{\nu}})(\omega)$$

$$= \lambda^*ln(1 + inc_{\frac{t_0+1}{\nu}})(\omega) + {}^* \textstyle\sum_{t+1\notin d(\omega)} d^*ln(X^{\delta,\nu}_t)(\omega),\ (t_0 + 1 \in d(\omega))$$

$$= \lambda^*ln(\eta - \tfrac{(\eta-1)\eta\delta}{\nu} - \tfrac{ic\eta}{\nu}) + {}^* \textstyle\sum_{t+1\notin d(\omega)} d^*ln(X^{\delta,\nu}_t)(\omega)$$

and, hence;

$$X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega) = (\eta - \tfrac{(\eta-1)\eta\delta}{\nu} - \tfrac{ic\eta}{\nu})^{\lambda*}exp({}^* \textstyle\sum_{t+1\notin d(\omega)} d^*ln(X^{\delta,\nu}_t))(\omega),\ (*****)$$

We observe that $S(\omega)=z$, mod η, iff $R(\omega)={}^*\sum_{t\notin d(\omega)}\omega_t=z$, mod η, and, that, for $V_0\cup V_1\subset {}^*\mathcal{Z}\cap[1,\kappa]$, with ${}^*Card(V_0)={}^*Card(V_1)$, we have that;

$$ {}^*\sum_{\omega\in\overline{\Omega}_\kappa,S(\omega)=z,mod\eta,d(\omega)=V_0}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega)={}^*\sum_{\omega\in\overline{\Omega}_\kappa,S(\omega)=z,mod\eta,d(\omega)=V_1}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega) $$

It follows, using $(***)$ and $(*****)$, that;

$$ {}^*\sum_{\omega\in\overline{\Omega}_\kappa,S(\omega)=z,mod\eta}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega) $$

$$ ={}^*\sum_{\lambda=0}^{\kappa}C^{\kappa}_{\lambda}{}^*\sum_{S(\omega)=z,mod\eta,d(\omega)={}^*\mathcal{Z}\cap(\kappa-\lambda,\kappa]}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega) $$

$$ ={}^*\sum_{\lambda=0}^{\kappa}C^{\kappa}_{\lambda}(\eta-\tfrac{(\eta-1)\eta\delta}{\nu}-\tfrac{ic\eta}{\nu})^{\lambda}{}^*\sum_{S(\omega)=z,mod\eta,d(\omega)={}^*\mathcal{Z}\cap(\kappa-\lambda,\kappa]}{}^*exp({}^*\sum_{t+1\in[1,\kappa-\lambda]}d^*ln(X^{\delta,\nu}_t))(\omega) $$

$$ ={}^*\sum_{\lambda=0}^{\kappa}C^{\kappa}_{\lambda}(\eta-\tfrac{(\eta-1)\eta\delta}{\nu}-\tfrac{ic\eta}{\nu})^{\lambda}{}^*\sum_{\omega\in\overline{\Omega}^{\eta-1}_{\kappa-\lambda},S(\omega)=z,mod\eta}{}^*exp({}^*\sum_{t+1\in[1,\kappa-\lambda]}d^*ln(X^{\delta,\nu}_t))(\omega) $$

$$ ={}^*\sum_{\lambda=0}^{\kappa}C^{\kappa}_{\lambda}(\eta-\tfrac{(\eta-1)\eta\delta}{\nu}-\tfrac{ic\eta}{\nu})^{\lambda}{}^*\sum_{\omega\in\overline{\Omega}^{\eta-1}_{\kappa-\lambda},S(\omega)=z,mod\eta}{}^*exp({}^*\sum_{t=1}^{\kappa-\lambda}({}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}\tfrac{(-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_j\overline{\omega^j_t})^k}{k}))(\omega $$

where we have naturally identified;

$$ \{\omega\in\overline{\Omega}_\kappa:S(\omega)=z,d(\omega)={}^*\mathcal{Z}\cap(\kappa-\lambda,\kappa]\} $$

with $\{\omega\in\overline{\Omega}^{\eta-1}_{\kappa-\lambda}:S(\omega)=z\}$, for

$$ \overline{\Omega}^{\eta-1}_{\kappa-\lambda}=\{\omega:\omega(i)\in{}^*\mathcal{Z}\cap[1,\eta-1],1\leq i\leq\kappa-\lambda\} $$

using the obvious restriction of the operator S and the relevant random variables. We simplify the term;

$$ {}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}\frac{(-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_j\overline{\omega^j_t})^k}{k} $$

Using the definition of s_j and Inc above, we can write;

$$ s_j=s_{1,j}+is_{2,j} $$

where;

$$ s_{1,j}=((\tfrac{\overline{A}^{-1}\overline{E}}{1-C}+Id)\overline{A}^{-1}(Inc_1))_j $$

$$ s_{2,j}=((\tfrac{\overline{A}^{-1}\overline{E}}{1-C}+Id)\overline{A}^{-1}(Inc_2))_j $$

and Inc_1, Inc_2 are the vectors of length $\eta-1$, given by $(\frac{\delta\eta}{\nu}-1,\ldots,\frac{\delta\eta}{\nu}-1)$ and $(\frac{\eta\lambda_1}{\nu},\ldots,\frac{\eta\lambda_{\eta-1}}{\nu})$. Using the binomial theorem, and the absolute convergence of the series, we have, for $\omega\in\overline{\Omega}_\kappa$, with $\omega_t\neq\eta$, that;

$$^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}\frac{(-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_j\overline{\omega_t^j})^k}{k}(\omega)$$

$$={}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}(\frac{(-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^k}{k}+\frac{(-1)^{k+1*}\sum_{s=1}^{k}C_s^k({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-s}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j})^s}{k})(\omega)$$

$$={}^*ln(1+\frac{\delta\eta}{\nu}-1)+{}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}(\frac{(-1)^{k+1*}\sum_{s=1}^{k}C_s^k({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-s}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j})^s}{k})(\omega)$$

$$={}^*ln(\frac{\delta\eta}{\nu})+{}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}(\frac{(-1)^{k+1*}\sum_{s=1}^{k}C_s^k({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-s}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j})^s}{k})(\omega)$$

It follows that, for $\omega\in\overline{\Omega}_{\kappa-\lambda}^{\eta-1}$;

$$^*exp({}^*\sum_{t=1}^{\kappa-\lambda}({}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}\frac{(-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_j\overline{\omega_t^j})^k}{k}))(\omega)$$

$$=(\frac{\delta\eta}{\nu})^{\kappa-\lambda*}exp({}^*\sum_{t=1}^{\kappa-\lambda*}\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}(\frac{(-1)^{k+1*}\sum_{s=1}^{k}C_s^k({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-s}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j})^s}{k}))(\omega)$$

$$=(\frac{\delta\eta}{\nu})^{\kappa-\lambda*}exp(R_{\kappa-\lambda})(\omega)$$

where we have abbreviated the term on the right with the random variable $R_{\kappa-\lambda}$. It follows that;

$$\frac{\sqrt{\eta}}{\eta^\kappa}{}^*\sum_{\omega\in\overline{\Omega}_\kappa,S(\omega)=z,mod\eta}X^{\delta,\nu}_{\frac{\kappa}{\nu}}(\omega)$$

$$=\frac{\sqrt{\eta}}{\eta^\kappa}{}^*\sum_{\lambda=0}^{\kappa}C_\lambda^\kappa(\eta-\frac{(\eta-1)\eta\delta}{\nu}-\frac{ic\eta}{\nu})^\lambda(\frac{\delta\eta}{\nu})^{\kappa-\lambda*}\sum_{\omega\in\overline{\Omega}_{\kappa-\lambda}^{\eta-1},S(\omega)=z,mod\eta}{}^*exp(R_{\kappa-\lambda})(\omega)$$

$$=\sqrt{\eta}^*\sum_{\lambda=0}^{\kappa}C_\lambda^\kappa(1-\frac{(\eta-1)\delta}{\nu}-\frac{ic}{\nu})^\lambda(\frac{\delta}{\nu})^{\kappa-\lambda*}\sum_{\omega\in\overline{\Omega}_{\kappa-\lambda}^{\eta-1},S(\omega)=z,mod\eta}{}^*exp(R_{\kappa-\lambda})(\omega)$$

$$=\sqrt{\eta}^*\sum_{\lambda=0}^{\kappa}C_\lambda^\kappa(1-\frac{(\eta-1)\delta}{\nu}-\frac{ic}{\nu})^\lambda(\frac{\delta}{\nu})^{\kappa-\lambda}(\eta-1)^{\kappa-\lambda}E_{\eta-1}^{\kappa-\lambda}(1_z^{\kappa-\lambda*}exp(R_{\kappa-\lambda}))$$

$$=\sqrt{\eta}^*\sum_{\lambda=0}^{\kappa}C_\lambda^\kappa(1-\frac{(\eta-1)\delta}{\nu}-\frac{ic}{\nu})^\lambda(\frac{\delta(\eta-1)}{\nu})^{\kappa-\lambda}E_{\eta-1}^{\kappa-\lambda}(1_z^{\kappa-\lambda*}exp(R_{\kappa-\lambda}))$$

where, $E_{\eta-1}^{\kappa-\lambda}$ refers to the nonstandard expectation in the probability space $\overline{\Omega}_{\kappa-\lambda}^{\eta-1}$ and $1_z^{\kappa-\lambda}$ denotes the indicator function, supported on $\{\omega\in\overline{\Omega}_{\kappa-\lambda}^{\eta-1},S(\omega)=z,mod\eta\}$.

Our aim now is to compute this expectation, by approximating the random variables $R_{\kappa-\lambda}$, using the central limit theorem. We observe that $s_{2,j}=\frac{\eta t_{2,j}}{\nu}$, where;

$$t_{2,j}=((\frac{\overline{A}^{-1}\overline{E}}{1-C}+Id)\overline{A}^{-1}(Inc_2'))_j$$

and Inc_2' is the vector of length $\eta-1$ given by $(\lambda_1,\ldots,\lambda_{\eta-1})$. We can see that $t_{2,j}$ only depends on η. We approximate;

$$R^1_{\kappa-\lambda}={}^*\sum_{t=1}^{\kappa-\lambda}{}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}(\frac{(-1)^{k+1}{}^*\sum_{s=1}^{1}C^k_s({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-s}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j})^s}{k})$$

$$={}^*\sum_{t=1}^{\kappa-\lambda}{}^*\sum_{k\in{}^*\mathcal{Z}_{\geq 1}}((-1)^{k+1}({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})^{k-1}({}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j}))$$

$$={}^*\sum_{t=1}^{\kappa-\lambda}\frac{1}{1+({}^*\sum_{j=1}^{\eta-1}s_{1,j}\overline{\omega_t^j})}{}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j}$$

$$={}^*\sum_{t=1}^{\kappa-\lambda}\frac{1}{1+(\frac{\delta\eta}{\nu}-1)}{}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j}$$

$$={}^*\sum_{t=1}^{\kappa-\lambda}\frac{\nu}{\delta\eta}{}^*\sum_{j=1}^{\eta-1}is_{2,j}\overline{\omega_t^j}$$

$$={}^*\sum_{t=1}^{\kappa-\lambda}{}^*\sum_{j=1}^{\eta-1}i\frac{t_{2,j}}{\delta}\overline{\omega_t^j}$$

......

□

Definition 0.64. *We fix $c\in{}^*\mathcal{N}$, with c odd and infinite. We let $\overline{\mathcal{R}_{\eta c}}$ be as in Definition 0.3, replacing η by ηc. Let $f\in S(\mathcal{R})$, then we define the contraction $f^{\sqrt{c}}\in V(\overline{\mathcal{R}_{\eta c}})$ by;*

$$f^{\sqrt{c}}(\tfrac{j}{\sqrt{\eta c}})={}^*f(\tfrac{j}{\sqrt{\eta}}),\ -\tfrac{(\eta c-1)}{2}\leq j\leq\tfrac{(\eta c-1)}{2}$$

Let $f\in V(\overline{\mathcal{R}_\eta})$, then we define the extension $f_{\sqrt{c}}\in V(\overline{\mathcal{R}_{\eta c}})$ by;

$$f_{\sqrt{c}}(x)=f(\tfrac{x}{\sqrt{c}}),\ \textit{for } x\in\overline{\mathcal{R}_{\eta c}}\setminus[-\tfrac{(\eta c-1)}{2\sqrt{\eta c}},-\tfrac{(\eta c-c)}{2\sqrt{\eta c}})$$

$$f_{\sqrt{c}}(x)=f(-\tfrac{(\eta-1)}{2\sqrt{\eta}}),\ \textit{for } x\in[-\tfrac{(\eta c-1)}{2\sqrt{\eta c}},-\tfrac{(\eta c-c)}{2\sqrt{\eta c}})$$

Remarks 0.65. *We remark that $f_{\sqrt{c}}$ is well defined and measurable. For suppose that;*

$$-\tfrac{(\eta c-c)}{2\sqrt{\eta c}}\leq x<\tfrac{(\eta c+1)}{2\sqrt{\eta c}}$$

then;

$-\frac{(\eta-1)}{2\sqrt{\eta}} \leq \frac{x}{\sqrt{c}} < \frac{(\eta+\frac{1}{c})}{2\sqrt{\eta}} < \frac{(\eta+1)}{2\sqrt{\eta}}$

If $-\frac{(\eta c-c)}{2} \leq j \leq \frac{(\eta c-1)}{2}$, *then the interval;*

$\frac{1}{\sqrt{c}}[\frac{j}{\sqrt{\eta c}}, \frac{j+1}{\sqrt{\eta c}}) = [\frac{j}{c\sqrt{\eta}}, \frac{j+1}{c\sqrt{\eta}})$

can be written in the form;

$\frac{r}{\sqrt{\eta}} + [\frac{a}{\sqrt{\eta}}, \frac{b}{\sqrt{\eta}}),\ 0 \leq a \leq b \leq 1$

$\subset [\frac{r}{\sqrt{\eta}}, \frac{r+1}{\sqrt{\eta}})$

where $r \geq -\frac{(\eta-1)}{2}$, $j = rc + d$, $0 \leq d < c$, $a = \frac{d}{c}$, $b = \frac{d+1}{c}$.

Lemma 0.66. *Let* $f \in S(\mathcal{R})$, *then;*

$\sqrt{c}F_{\eta c}(f^{\sqrt{c}}) \simeq (F_\eta(f_\eta))_{\sqrt{c}}$ *on* $\overline{\mathcal{R}_{\eta c}}$

where we have denoted the discrete Fourier transforms on $\{\overline{\mathcal{R}}_\eta, \overline{\mathcal{R}}_{\eta c}\}$, *by* $\{F_\eta, F_{\eta c}\}$ *respectively. Moreover, there exists a constant* $H \in \mathcal{R}$, *such that;*

$|\sqrt{c}F_{\eta c}(f^{\sqrt{c}})(y) - (F_\eta(f_\eta))_{\sqrt{c}}(y)| \leq \frac{H}{\sqrt{\eta}}$, $(y \in \overline{\mathcal{R}_{\eta c}})$

Proof. We have, using the definition of $F_{\eta c}$ and $f \in S(\mathcal{R})$, that, for $-\frac{(\eta c-1)}{2} \leq m \leq \frac{\eta c-1}{2}$;

$\sqrt{c}F_{\eta c}(f^{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})$

$= \sqrt{c}\frac{1}{\sqrt{\eta c}}{}^*\sum_{-\frac{(\eta c-1)}{2} \leq j \leq \frac{\eta c-1}{2}} f^{\sqrt{c}}(\frac{j}{\sqrt{\eta c}})exp_{\eta c}(-2\pi i(\frac{m}{\sqrt{\eta c}})(\frac{j}{\sqrt{\eta c}}))$

$= \frac{1}{\sqrt{\eta}}{}^*\sum_{-\frac{(\eta c-1)}{2} \leq j \leq \frac{\eta c-1}{2}} {}^*f(\frac{j}{\sqrt{\eta}}){}^*exp(-2\pi i\frac{mj}{\eta c})$

$\simeq \frac{1}{\sqrt{\eta}}{}^*\sum_{-\frac{(\eta-1)}{2} \leq j \leq \frac{\eta-1}{2}} {}^*f(\frac{j}{\sqrt{\eta}}){}^*exp(-2\pi i\frac{mj}{\eta c}) = L_1$, $(\dagger)$

Similarly, using the definition of F_η, we have, for $-\frac{(\eta c-(c+2))}{2} \leq m \leq \frac{\eta c-1}{2}$;

$(F_\eta(f_\eta))_{\sqrt{c}}(\frac{m}{\sqrt{\eta c}}) = F_\eta(f_\eta)(\frac{m}{c\sqrt{\eta}})$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{\eta-1}{2}}{}^{*}f(\frac{j}{\sqrt{\eta}})exp_{\eta}(-2\pi i(\frac{m}{c\sqrt{\eta}})(\frac{j}{\sqrt{\eta}}))$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{\eta-1}{2}}{}^{*}f(\frac{j}{\sqrt{\eta}})^{*}exp(-2\pi i(\frac{[\frac{m}{c}]}{\sqrt{\eta}})(\frac{j}{\sqrt{\eta}})) = L_2$

Transferring the result of the Mean Value Theorem on $\mathcal{R}$, and, using the fact that, for $f \in S(\mathcal{R})$, there exists $G \in \mathcal{R}_{>0}$, such that $|f_\eta(x)| \leq \frac{G}{|x_\eta|^3}$, for $x \in \overline{\mathcal{R}_\eta}$, see the method of Lemma 0.21, we have that;

$|L_1-L_2| \leq \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{\eta-1}{2}}|{}^{*}f(\frac{j}{\sqrt{\eta}})||{}^{*}exp(-2\pi i(\frac{(\frac{m}{c})}{\sqrt{\eta}})(\frac{j}{\sqrt{\eta}}))-{}^{*}exp(-2\pi i(\frac{[\frac{m}{c}]}{\sqrt{\eta}})(\frac{j}{\sqrt{\eta}}))|$

$\leq \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq\frac{\eta-1}{2}}|{}^{*}f(\frac{j}{\sqrt{\eta}})|E\frac{|-2\pi i(\frac{j}{\sqrt{\eta}})|}{\sqrt{\eta}}$ $(E \in \mathcal{R}_{>0})$

$= \frac{2\pi E}{\sqrt{\eta}}{}^{*}\sum_{0\leq|j|\leq[\sqrt{\eta}]}|{}^{*}f(\frac{j}{\sqrt{\eta}})|\frac{|(\frac{j}{\sqrt{\eta}})|}{\sqrt{\eta}} + \frac{2\pi E}{\sqrt{\eta}}{}^{*}\sum_{[\sqrt{\eta}]<|j|\leq\frac{\eta-1}{2}}|{}^{*}f(\frac{j}{\sqrt{\eta}})|\frac{|(\frac{j}{\sqrt{\eta}})|}{\sqrt{\eta}}$

$\leq \frac{2\pi EF}{\eta^{\frac{3}{2}}}{}^{*}\sum_{0\leq|j|\leq[\sqrt{\eta}]}|j|+\frac{2\pi E}{\sqrt{\eta}}{}^{*}\sum_{[\sqrt{\eta}]<|j|\leq\frac{\eta-1}{2}}\frac{G}{(|\frac{j}{\sqrt{\eta}}|)^3}\frac{|(\frac{j}{\sqrt{\eta}})|}{\sqrt{\eta}}$ $(\{F,G\} \subset \mathcal{R}_{>0})$

$\leq \frac{4\pi EF}{\eta^{\frac{3}{2}}}[\sqrt{\eta}][\sqrt{\eta}+1] + \frac{2\pi EG}{\sqrt{\eta}}{}^{*}\sum_{[\sqrt{\eta}]<|j|\leq\frac{\eta-1}{2}}\frac{1}{(|\frac{j}{\sqrt{\eta}}|)^2}\frac{1}{\sqrt{\eta}}$

$\leq \frac{4\pi EF}{\eta^{\frac{3}{2}}}[\sqrt{\eta}][\sqrt{\eta}+1] + \frac{2}{\sqrt{\eta}}(2\pi EG)(\frac{\sqrt{\eta}}{[\sqrt{\eta}]} - \frac{2\sqrt{\eta}}{\eta-1}) \simeq 0$, $(\dagger\dagger)$

We also have to check the calculation for the endpoints $-\frac{(\eta c-1)}{2} \leq m \leq -\frac{(\eta c-c)}{2}$, the details are left to the reader. For the second part, using the calculation $(\dagger\dagger)$, we have that, $|L_1 - L_2| \leq \frac{H_1}{\sqrt{\eta}}$, for some $H_1 \in \mathcal{R}_{>0}$. Considering the step $(\dagger)$, we can clearly find $H_2 \in \mathcal{R}_{>0}$, such that;

$|\sqrt{c}F_{\eta c}(f^{\sqrt{c}})(y) - L_1| \leq \frac{H_2}{\eta}$, $y \in \overline{\mathcal{R}_{\eta c}}$

Combining these inequalities, we obtain the result, taking $H = H_1 + 1$, as $\frac{H_2}{\sqrt{\eta}} < 1$

□

Lemma 0.67. *Let $f \in V(\mathcal{R}_\eta)$, with f bounded, S-continuous, and for which there exists a constant $C \in \mathcal{R}$, with;*

$|f(y)| \leq \frac{C}{|y|^2}$, $|y| \geq 1$ $(*)$

then for $-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}$, we have that;

$\frac{1}{\sqrt{c}}F_{\eta c}(f_{\sqrt{c}})(\frac{m}{\sqrt{\eta c}}) \simeq F_{\eta}(f)(\frac{m}{\sqrt{\eta}})$

Proof. We again calculate, taking into account endpoints;

$\frac{1}{\sqrt{c}}F_{\eta c}(f_{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{c}}\frac{1}{\sqrt{\eta c}}{}^{*}\sum_{-\frac{(\eta c-1)}{2}\leq j\leq \frac{\eta c-1}{2}} f_{\sqrt{c}}(\frac{j}{\sqrt{\eta c}})exp_{\eta c}(-2\pi i(\frac{m}{\sqrt{\eta c}})(\frac{j}{\sqrt{\eta c}}))$

$\simeq \frac{1}{c}\frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta c-c)}{2}\leq j\leq \frac{\eta c-1}{2}} f(\frac{j}{c\sqrt{\eta}})^{*}exp(-2\pi i(\frac{mj}{\eta c})) = L_1$

$F_{\eta}(f)(\frac{m}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq \frac{\eta-1}{2}} f(\frac{j}{\sqrt{\eta}})exp_{\eta}(-2\pi i(\frac{m}{\sqrt{\eta}})(\frac{j}{\sqrt{\eta}}))$

$= \frac{1}{\sqrt{\eta}}{}^{*}\sum_{-\frac{(\eta-1)}{2}\leq j\leq \frac{\eta-1}{2}} f(\frac{j}{\sqrt{\eta}})^{*}exp(-2\pi i(\frac{mj}{\eta})) = L_2$

Suppose, for contradiction, that there exists $\epsilon \in \mathcal{R}_{>0}$, with $|L_1 - L_2| > \epsilon$, then, using condition $(*)$, we can see that there exists $n \in \mathcal{N}_{>0}$, $\delta \in \mathcal{R}_{>0}$, such that $|M_1 - M_2| > \delta$, where;

$M_1 = \frac{1}{c\sqrt{\eta}}{}^{*}\sum_{|j|\leq nc\sqrt{\eta}} f(\frac{j}{c\sqrt{\eta}})^{*}exp(-2\pi i(\frac{mj}{\eta c}))$

$M_2 = \frac{1}{\sqrt{\eta}}{}^{*}\sum_{|j|\leq n\sqrt{\eta}} f(\frac{j}{\sqrt{\eta}})^{*}exp(-2\pi i(\frac{mj}{\eta})$

Let $\overline{\mathcal{R}_{\eta,c}}$ be the refinement of $\overline{\mathcal{R}_{\eta}}$, generated by the intervals $[\frac{i}{\sqrt{\eta c}}, \frac{i+1}{\sqrt{\eta c}})$ and let $g \in V(\overline{\mathcal{R}_{\eta,c}} \cap {}^{*}[-n,n])$, be defined by;

$g(y) = f(y)^{*}exp(-2\pi i\frac{m}{\sqrt{\eta}}\frac{[\sqrt{\eta}cy]}{c\sqrt{\eta}});$

We have that g is S-continuous and bounded, and hence S-integrable with respect to the measures $\{\mu_{\eta}, \mu_{\eta,c}\}$ on $\overline{\mathcal{R}_{\eta,c}} \cap {}^{*}[-n,n]$, where μ_{η} was defined in Definition 0.3, and $\mu_{\eta,c}([\frac{i}{\sqrt{\eta c}}, \frac{i+1}{\sqrt{\eta c}})) = \frac{1}{\sqrt{\eta c}}$. The result now follows by observing that $({}^{\circ}g)$ is well-defined, continuous and bounded on $[-n,n]$, see Theorem 4.5.10 of [11], and;

$M_1 = \int_{\overline{\mathcal{R}_{\eta,c}}\cap^{*}[-n,n]} g d\mu_{\eta,c} \simeq \int_{\overline{\mathcal{R}_{\eta}}\cap^{*}[-n,n]} g d\mu_{\eta} = M_2 \simeq \int_{[-n,n]}({}^{\circ}g)d\mu$

where μ is Lebesgue measure, see Theorem 3.12 of [6].

□

Lemma 0.68. *Let $g \in S(R)$, then $\mathcal{F}_\eta(g_\eta)$ satisfies the hypotheses of Lemma 0.67.*

Proof. Using Lemma 0.13 and Lemma 0.34, we have that $\mathcal{F}_\eta(g_\eta)$ is bounded and satisfies the condition $(*)$ of Lemma 0.67. It remains to check S-continuity. If $\{y, y'\} \subset \overline{\mathcal{R}_\eta}$, with y and y' infinite, $y \simeq y'$, then;

$$\mathcal{F}_\eta(g_\eta)(y) \simeq \mathcal{F}_\eta(g_\eta)(y') \simeq 0$$

using the condition $(*)$. If y and y' are finite, $y \simeq y'$, with $|y - y'| = \delta \simeq 0$, then, we compute, similarly to Lemma 0.66, using the Mean Value Theorem, and $g \in S(\mathcal{R})$;

$$|\mathcal{F}_\eta(g_\eta(y)) - \mathcal{F}_\eta(g_\eta(y'))|$$

$$= |\int_{\mathcal{R}_\eta} g_\eta(x) exp_\eta(-2\pi i y x) d\mu_\eta(x) - \int_{\mathcal{R}_\eta} g_\eta(x) exp_\eta(-2\pi i y' x) d\mu_\eta(x)|$$

$$\leq \int_{\mathcal{R}_\eta} |g_\eta(x)||exp_\eta(-2\pi i y x) - exp_\eta(-2\pi i y' x)| d\mu_\eta(x)$$

$$\leq \int_{\mathcal{R}_\eta} |g_\eta(x)|| - 2\pi i x_\eta|\delta d\mu_\eta(x)$$

$$\leq \delta \int_{|x| \leq 1} D| - 2\pi i x_\eta| d\mu_\eta(x) + \int_{|x|>1} \frac{E}{|x_\eta|^3}| - 2\pi i x_\eta|\delta d\mu_\eta(x)$$

$$= \delta F + \delta \int_{|x|>1} \frac{2\pi E}{|x_\eta|^2} d\mu_\eta(x) \simeq 0$$

where $\{D, E, F\} \subset \mathcal{R}_{>0}$.

□

Lemma 0.69. *Let $f \in S(\mathcal{R})$, $n \in \mathcal{Z}_{>0}$, $g \in V(\overline{\mathcal{R}_\eta})$, defined by;*

$$g(y) = {}^*exp(-2\pi i n \frac{[\sqrt{\eta} y]^2}{\eta}), \ (y \in \overline{\mathcal{R}_\eta})$$

then, for $-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}$;

$$F_{\eta c}^{-1}(g_{\sqrt{c}} F_{\eta c}(f_\eta^{\sqrt{c}}))(\frac{m}{\sqrt{\eta c}}) \simeq F_\eta^{-1}(g F_\eta(f_\eta))(\frac{m}{\sqrt{\eta}})$$

Proof. By Lemma 0.66, we have that;

$$\sqrt{c} F_{\eta c}(f^{\sqrt{c}}) = (F_\eta(f_\eta))_{\sqrt{c}} + \epsilon \text{ on } \overline{\mathcal{R}_{\eta c}}$$

where $|\epsilon| \leq \frac{H}{\sqrt{\eta}}$. Therefore;

$g_{\sqrt{c}}F_{\eta c}(f^{\sqrt{c}}) = \frac{g_{\sqrt{c}}}{\sqrt{c}}(F_\eta(f_\eta))_{\sqrt{c}} + \frac{\epsilon g_{\sqrt{c}}}{\sqrt{c}}$

$g_{\sqrt{c}}F_{\eta c}(f^{\sqrt{c}}) = \frac{1}{\sqrt{c}}(F_\eta(f_\eta)g)_{\sqrt{c}} + \frac{\epsilon g_{\sqrt{c}}}{\sqrt{c}}$

We have, for $-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}$;

$F_{\eta c}^{-1}((g_{\sqrt{c}}F_{\eta c}(f^{\sqrt{c}})))(\frac{m}{\sqrt{\eta c}}) = F_{\eta c}^{-1}(\frac{1}{\sqrt{c}}(F_\eta(f_\eta)g)_{\sqrt{c}})(\frac{m}{\sqrt{\eta c}}) + F_{\eta c}^{-1}(\frac{\epsilon g_{\sqrt{c}}}{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})$

We compute;

$F_{\eta c}^{-1}(\frac{\epsilon g_{\sqrt{c}}}{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})$

$= \frac{\epsilon}{\sqrt{c}}\frac{1}{\sqrt{\eta c}}{}^*\sum_{-\frac{(\eta c-1)}{2}\leq j\leq\frac{(\eta c-1)}{2}} g_{\sqrt{c}}(\frac{j}{\sqrt{\eta c}})^* exp(2\pi i\frac{m}{\sqrt{\eta c}}\frac{j}{\sqrt{\eta c}})$

$= \frac{\epsilon}{\sqrt{\eta c}}{}^*\sum_{-\frac{(\eta c-1)}{2}\leq j\leq\frac{(\eta c-1)}{2}} g(\frac{j}{\sqrt{\eta c}})^* exp(2\pi i\frac{mj}{\eta c})$

$= \frac{\epsilon}{\sqrt{\eta c}}{}^*\sum_{0\leq k\leq \eta c-1} g(\frac{k-\frac{(\eta c-1)}{2}}{\sqrt{\eta c}})^* exp(2\pi i\frac{m(k-\frac{(\eta c-1)}{2})}{\eta c})$

$= \frac{\epsilon}{\sqrt{\eta c}}{}^*\sum_{0\leq k\leq \eta c-1}{}^* exp(\frac{-2\pi i n[\frac{k-\frac{(\eta c-1)}{2}}{c}]^2}{\eta})^* exp(2\pi i\frac{m(k-\frac{(\eta c-1)}{2})}{\eta c})$

$= \frac{\epsilon}{\sqrt{\eta c}}{}^*\sum_{0\leq r\leq \eta-1}{}^*\sum_{0\leq s\leq c-1}{}^* exp(\frac{-2\pi i n}{\eta}[\frac{rc+s-\frac{(\eta c-1)}{2}}{c}]^2)^* exp(\frac{2\pi i m}{\eta}\frac{(rc+s-\frac{(\eta c-1)}{2})}{c})$

$= \frac{\epsilon}{\sqrt{\eta c}}{}^*\sum_{0\leq s\leq c-1}{}^*\sum_{0\leq r\leq \eta-1}{}^* exp(\frac{-2\pi i n}{\eta}(r-\frac{(\eta-1)}{2})^2)^* exp(\frac{2\pi i m}{\eta}((r-\frac{(\eta-1)}{2})))$

Hence, $|F_{\eta c}^{-1}(\frac{\epsilon g_{\sqrt{c}}}{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})| \leq \frac{\epsilon}{\sqrt{\eta c}}c\sqrt{\eta} = \epsilon \simeq 0$, by the same argument as at the end of Lemma 0.39.

It follows, using Lemma 0.67, for $-\frac{(\eta-1)}{2} \leq m \leq \frac{(\eta-1)}{2}$ that;

$F_{\eta c}^{-1}((g_{\sqrt{c}}F_{\eta c}(f^{\sqrt{c}})))(\frac{m}{\sqrt{\eta c}})$

$\simeq F_{\eta c}^{-1}(\frac{1}{\sqrt{c}}(F_\eta(f_\eta)g)_{\sqrt{c}})(\frac{m}{\sqrt{\eta c}})$

$\simeq F_\eta^{-1}(gF_\eta(f_\eta))(\frac{m}{\sqrt{\eta}})$

as required. □

Lemma 0.70. *There exists a unique $K_c \in V(\overline{\mathcal{R}_{\eta c}})$, with;*

$\mathcal{F}_{\eta c}(K_c)(y) = -\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2}$, $(y \in \overline{\mathcal{R}_{\eta c}})$

Moreover, for $\gamma > 0$, $\gamma \in \mathcal{R}$;

$$\mathcal{F}_{\eta c}(K_c) \simeq \frac{-4\pi^2 i[\sqrt{\eta} y]^2}{\eta c} \simeq \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}, \ (y \in \overline{\mathcal{R}_{\eta c}}), \ |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c}$$

Explicitly;

$$|\mathcal{F}_{\eta c}(K_c) - \frac{-4\pi^2 i[\sqrt{\eta} y]^2}{\eta c}| \leq 4\pi^2(\frac{1}{\eta c} + \frac{1}{c^{\frac{1}{2}}}) \ (y \in \overline{\mathcal{R}_{\eta c}})$$

and;

$$|\frac{-4\pi^2 i[\sqrt{\eta} y]^2}{\eta c} - \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}|$$

$$\leq \frac{4\pi^2}{\eta} + \frac{12\pi^2}{\eta c} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma c^{\frac{1}{2}}} + \frac{8\pi^2}{\eta^\gamma}, \ |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c}$$

and;

$$|\mathcal{F}_{\eta c}(K_c) - \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}|$$

$$\leq \frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}) + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma}, \ (|y| \leq |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c})$$

For the convolution equation;

$$\frac{\partial F}{\partial t} - K_c * F = 0 \ \textit{on} \ \overline{\mathcal{R}_{\eta c}} \times \overline{\mathcal{T}_\nu}$$

with initial condition $f \in V(\overline{\mathcal{R}_{\eta c}})$, *we have that;*

$$\mathcal{F}_{\eta c}(F)(y,t) = (1 - \frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2 \nu})^{[\nu t]}\mathcal{F}_{\eta c}(f)(y) \ (*)$$

(compare Lemma 0.33) $(y,t) \in (\overline{\mathcal{R}_{\eta c}} \times \overline{\mathcal{T}_\nu})$

For $\nu \geq \eta^5$, *and finite* $t \in \overline{\mathcal{T}_\nu}$, *we have, for* $(y \in \overline{\mathcal{R}_{\eta c}}, |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c})$, *that;*

$$(1 - \frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2 \nu})^{[\nu t]} \simeq {}^*exp(\frac{-4\pi^2 it[\sqrt{\eta} y]^2}{\eta c}) \simeq {}^*exp(\frac{-4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}), \ (**)$$

Proof. For the first claim, using Lemma 0.31, there exists a unique $K \in V(\overline{\mathcal{R}}_{\eta c})$, with;

$$\mathcal{F}_{\eta c}(K)(y) = \frac{-4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c}$$

Taking $K_c = \frac{K}{c}$ gives the result. For the first part of the second claim, let $y = \frac{j}{\sqrt{\eta c}}$, for $j \in {}^*\mathcal{Z}$, $-\frac{(\eta c-1)}{2} \leq j < \frac{(\eta c-1)}{2}$, then;

$\mathcal{F}_{\eta c}(K)(y) = \frac{-4\pi^2 ij^2}{\eta c^2}$

Let $\sqrt{\eta}y = \frac{j}{\sqrt{c}} = k + \epsilon$, where $k \in {}^*\mathcal{Z}$ and $0 \leq \epsilon < 1$, then;

$\frac{-4\pi^2 i[\sqrt{\eta}y]^2}{\eta c} = \frac{-4\pi^2 ik^2}{\eta c} = \frac{-4\pi^2 i(\frac{j}{\sqrt{c}}-\epsilon)^2}{\eta c}$

$= \frac{-4\pi^2 i(\frac{j^2}{c}-\frac{2\epsilon j}{\sqrt{c}}+\epsilon^2)}{\eta c}$

Therefore;

$|\mathcal{F}_{\eta c}(K)(y) - (\frac{-4\pi^2 i[\sqrt{\eta}y]^2}{\eta c})| \leq 4\pi^2|\frac{2\epsilon j}{\eta c^{\frac{3}{2}}} - \frac{\epsilon^2}{\eta c}|$

$\leq \frac{8\pi^2|j|}{\eta c^{\frac{3}{2}}} + \frac{4\pi^2}{\eta c}$ (as $|\epsilon| < 1$)

$\leq \frac{8\pi^2 \eta c}{2\eta c^{\frac{3}{2}}} + \frac{4\pi^2}{\eta c}$

$= \frac{4\pi^2}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta c} \simeq 0$, $|j| \leq \frac{\eta c}{2}$

as η and c are infinite.

For the second part of the second claim, again let $y = \frac{j}{\sqrt{\eta c}}$, for $j \in {}^*\mathcal{Z}$, $-\frac{(\eta c-1)}{2} \leq j < \frac{(\eta c-1}{2)}$. Then it is sufficient to prove that;

$\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} \simeq \frac{[\sqrt{\eta}y]^2}{\eta c}$

$\frac{[\frac{j}{c}]^2}{\eta} \simeq \frac{[\frac{j}{\sqrt{c}}]^2}{\eta c}$

Let $\frac{j}{\sqrt{c}} = k+\epsilon$, where $k \in {}^*\mathcal{Z}$, $0 \leq \epsilon < 1$, then sufficient to prove that;

$\frac{[\frac{k+\epsilon}{\sqrt{c}}]^2}{\eta} \simeq \frac{k^2}{\eta c}$

We have that $\frac{[\frac{k+\epsilon}{\sqrt{c}}]^2}{\eta} \simeq \frac{[\frac{k}{\sqrt{c}}]^2}{\eta} + \frac{2\epsilon k}{c\eta} + \frac{\epsilon^2}{c\eta} \simeq \frac{[\frac{k}{\sqrt{c}}]^2}{\eta}$

as $|k| \leq \frac{\eta c-1}{2\sqrt{c}} \leq \eta\sqrt{c}$, $|\epsilon| < 1$ and $\{c, \eta\}$ are infinite.

To determine when $\frac{[\frac{k}{\sqrt{c}}]^2}{\eta} \simeq \frac{k^2}{\eta c}$, let $\frac{k}{\sqrt{c}} = m + \delta$, where $m \in {}^*\mathcal{Z}$, $0 \leq \delta < 1$. Then;

$$\frac{[\frac{k}{\sqrt{c}}]^2}{\eta} = \frac{m^2}{\eta}$$

$$\frac{k^2}{\eta c} = \frac{(m+\delta)^2}{\eta} = \frac{m^2+2\delta m+\delta^2}{\eta} \simeq \frac{m^2+2\delta m}{\eta}$$

So, as $\delta < 1$, it is sufficient to determine the conditions when $\frac{2m}{\eta} \simeq 0$. This holds for $|m| < \eta^{1-\gamma}$, therefore, $|k| \leq \sqrt{c}\eta^{1-\gamma}$, therefore $|j| \leq c\eta^{1-\gamma}$, therefore $|y| \leq \sqrt{c}\eta^{\frac{1}{2}-\gamma}$.

Explicitly, letting $y = \frac{j}{\sqrt{\eta c}}$, $\frac{j}{\sqrt{c}} = k + \epsilon$, with $0 < \epsilon < 1$, so that $\frac{j}{c} = \frac{k+\epsilon}{\sqrt{c}}$, $\frac{(k+\epsilon)}{\sqrt{c}} = [\frac{(k+\epsilon)}{\sqrt{c}}] + \theta$, with $0 < \theta < 1$, we have;

$$|\frac{-4\pi^2 i[\sqrt{\eta}y]^2}{\eta c} - \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}|$$

$$= \frac{4\pi^2}{\eta}|\frac{[\frac{j}{\sqrt{c}}]^2}{c} - [\frac{j}{c}]^2|$$

$$= \frac{4\pi^2}{\eta}|\frac{k^2}{c} - [\frac{(k+\epsilon)}{\sqrt{c}}]^2|$$

$$= \frac{4\pi^2}{\eta}|\frac{k^2}{c} - (\frac{(k+\epsilon)}{\sqrt{c}} - \theta)^2|$$

$$= \frac{4\pi^2}{\eta}|\frac{k^2}{c} - (\frac{(k+\epsilon)^2}{c} - \frac{2\theta(k+\epsilon)}{\sqrt{c}} + \theta^2)|$$

$$\leq \frac{4\pi^2}{\eta} + \frac{8\pi^2(k+\epsilon)}{\eta\sqrt{c}} + \frac{4\pi^2}{\eta}(\frac{2k\epsilon}{c} + \frac{\epsilon^2}{c}), \text{ as } |\theta| < 1$$

$$\leq \frac{4\pi^2}{\eta} + \frac{8\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2 k}{\eta\sqrt{c}} + \frac{8\pi^2 k}{\eta c} + \frac{4\pi^2}{c\eta}, \text{ as } |\epsilon| < 1$$

$$\leq \frac{4\pi^2}{\eta} + \frac{8\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta\sqrt{c}}(\frac{|j|}{\sqrt{c}+1}) + \frac{8\pi^2}{\eta c}(\frac{|j|}{\sqrt{c}+1}) + \frac{4\pi^2}{c\eta}, \text{ as } |k| \leq \frac{|j|}{\sqrt{c}} + 1$$

$$\leq \frac{4\pi^2}{\eta} + \frac{12\pi^2}{\eta c} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2|j|}{\eta c^{\frac{3}{2}}} + \frac{8\pi^2|j|}{\eta c}$$

$$\leq \frac{4\pi^2}{\eta} + \frac{12\pi^2}{\eta c} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2\eta^{1-\gamma}}{\eta c^{\frac{3}{2}}} + \frac{8\pi^2\eta^{1-\gamma}c}{\eta c} \quad (|j| \leq \eta^{1-\gamma}c,\ |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c})$$

$$= \frac{4\pi^2}{\eta} + \frac{12\pi^2}{\eta c} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma c^{\frac{1}{2}}} + \frac{8\pi^2}{\eta^\gamma}$$

It follows that;

$$|\mathcal{F}_{\eta c}(K_c) - \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}|$$

$\leq 4\pi^2(\frac{1}{\eta c}+\frac{1}{c^{\frac{1}{2}}})+\frac{4\pi^2}{\eta}+\frac{12\pi^2}{\eta c}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma c^{\frac{1}{2}}}+\frac{8\pi^2}{\eta^\gamma}$

$=\frac{16\pi^2}{\eta c}+(4\pi^2+\frac{8\pi^2}{\eta^\gamma})\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma}$

The second claim $(**)$ follows from the second part of the first claim, together with the fact that $^*exp(itx)$ is bounded and S-continuous on $^*\mathcal{R}$, for finite t

We have that;

$(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}$

$\simeq {}^*exp(\frac{-4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2})$

by Lemma 0.33. Using the first part of the Lemma, that;

$\frac{-4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2}\simeq\frac{-4\pi^2 i[\sqrt{\eta}y]^2}{\eta c}$

therefore,

$\frac{-4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2}\simeq\frac{-4\pi^2 it[\sqrt{\eta}y]^2}{\eta c}$

for finite t, we obtain, as exp bounded, S-continuous, that;

${}^*exp(\frac{-4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2})$

$\simeq {}^*exp(\frac{-4\pi^2 it[\sqrt{\eta}y]^2}{\eta c})$

$\square$

Definition 0.71. *We define;*

$h_c(t,y)={}^*exp(\frac{-4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2})$

$g_{\sqrt{c}}(t,y)={}^*exp(\frac{-4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta})$, $(\overline{\mathcal{R}_{\eta c}}\times\overline{\mathcal{T}_\nu})$

$g(t,y)={}^*exp(\frac{-4\pi^2 it[\sqrt{\eta}y]^2}{\eta})$, $(\overline{\mathcal{R}_\eta}\times\overline{\mathcal{T}_\nu})$

Lemma 0.72. *Let $f\in\mathcal{S}(\mathcal{R})$, then, for $-\frac{(\eta-1)}{2}\leq m\leq\frac{(\eta-1)}{2}$, and finite $t\in\overline{\mathcal{T}}_\nu$;*

$\mathcal{F}_{\eta c}^{-1}(g_{t,\sqrt{c}}\mathcal{F}_{\eta c}(f_\eta^{\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$ (1)

$\simeq \mathcal{F}_\eta^{-1}(g_t\mathcal{F}_\eta(f_\eta))(\frac{m}{\sqrt{\eta}})$ (2)

$\simeq \mathcal{F}_{\eta c}^{-1}(h_{t,c}\mathcal{F}_{\eta c}(f_\eta^{\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$ (3)

$\simeq G_c(t,\frac{y}{\sqrt{c}})$, $y \in (-\sqrt{\eta},\sqrt{\eta})$ (4)

$\simeq G(t,y)$, $y \in (-\sqrt{\eta},\sqrt{\eta})$ (5)

where G_c is the nonstandard solution to;

$\frac{\partial G_c}{\partial t} - K_c * G_c = 0$ *on* $(\overline{\mathcal{R}_{\eta c}} \times \overline{\mathcal{T}_\nu})$ *with initial condition* $f_\eta^{\sqrt{c}}$

and G is the nonstandard solution to;

$\frac{\partial G}{\partial t} - K * G = 0$ *on* $(\overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu})$ *with initial condition* f_η.

Proof. For the equality of (1) and (2), extend Lemma 0.69 for integral t, for the equality of (1) and (3), use Lemma 0.70 and the fact that $\mathcal{F}_{\eta c}(f_\eta^{\sqrt{c}})$ is rapidly decreasing. For the equality of (3) and (4), use $(*), (**)$, the fact that $\mathcal{F}_{\eta c}(f_\eta^{\sqrt{c}})$ is rapidly decreasing, and generalise the nonstandard inversion theorem $(+)$ of Lemma 0.33 to the new convolution equation. (5) = (2), again apply the inversion theorem to $(+)$ of Lemma 0.33.

□

Lemma 0.73. *For $g \in S(\mathcal{R})$, with corresponding $g_\eta \in V(\mathcal{R}_\eta)$, we have that;*

$\int_{\overline{\mathcal{R}_{\eta c}}} g_\eta^{\sqrt{c}} d\mu_{\eta c} \simeq \frac{1}{\sqrt{c}} \int_{\overline{\mathcal{R}_\eta}} g_\eta d\mu_\eta$

$\int_{\overline{\mathcal{R}_{\eta c}}} |g_\eta^{\sqrt{c}}| d\mu_{\eta c} \simeq \frac{1}{\sqrt{c}} \int_{\overline{\mathcal{R}_\eta}} |g_\eta| d\mu_\eta$

for $0 \leq c < \eta^{\frac{2}{3}}$

Proof. We have that;

$\int_{\overline{\mathcal{R}_{\eta c}}} g_\eta^{\sqrt{c}} d\mu_{\eta c}$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{-\eta c \leq j \leq \eta c - 1} g_\eta^{\sqrt{c}}(\frac{j}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{-\eta c \leq j \leq \eta c - 1} g^*(\frac{j}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{c}} (\frac{1}{\sqrt{\eta}} {}^* \sum_{m=-c}^{c-1} {}^* \sum_{j=m\eta}^{(m+1)\eta} g^*(\frac{j}{\sqrt{\eta}}) - \frac{1}{\sqrt{\eta}} g^*(\frac{\eta c}{\sqrt{\eta}}))$

$= \frac{1}{\sqrt{c}} (\frac{1}{\sqrt{\eta}} {}^* \sum_{j=-(\eta-1)}^{j=(\eta-1)} g^*(\frac{j}{\eta}) + \frac{1}{\sqrt{\eta}} {}^* \sum_{m=1}^{c-1} {}^* \sum_{j=\eta m}^{\eta(m+1)} g^*(\frac{j}{\sqrt{\eta}})$

$+ \frac{1}{\sqrt{\eta}} {}^* \sum_{m=-c}^{-2} {}^* \sum_{j=\eta m}^{\eta(m+1)} g^*(\frac{j}{\sqrt{\eta}}) - \frac{1}{\sqrt{\eta}} g^*(\frac{\eta c}{\sqrt{\eta}})$

$= \frac{1}{\sqrt{c}} (\int_{\overline{\mathcal{R}_\eta}} g_\eta d\mu_\eta - \frac{1}{\sqrt{\eta}} g^*(-\sqrt{\eta} - \frac{1}{\sqrt{\eta}} g^*(\sqrt{\eta}) + \epsilon)$

where;

$|\epsilon| \leq \frac{1}{\sqrt{\eta}} (c-1)^* \sum_{\eta}^{2\eta} |g^*(\frac{j}{\sqrt{\eta}})| + \frac{1}{\sqrt{\eta}} ((c-2)+1)^* \sum_{-2\eta}^{-\eta} |g^*(\frac{j}{\sqrt{\eta}})|)$

$\leq \frac{2(c-1)}{\sqrt{\eta}} {}^* \sum_{j=\eta}^{2\eta} \frac{D\eta}{j^2}$

$= 2\sqrt{\eta}(c-1)^* \sum_{j=\eta}^{2\eta} \frac{D}{j^2}$

$\leq 2\sqrt{\eta}(c-1) \int_{\eta-1}^{2\eta} \frac{D}{x^2} dx$ (by transfer)

$= 2\sqrt{\eta}(c-1)[\frac{-D}{x}]_{\eta-1}^{2\eta}$

$= 2\sqrt{\eta}(c-1)(\frac{D}{\eta-1} - \frac{D}{2\eta})$

Then;

$\frac{|\epsilon|}{\sqrt{c}} \leq \frac{2\sqrt{\eta}(c-1)}{\sqrt{c}} (\frac{D}{\eta-1} - \frac{D}{2\eta})$

$\leq 2\sqrt{\eta c} (\frac{D}{\eta-1} - \frac{D}{2\eta})$

$\leq \frac{4D\sqrt{\eta c}}{\eta}$

$= \frac{4D\sqrt{c}}{\sqrt{\eta}}$

$\simeq 0$, for $0 \leq c < \eta^{\frac{2}{3}}$

The second part is similar.

□

Definition 0.74. *Suppose $f \in S(\mathcal{R})$, then let f^c be defined by;*

$f^{c}(k\frac{c}{\sqrt{\eta}}+\frac{l}{\sqrt{\eta}})=f^{*}(k\frac{c}{\sqrt{\eta}})$ *for* $0\leq l\leq c-1$, $-\eta\leq k\leq\eta-1$

Lemma 0.75. *Suppose* $f\in S(\mathcal{R})$, *with;*

$f^{c}(-\frac{\sqrt{\eta}c}{2}+k\frac{c}{\sqrt{\eta}}+\frac{l}{\sqrt{\eta}})=f^{c}(-\frac{\sqrt{\eta}c}{2}+k\frac{c}{\sqrt{\eta}})$

for $0\leq l\leq c-1$, $0\leq k\leq\eta-1$

then;

$((f^{c})^{\sqrt{c}})(-\frac{\sqrt{\eta c}}{2}+\frac{kc}{\sqrt{\eta c}})=((f^{c})^{\sqrt{c}})(-\frac{\sqrt{\eta c}}{2}+\frac{kc}{\sqrt{\eta c}}+\frac{l}{\sqrt{\eta c}})$

for $0\leq l\leq c-1$, $0\leq k\leq\eta-1$

Proof. We have, using Definition 0.74 and Definition 0.64 that, for $0\leq l\leq c-1$, and $0\leq k\leq\eta-1$;

$(f^{c})^{\sqrt{c}}(-\frac{\sqrt{\eta c}}{2}+\frac{kc}{\sqrt{\eta c}}+\frac{l}{\sqrt{\eta c}})$

$=(f^{c})(-\frac{\sqrt{\eta c}\sqrt{c}}{2}+\frac{kc\sqrt{c}}{\sqrt{\eta c}}+\frac{l\sqrt{c}}{\sqrt{\eta c}})$

$=(f^{c})(-\frac{\sqrt{\eta}c}{2}+\frac{kc}{\sqrt{\eta}}+\frac{l}{\sqrt{\eta}})$

$=(f^{c})(-\frac{\sqrt{\eta}c}{2}+\frac{kc}{\sqrt{\eta}})$

$=(f^{c})^{\sqrt{c}}(-\frac{\sqrt{\eta c}}{2}+\frac{kc}{\sqrt{\eta c}})$

□

Lemma 0.76. *Suppose* $g\in V(\overline{\mathcal{R}_{\eta c}})$, *with;*

$g(-\sqrt{\frac{\eta c}{2}}+\frac{kc}{\sqrt{\eta c}}+\frac{l}{\sqrt{\eta c}})=g(-\sqrt{\frac{\eta c}{2}}+\frac{kc}{\sqrt{\eta c}})$

for $0\leq l\leq c-1$, $0\leq k\leq\eta-1$

then $g=h_{\sqrt{c}}$, *for some* $h\in V(\overline{\mathcal{R}_{\eta}})$

Proof. Define h by;

$h(\frac{k}{\sqrt{\eta}})=g(k\frac{\sqrt{c}}{\sqrt{\eta}})=g(\frac{kc}{\sqrt{\eta c}})$

for $-\eta \leq k \leq \eta - 1$,
for $-\eta \leq k \leq \eta - 1$

then;

$h_{\sqrt{c}}(\frac{kc}{\sqrt{\eta c}} + \frac{l}{\sqrt{\eta c}})$, $0 \leq k \leq \eta - 1$, $0 \leq l \leq c - 1$

$= h(\frac{k\sqrt{c}}{\sqrt{\eta c}} + \frac{l}{\sqrt{\eta c}})$

$= h(\frac{k\sqrt{c}}{\sqrt{\eta c}})$ as $|\frac{c-1}{c}| < 1$, as $h \in V(\overline{\mathcal{R}_\eta})$

$= h(\frac{k}{\sqrt{\eta}})$

$= g(\frac{kc}{\sqrt{\eta c}})$

$= g(\frac{kc}{\sqrt{\eta c}} + \frac{l}{\sqrt{\eta c}})$

□

Lemma 0.77. *Suppose $f \in S(\mathcal{R})$, then $(f^c)^{\sqrt{c}} = g_{\sqrt{c}}$*

for some $g \in V(\overline{\mathcal{R}_\eta})$

Proof. Using the fact that $((f^c)^{\sqrt{c}})$ satisfies the conditions of Lemma 0.76.

□

Lemma 0.78. *Let $f \in S(\mathcal{R})$, and $\{(f^c)^{\sqrt{c}}, g_{\sqrt{c}}\}$, as in 2 previous lemmas. Then;*

$g(\frac{i}{\sqrt{\eta}}) = (f^*)(\frac{ci}{\sqrt{\eta}})$, *for* $-\eta \leq i \leq \eta - 1$

$g_{\sqrt{c}}(\frac{l}{\sqrt{\eta c}}) = (f^*)(\frac{ci}{\sqrt{\eta}})$, $l = ci + j$, $-\eta \leq i \leq \eta - 1$, $0 \leq j \leq c - 1$

Proof. We have that;

$(f^c)^{\sqrt{c}}(\frac{k}{\sqrt{\eta c}}) = f^c(\frac{k\sqrt{c}}{\sqrt{\eta c}}) = f^c(\frac{k}{\sqrt{\eta}})$

for $-\eta c \leq k \leq \eta c - 1$. Let $k = ci + j$, $0 \leq j \leq c - 1$, $-\eta \leq i \leq \eta - 1$, then by f^c in Definition 0.74;

$f^c(\frac{k}{\sqrt{\eta}}) = f^*(\frac{ci}{\sqrt{\eta}})$

Then, by the definition of g in Lemma 0.76 and Lemma 0.78;

$g(\frac{i}{\sqrt{\eta}})$

$= (f^c)^{\sqrt{c}}(i\sqrt{\frac{c}{\sqrt{\eta}}})$

$= (f^c)^{\sqrt{c}}(\frac{\sqrt{c}i}{\sqrt{\eta}})$

$= (f^c)^{\sqrt{c}}(\frac{\sqrt{c}i\sqrt{c}}{\sqrt{\eta c}})$

$= (f^c)^{\sqrt{c}}(\frac{ci}{\sqrt{\eta c}}) = f^*(\frac{ci}{\sqrt{\eta}})$ (double squash, $c = (\sqrt{c})^2$)

For the second part, we have that;

$g_{\sqrt{c}}(\frac{l}{\sqrt{\eta c}})$

$= (f^c)^{\sqrt{c}}(\frac{l}{\sqrt{\eta c}})$

$= (f^c)(\frac{l\sqrt{c}}{\sqrt{\eta c}})$

$= (f^c)(\frac{l}{\sqrt{\eta}})$

$= (f^c)(\frac{ci+j}{\sqrt{\eta}})$

$= (f^c)(\frac{ci}{\sqrt{\eta}})$

$= (f^*)(\frac{ci}{\sqrt{\eta}})$

□

Lemma 0.79. *Let* $g_{\sqrt{c}}$ *be as in Lemma 0.78, then;*

$|\int_{\overline{\mathcal{R}_{\eta c}}} g_{\sqrt{c}} d\mu_{\eta c}| \leq \frac{2(M+D)}{\sqrt{c}}$

$|\mathcal{F}_{\eta c}(g_{\sqrt{c}})| \leq \frac{2(M+D)}{\sqrt{c}}$

where $\{M, D\} \subset \mathcal{R}$

Proof. We have that;

$|\int_{\overline{\mathcal{R}_{\eta c}}} g_{\sqrt{c}} d\mu_{\eta c}|$

$= |\frac{1}{\sqrt{\eta c}} {}^*\sum_{j=-\eta c}^{\eta c-1} g_{\sqrt{c}}(\frac{j}{\sqrt{\eta c}})|$

$\leq \frac{1}{\sqrt{\eta c}} {}^*\sum_{j=-\eta c}^{\eta c-1} |g(\frac{j}{\sqrt{\eta c}})|$

$= \frac{\sqrt{c}}{\sqrt{\eta}} {}^*\sum_{j=-\eta}^{\eta-1} |g(\frac{j}{\sqrt{\eta}})|$, as $g \in V(\overline{\mathcal{R}_\eta})$

$= \frac{\sqrt{c}}{\sqrt{\eta}} {}^*\sum_{j=-\eta}^{\eta-1} |f^*(\frac{cj}{\sqrt{\eta}})|$, by the computation of g in 0.78

$= \frac{\sqrt{c}}{\sqrt{\eta}}({}^*\sum_{|j| \leq \frac{\sqrt{\eta}}{c}} |f^*(\frac{cj}{\sqrt{\eta}})|)$

$+ \frac{\sqrt{c}}{\sqrt{\eta}}({}^*\sum_{\frac{\sqrt{\eta}}{c} \leq j \leq \eta} |f^*(\frac{cj}{\sqrt{\eta}})|)$

We have that;

$|\frac{\sqrt{c}}{\sqrt{\eta}}({}^*\sum_{|j| \leq \frac{\sqrt{\eta}}{c}} f^*(\frac{cj}{\sqrt{\eta}}))|$

$\leq \frac{M\sqrt{c}}{\sqrt{\eta}} \frac{2\sqrt{\eta}}{c}$, $M \in \mathcal{R}$, as f^* is bounded, c is infinite.

$= \frac{2M}{\sqrt{c}}$

As $f \in S(\mathcal{R})$, we have that;

$|f^*(\frac{cj}{\sqrt{\eta}})| \leq \frac{D\eta}{c^2 j^2}$, for $|j| \geq \frac{\sqrt{\eta}}{c}$

so that;

$\frac{\sqrt{c}}{\sqrt{\eta}}({}^*\sum_{\frac{\sqrt{\eta}}{c} \leq j \leq \eta} |f^*(\frac{cj}{\sqrt{\eta}})|)$

$\leq \frac{\sqrt{c}}{\sqrt{\eta}}({}^*\sum_{\frac{\sqrt{\eta}}{c} \leq j \leq \eta} \frac{D\eta}{c^2 j^2})$

$= \frac{D\sqrt{c}\sqrt{\eta}}{c^2} {}^*\sum_{\frac{\sqrt{\eta}}{c} \leq j \leq \eta} \frac{1}{j^2}$

$\leq \frac{D\sqrt{c}\sqrt{\eta}}{c^2}(\int_{[\frac{\sqrt{\eta}}{c}]-1}^{\eta} \frac{dx}{x^2})$ (by transfer)

$= \frac{D\sqrt{c}\sqrt{\eta}}{c^2}([\frac{-1}{x}]_{[\frac{\sqrt{\eta}}{c}]-1}^{\eta})$

$= \frac{D\sqrt{c}\sqrt{\eta}}{c^2}(\frac{1}{([\frac{\sqrt{\eta}}{c}]-1)} - \frac{1}{\eta})$

$\leq \frac{D\sqrt{c}\sqrt{\eta}}{c^2}(\frac{1}{\frac{\sqrt{\eta}}{c}-\epsilon-1})$, with $|\epsilon| \leq 1$

$\leq \frac{D\sqrt{c}\sqrt{\eta}}{c^2}\frac{2}{\frac{\sqrt{\eta}}{c}}$

$\leq \frac{D\sqrt{c}\sqrt{\eta}}{c^2}\frac{2}{\frac{\sqrt{\eta}}{c}}$

$= \frac{2D}{\sqrt{c}}$

For the second part, we have, for $y \in \overline{\mathcal{R}_{\eta c}}$ that;

$|\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)|$

$= |\int_{\overline{\mathcal{R}_{\eta c}}} g_{\sqrt{c}}(x) exp_{\eta c}(-2\pi i x y) d\mu_{\eta c}(x)|$

$\leq \int_{\overline{\mathcal{R}_{\eta c}}} |g_{\sqrt{c}}(x)| d\mu_{\eta c}(x)$

$\leq \frac{2(M+D)}{\sqrt{c}}$

as above.

□

Lemma 0.80. *Suppose $f \in S(\mathcal{R})$, then $\mathcal{F}_{\eta c}((f^c)^{\sqrt{c}}) = \mathcal{F}_{\eta c}(g_{\sqrt{c}}) \leq ?$*

(obtain bound)

Proof. Adapt proof of Lemma 0.3 in [10].

□

Lemma 0.81. *For c divisible by 4, $\frac{\eta c}{4} \leq |r| \leq \eta c$, we have that;*

$0 < \{\mu x : x > 0, cos_{\eta c}(\frac{2\pi r x}{\sqrt{\eta c}}) = 0\} \leq \frac{1}{\sqrt{\eta c}}$

Proof. We have that;

${}^*cos(2\pi(\frac{(\frac{\eta c}{4})}{\sqrt{\eta c}})\frac{1}{\sqrt{\eta c}})$

$= cos_{\eta c}(2\pi(\frac{(\frac{\eta c}{4})}{\sqrt{\eta c}})\frac{1}{\sqrt{\eta c}}) = 0$

when c is divisible by 4 and η is prime, as $\frac{\eta c}{4} \in {}^*\mathcal{Z}$, the result is then clear.

□

Lemma 0.82. *For $0 < \gamma < \frac{1}{2}$, $\gamma \in \mathcal{R}$, $\sqrt{c}(\eta^{\frac{1}{2}-\gamma}) \leq |y| \leq \frac{\sqrt{\eta c}}{4}$;*

$\mathcal{F}_{\eta c}(g_{\eta,\sqrt{c}})(y) \leq ?$

Proof. Adapt proof of Lemma 0.3 in [10].....

We first prove that;

$|\int_{(|x|>n\sqrt{c})\cap\overline{\mathcal{R}_{\eta c}}} g_{\sqrt{c}} exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)| \leq \frac{E}{c^{\frac{3}{2}}} \simeq 0$

for given $n \in {}^*\mathcal{N}$ and $E \in \mathcal{R}$.

We have that;

$|\int_{(|x|>n\sqrt{c})\cap\overline{\mathcal{R}_{\eta c}}} g_{\sqrt{c}} exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq \frac{1}{\sqrt{\eta c}} {}^*\sum_{k=[n\sqrt{\eta c}]}^{\eta c-1} |g_{\sqrt{c}}(\frac{k}{\sqrt{\eta c}})|$ $(\frac{k}{\sqrt{\eta c}} = n\sqrt{c})$

$= \frac{1}{\sqrt{\eta c}} {}^*\sum_{k=[n\sqrt{\eta c}]}^{\eta c-1} |g(\frac{k}{\sqrt{\eta c}})|$

$\simeq \frac{c}{\sqrt{\eta c}} {}^*\sum_{i=\frac{[n\sqrt{\eta c}]}{c}}^{\eta-1} g(\frac{i}{\sqrt{\eta}})$ (as $g \in V(\overline{\mathcal{R}_\eta})$)

$\simeq \frac{\sqrt{c}}{\sqrt{\eta}} {}^*\sum_{i=[n\sqrt{\eta}]}^{\eta-1} f^*(\frac{ci}{\sqrt{\eta}})$ (By the property of g in Lemma 0.78

$\leq \frac{\sqrt{c}}{\sqrt{\eta}} {}^*\sum_{i=[n\sqrt{\eta}]}^{\eta-1} D\eta c^2 i^2$, as $f \in S(\mathcal{R})$

$= \frac{\sqrt{c}}{\sqrt{\eta}} \frac{D\eta}{c^2} {}^*\sum_{i=[n\sqrt{\eta}]}^{\eta-1} \frac{1}{i^2}$

$\leq \frac{D\sqrt{\eta}}{c^{\frac{3}{2}}} \int_{[n\sqrt{\eta}]-1}^{\eta-1} \frac{dx}{x^2}$ (by transfer)

$= \frac{D\sqrt{\eta}}{c^{\frac{3}{2}}} [\frac{-1}{x}]_{[n\sqrt{\eta}]-1}^{\eta-1}$

$\leq \frac{D\sqrt{\eta}}{c^{\frac{3}{2}}} (\frac{1}{[n\sqrt{\eta}]-1})$

$\leq \frac{2D\sqrt{\eta}}{c^{\frac{3}{2}}[n\sqrt{\eta}]}$

$\simeq \frac{2D}{nc^{\frac{3}{2}}}$

which gives the result, taking $E = \frac{2D}{n}$.

....
We have that;

$\int_{[-n\sqrt{c},n\sqrt{c}]} g_{\sqrt{c}} exp_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$

$= \int_{[-n\sqrt{c},n\sqrt{c}]} g_{\sqrt{c}} cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{[-n\sqrt{c},n\sqrt{c}]} g_{\sqrt{c}} sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \int_{[-n\sqrt{c},n\sqrt{c}]} Re(g_{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+i \int \int_{[-n\sqrt{c},n\sqrt{c}]} Im(g_{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{[-n\sqrt{c},n\sqrt{c}]} Re(g_{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+ \int_{[-n\sqrt{c},n\sqrt{c}]} Im(g_{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

We have;

$\int_{[-n\sqrt{c},n\sqrt{c}]} Re(g_{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{|\frac{l}{\sqrt{\eta c}}| \leq n\sqrt{c}} Re(f^c)(\frac{l}{\sqrt{\eta}}) cos_{\eta c}(\frac{2\pi lk}{\eta c})$, $(y = \frac{k}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{|l| \leq n\sqrt{\eta c}} Re(f^c)(\frac{l}{\sqrt{\eta}}) cos_{\eta c}(\frac{2\pi lk}{\eta c})$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{0 \leq j \leq c-1} ({}^* \sum_{|i| \leq n\sqrt{\eta}} Re(f^c)((\frac{ci+j}{\sqrt{\eta}}) cos_{\eta c}(\frac{2\pi(ci+j)k}{\eta c}))$, $(l = ci+j)$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{0 \leq j \leq c-1} ({}^* \sum_{|i| \leq n\sqrt{\eta}} Re(f^*)(\frac{ci}{\sqrt{\eta}}) cos_{\eta c}(\frac{2\pi(ci+j)k}{\eta c}))$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{0 \leq j \leq c-1} ({}^* \sum_{|i| \leq n\sqrt{\eta}} Re(f^*)(\frac{ci}{\sqrt{\eta}}) \theta_{j,k}(\frac{ci}{\sqrt{\eta}}))$

where $\theta_{j,k}(\frac{ci}{\sqrt{\eta}}) = cos_{\eta c}(\frac{2\pi(ci+j)k}{\eta c})$

We compute an upper bound, for given j, k of;

$\frac{1}{\sqrt{\eta}} {}^* \sum_{|i| \leq n\sqrt{\eta}} Re(f^*)(\frac{ci}{\sqrt{\eta}}) \theta_{j,k}(\frac{ci}{\sqrt{\eta}})$

by transfer of the result for;

$\frac{1}{m} {}^* \sum_{|i| \leq nm} Re(f)(\frac{vi}{m}) \theta_{j,k}(\frac{vi}{m})$

$= \frac{1}{m} {}^* \sum_{|i| \leq nm} Re(f_v)(\frac{i}{m}) \theta_{j,k,v}(\frac{i}{m})$

where $\{v,m\}\subset\mathcal{R}_{>0}$, $n\in\mathcal{N}$, $\{f,f_v\}\subset S(\mathcal{R})$ and $\theta_{j,k}(\frac{vi}{m})=cos_{m^2v}(\frac{2\pi(vi+j)k}{m^2v})$
(v corresponds to c,m to $\sqrt{\eta}$)

We have that;

$cos_{m^2v}(\frac{2\pi(vi+j)k}{m^2v})=0$

iff $cos_{m^2v}(\frac{2\pi jk}{m^2v}+\frac{2\pi vik}{m^2v})=0$

$cos_{m^2v}(\frac{2\pi jk}{m^2v}+\frac{2\pi vxk}{mv})=0$, $(x=\frac{i}{m})$

iff $cos_{m^2v}(\frac{2\pi jk}{m^2v}+\frac{2\pi xk}{m})=0$

$cos(\frac{2\pi jk}{m^2v}+\frac{2\pi xk}{m})=0$

iff $\frac{2\pi jk}{m^2v}+\frac{2\pi kx}{m}=\frac{\pi}{2}+t\pi$

iff $x=\frac{(t+\frac{1}{2})m}{2k}-\frac{j}{mv}$;

$t\in\mathcal{Z}$, $\{v,m\}\subset\mathcal{R}_{>0}$

$0\leq j\leq v-1$

$\sqrt{v}(m^{1-2\gamma})\leq|y|\leq\frac{\sqrt{v}m}{4}$, $0<\gamma<\frac{1}{2}$, $\gamma\in\mathcal{R}$

$y=\frac{k}{m\sqrt{v}}$

$vm^{2(1-\gamma)}\leq k\leq\frac{m^2v}{4}$

We have that $vm^{2(1-\gamma)}\leq\frac{m^2}{2}$ $(*)$

iff $vm^2m^{-2\gamma}\leq\frac{m^2}{2}$

iff $v\leq\frac{m^{2\gamma}}{2}$

Fix $\gamma_0\in\mathcal{R}$, with $0<\gamma_0<\frac{1}{2}$, so that $(*)$ holds for this choice of v, γ_0 close to $\frac{1}{2}$.

We consider the case when $vm^{2(1-\gamma_0)}\leq k<\frac{m^2}{2}\leq\frac{m^2v}{4}$, $(v\geq 2)$

For $k < \frac{m^2}{2}$, we have that the zero spacing $\frac{m}{2k} > \frac{1}{m}$

Let $\epsilon > 0$. As $Re(f)|_{[-n-\epsilon,n+\epsilon]}$ is continuous, without loss of generality, it has finitely many zeroes at $\{x_1,\ldots,x_{a(n)}\}$, with $-n \leq x_1 \leq \ldots \leq x_{i(n)} \leq \ldots \leq x_{a(n)} \leq n$. As $Re(f)|_{[x_i,x_{i+1}]}$ is differentiable, it has finitely many maxima and minima, $\{x_{i,1},\ldots,x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$. It follows that $Re(f)|_{[x_{i,j},x_{i,j+1}]}$ is monotone for $1 \leq j \leq b(i)-1$.

Without loss of generality, there are five cases to consider, $Re(f_v) \equiv 0$ on $[x_{i,j},x_{i,j+1}]$, $Re(f_v) > 0$ on $[x_{i,j},x_{i,j+1}]$, with $Re(f_v)(x_{i,j}) < Re(f_v)(x_{i,j+1})$, $Re(f_v) > 0$ on $[x_{i,j},x_{i,j+1}]$, with $Re(f_v)(x_{i,j}) > Re(f_v)(x_{i,j+1})$, $Re(f_v)|_{[x_{i,j},x_{i,j+1}]} > 0$, with $Re(f_v)(x_{i,j}) = 0$, and $Re(f_v)|_{[x_{i,j},x_{i,j+1}]} > 0$, with $Re(f_v)(x_{i,j+1}) = 0$. The cases for $Re(f_v)|_{[x_{i,j},x_{i,j+1}]} \leq 0$ follow by considering $-Re(f_v)$.

Let $\{r_{ghd} : 1 \leq d \leq e(g,h)\}$ enumerate the zeroes of $cos(\frac{2\pi jk}{m^2 v} + \frac{2\pi xk}{m})$ on $[x_{g,h},x_{g,h+1}]$, then;

$\frac{1}{m} {}^*\sum_{|i|\leq nm} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

$= \frac{1}{m} {}^*\sum_{g=1}^{a(n)-1} {}^*\sum_{i=[nmx_g]}^{[nmx_{g+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

$= \frac{1}{m} {}^*\sum_{g=1}^{a(n)-1} {}^*\sum_{h=1}^{b(g)-1} {}^*\sum_{i=[nmx_{g,h}]}^{[nmx_{g,h+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

$= \frac{1}{m} {}^*\sum_{g=1}^{a(n)-1} {}^*\sum_{h=1}^{b(g)-1} {}^*\sum_{d=1}^{e(g,h)-1} {}^*\sum_{i=[nmr_{ghd}]}^{[nmr_{ghd+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

We compute ${}^*\sum_{d=1}^{e(g,h)-1} {}^*\sum_{i=[nmr_{ghd}]}^{[nmr_{ghd+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

Let $\theta_{g,h}(d) = {}^*\sum_{i=[nmr_{ghd}]}^{[nmr_{ghd+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

We consider Case 3, $Re(f_v) > 0$ on $[x_{g,h},x_{g,(h+1)}]$, with $Re(f_v)(x_{g,h}) > Re(f_v)(x_{g,(h+1)})$. Assume, without loss of generality, that $\theta_{j,k,v}|_{[r_{gh1},r_{gh2}]} > 0$, then, as $Re(f_v)|_{[x_{g,h},x_{g,(h+1)}]} > 0$, and;

$\frac{\theta_{j,k,v}|_{[r_{ghd},r_{ghd+1}]}}{|\theta_{j,k,v}|_{[r_{ghd},r_{ghd+1}]}|} = -\frac{\theta_{j,k,v}|_{[r_{ghd+1},r_{ghd+2}]}}{|\theta_{j,k,v}|_{[r_{ghd+1},r_{ghd+2}]}|}$, $1 \leq d \leq e(g,h)-2$

the sequence $\{\theta_{g,h}(d) : 1 \leq d \leq e(g,h)\}$ is alternating. As $Re(f_v)|_{[x_{g,h},x_{g,(h+1)}]}$ is decreasing, we have that;

$|\theta_{g,h}(d)| \leq |\theta_{g,h}(d+1)|$, for $1 \leq d \leq e(g,h)-1$

so that sequence $\{\theta_{g,h}(d) : 1 \leq d \leq e(g,h)\}$ is decreasing in magnitude.

We have that $0 \leq \theta_{g,h}(1) \leq l_{g,h}$, $l_{g,h} \in \mathcal{R}$. Assume that $e(g,h)$ is even, then, as;

$\theta_{g,h}(2p) + \theta_{g,h}(2p+1) \leq 0$, for $1 \leq p \leq \frac{e(g,h)-2}{2}$

$\theta_{g,h}(e(g,h)) \leq 0$

so that ${}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d) \leq l_{g,h}$

and, as;

$\theta_{g,h}(2p-1) + \theta_{g,h}(2p) \geq 0$, for $1 \leq p \leq \frac{e(g,h)}{2}$

$0 \leq {}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d)$

therefore, $0 \leq {}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d) \leq l_{g,h}$

Assume that $e(g,h)$ is odd, then as;

$\theta_{g,h}(2p) + \theta_{g,h}(2p+1) \leq 0$, for $1 \leq p \leq \frac{e(g,h)-1}{2}$

so that;

${}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d) \leq l_{g,h}$

$\theta_{g,h}(2p-1) + \theta_{g,h}(2p) \geq 0$, for $1 \leq p \leq \frac{e(g,h)-1}{2}$

$\theta_{g,h}(e(g,h)) \geq 0$

and as;

$0 \leq {}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d)$

therefore;

$0 \leq {}^*\sum_{d=1}^{e(g,h)} \theta_{g,h}(d) \leq l_{g,h}$

We compute;

$|\theta_{g,h}(1)| = |^{*}\sum_{i=[nmr_{gh1}]}^{[nmr_{gh2}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})|$

$\leq {}^{*}\sum_{i=[nmr_{gh1}]}^{[nmr_{gh2}]} |Re(f_v)(\frac{i}{m})||\theta_{j,k,v}(\frac{i}{m})|$

$\leq D_v{}^{*}\sum_{i=[nmr_{gh1}]}^{[nmr_{gh2}]} 1$

$= D_v([nmr_{gh2}] - [nmr_{gh1}] + 1)$

so that;

$\frac{1}{m}{}^{*}\sum_{d=1}^{e(g,h)-1}{}^{*}\sum_{i=[nmr_{ghd}]}^{[nmr_{ghd+1}]} Re(f_v)(\frac{i}{m})\theta_{j,k,v}(\frac{i}{m})$

$\leq \frac{D_v}{m}([nmr_{gh2}] - [nmr_{gh1}] + 1)$

$= \frac{D_v}{m}([nm(\frac{(t_{gh2}+\frac{1}{2})m}{2k} - \frac{j}{mv})] - [nm(\frac{(t_{gh1}+\frac{1}{2})m}{2k} - \frac{j}{mv})])$

$= \frac{D_v}{m}([nm(\frac{(t_{gh1}+\frac{3}{2})m}{2k} - \frac{j}{mv})] - [nm(\frac{(t_{gh1}+\frac{1}{2})m}{2k} - \frac{j}{mv})])$

□

Lemma 0.83. *For δ infinitesimal, $|y| \leq \sqrt{c}\eta^{\delta}$, we have that;*

$|(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (-4\pi^2 i)\frac{[\sqrt{\eta}cy]^2}{\eta c^2}| \leq 4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{(1+c)}{c^{\frac{3}{2}}}) + \frac{1}{\eta} + \frac{1}{\eta c^2}) \simeq 0$

and $|(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta}cy]^2}{\eta c^2}|$

$\leq \frac{2(M+D)}{\sqrt{c}}4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{1+c}{c}) + \frac{1}{\eta} + \frac{1}{\eta c^2}) \simeq 0$

Proof. Let $\frac{\sqrt{\eta}y}{\sqrt{c}} = m + \delta$, with $m \in {}^{*}\mathcal{Z}$, $|\delta| < 1$

Then;

$\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}$

$= \frac{m^2}{\eta}$

$= \frac{(\frac{\sqrt{\eta}y}{\sqrt{c}} - \delta)^2}{\eta}$

Let $\frac{\sqrt{\eta c}y}{\eta c^2} = n + \delta'$, with $n \in {}^{*}\mathcal{Z}$, $|\delta'| < 1$

Then;

$\frac{[\sqrt{\eta c}y]^2}{\eta c^2}$

$= \frac{n^2}{\eta c^2}$

$= \frac{(\sqrt{\eta c}y-\delta')^2}{\eta c^2}$

We have that;

$|\frac{(-4\pi^2 i)[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$= 4\pi^2|\frac{(\sqrt{\frac{\eta y}{\sqrt{c}}}-\delta)^2}{\eta} - \frac{(\sqrt{\eta c}-\delta')^2}{\eta c^2}|$

$= 4\pi^2|\frac{(\frac{\eta y^2}{c}-\frac{2\delta\sqrt{\eta}y}{\sqrt{c}}+\delta^2)}{\eta} - \frac{(\eta c y^2-2\sqrt{\eta c}y\delta'+\delta'^2)}{\eta c^2}|$

$= 4\pi^2|(\frac{y^2}{c} - \frac{2\delta y}{\sqrt{\eta c}} + \frac{\delta^2}{\eta}) - (\frac{y^2}{c} - \frac{2\delta' y}{\sqrt{\eta}c^{\frac{3}{2}}} + \frac{\delta'^2}{\eta c^2})|$

$= 4\pi^2|\frac{2\delta' y}{\sqrt{\eta}c^{\frac{3}{2}}} - \frac{2\delta y}{\sqrt{\eta c}} + \frac{\delta^2}{\eta} - \frac{\delta'^2}{\eta c^2}|$

$\leq 4\pi^2(\frac{2|y|}{\sqrt{\eta}c^{\frac{3}{2}}} + \frac{2|y|}{\sqrt{\eta c}} + \frac{1}{\eta} + \frac{1}{\eta c^2})$

$\leq 4\pi^2(\frac{2\sqrt{c}\eta^{\delta}}{\eta^{\frac{1}{2}}c^{\frac{3}{2}}} + \frac{2\sqrt{c}\eta^{\delta}}{\sqrt{\eta c}} + \frac{1}{\eta} + \frac{1}{\eta c^2})$

$\leq 4\pi^2\frac{2\eta^{\delta-\frac{1}{2}}}{c} + \frac{2\eta^{\delta}}{\sqrt{\eta}} + \frac{1}{\eta} + \frac{1}{\eta c^2}) \simeq 0$

$= 4\pi^2(2\eta^{\delta-\frac{1}{2}}(1+\frac{1}{c}) + \frac{1}{\eta} + \frac{1}{\eta c^2})$

$= 4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{1+c}{c}) + \frac{1}{\eta} + \frac{1}{\eta c^2})$

We have that;

$|(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$\leq |(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)|(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$\leq \frac{2(M+D)}{\sqrt{c}}4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{1+c}{c}) + \frac{1}{\eta} + \frac{1}{\eta c^2}) \simeq 0$

using Lemma 0.79.

$\square$

Lemma 0.84. *If $f \in S(\mathcal{R})$, with the extra assumption that $Re(f)$ and $Im(f)$ and are real analytic. For $0 < \gamma < \frac{1}{2}$, $\gamma \in \mathcal{R}$, $\sqrt{c}(\eta^{\frac{1}{2}-\gamma}) \leq |y| \leq \frac{\sqrt{\eta c}}{4}$;*

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{\eta^{\gamma-\frac{1}{2}}}{2}+\frac{1}{\sqrt{\eta}})+\frac{E}{(n-1)\sqrt{c}}$

where;

$w(n) = Card(Re(f)'|_{[-n,n]} = 0)$, $a(n) = Card(Re(f)|_{[-n,n]} = 0)$, $w'(n) = Card(Im(f)'|_{[-n,n]} = 0)$, $a'(n) = Card(Im(f)|_{[-n,n]} = 0)$ $E = 2C$, $|f| \leq \frac{C}{|x|^2}$, $|x| > 1$.

Proof. We can assume that $Re(f)$ and $Im(f) \neq 0$. Let $n \in \mathcal{N}$, we first prove that;

$|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)| \leq \frac{E}{(n-1)\sqrt{c}} \simeq 0$, $(*)$

where $E \in \mathcal{R}$.

We have, using the definition of $f^{\sqrt{c}}$ in Definition 0.64, that;

$|\int_{(|x|\geq\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq \frac{1}{\sqrt{\eta c}} {}^*\sum_{|k|=[n\sqrt{\eta}]}^{\eta c-1} |f^{\sqrt{c}}(\frac{k}{\sqrt{\eta c}})|$

$= \frac{1}{\sqrt{\eta c}} {}^*\sum_{k=[n\sqrt{\eta}]}^{\eta c-1} |f^*|(\frac{k}{\sqrt{\eta}})$

As $f \in S(\mathcal{R})$, we have that, $|f^*|(x) \leq \frac{C}{|x|^2}$, for $|x| \geq 1$, $C \in \mathcal{R}$. It follows that;

$|f^*|(\frac{k}{\sqrt{\eta}}) \leq \frac{C}{|\frac{k}{\sqrt{\eta}}|^2} = \frac{C\eta}{|k|^2}$, for $|k| \geq [n\sqrt{\eta}]$

Then;

$\frac{1}{\sqrt{\eta c}} {}^*\sum_{k=[n\sqrt{\eta}]}^{\eta c-1} |f^*|(\frac{k}{\sqrt{\eta}})$

$\leq \frac{1}{\sqrt{\eta c}} {}^*\sum_{k=[n\sqrt{\eta}]}^{\eta c-1} \frac{C\eta}{|k|^2}$

$= \frac{2\eta C}{\sqrt{\eta c}} {}^* \sum_{k=[n\sqrt{\eta}]}^{\eta c-1} \frac{1}{k^2}$

$\leq \frac{2\eta C}{\sqrt{\eta c}} \int_{[n\sqrt{\eta}]-1}^{\eta c-1} \frac{dx}{x^2}$ (by transfer)

$= \frac{2C\sqrt{\eta}}{\sqrt{c}} [\frac{-1}{x}]_{[n\sqrt{\eta}]-1}^{\eta c-1}$

$= \frac{2C\sqrt{\eta}}{\sqrt{c}} (\frac{1}{[n\sqrt{\eta}]-1} - \frac{1}{\eta c-1})$

$\leq \frac{2C\sqrt{\eta}}{\sqrt{c}} \frac{1}{n\sqrt{\eta}-2}$

$\leq \frac{2C\sqrt{\eta}}{\sqrt{c}} \frac{1}{(n-1)\sqrt{\eta}}$

$= \frac{2C}{(n-1)\sqrt{c}}$

which gives the result $(*)$, taking $E = 2C$.

We have that;

$\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)$

$\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} exp_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$

$= \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} f^{\sqrt{c}} sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+i \int \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$-i \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$+ \int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}}) sin_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

We have;

$\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}}, \frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}}) cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)$

$= \frac{1}{\sqrt{\eta c}} {}^* \sum_{|\frac{l}{\sqrt{\eta c}}|<\frac{n\sqrt{\eta}}{\sqrt{\eta c}}} Re(f^{\sqrt{c}})(\frac{l}{\sqrt{\eta c}}) cos_{\eta c}(\frac{2\pi lk}{\eta c})$, $(y = \frac{k}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{\eta c}}{}^{*}\sum_{|l|<n\sqrt{\eta}} Re(f^{\sqrt{c}})(\frac{l}{\sqrt{\eta c}})cos_{\eta c}(\frac{2\pi lk}{\eta c})$

$= \frac{1}{\sqrt{\eta c}}{}^{*}\sum_{|l|<n\sqrt{\eta}} Re(f^{*})(\frac{l}{\sqrt{\eta}})cos_{\eta c}(\frac{2\pi lk}{\eta c})$

$= \frac{1}{\sqrt{\eta c}}{}^{*}\sum_{|l|<n\sqrt{\eta}} Re(f^{*})(\frac{l}{\sqrt{\eta}})\theta_{c,k}(\frac{l}{\sqrt{\eta}})$

where $\theta_{c,k}(\frac{l}{\sqrt{\eta}}) = cos_{\eta c}(\frac{2\pi lk}{\eta c})$

We compute an upper bound, for given $k \in {}^{*}\mathcal{Z}$, $n \in \mathcal{N}$, of;

$\frac{1}{\sqrt{\eta c}}{}^{*}\sum_{|l|<n\sqrt{\eta}} Re(f^{*})(\frac{l}{\sqrt{\eta}})\theta_{c,k}(\frac{l}{\sqrt{\eta}}))$

by transfer of the result for;

$\frac{1}{mv}{}^{*}\sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

where $\{v, m\} \subset \mathcal{R}_{>0}$, $r \in \mathcal{Z}$, $n \in \mathcal{N}$, and $\theta_{v^2,r}(\frac{l}{m}) = cos_{v^2m^2}(\frac{2\pi lr}{v^2m^2})$, ($v$ corresponds to $\sqrt{c}$,m to $\sqrt{\eta}$, r to k)

For $x \in \mathcal{R}$, we have that;

$\theta_{v^2,r}(x) = cos_{v^2m^2}(\frac{2\pi rx}{v^2m}) = 0$

iff $\frac{2\pi rx}{v^2m} = \frac{\pi}{2} + t\pi$, ($t \in \mathcal{Z}$

iff $x = \frac{\frac{\pi v^2m}{2}+\pi tv^2m}{2\pi r}$

iff $x = (\frac{1}{4r} + \frac{t}{2r})v^2m$, ($\sharp$)

$\{r, t\} \subset \mathcal{Z}$, $\{v, m\} \subset \mathcal{R}_{>0}$

With the assumption that $|\frac{r}{vm}| \leq \frac{vm}{4}$, we have that $|r| \leq \frac{v^2m^2}{4}$, $\frac{1}{|r|} \geq \frac{4}{v^2m^2}$, $\frac{v^2m}{2|r|} \geq \frac{2}{m} > \frac{1}{m}$, where $\frac{v^2m}{2|r|} = z_2 - z_1$, $\theta_{v^2,r}(z_1) = 0$ and $z_2 = \mu z(z > z_1 : \theta_{v^2,r}(z_2) = 0)$. Fix $\gamma_0 \in \mathcal{R}$, with $0 < \gamma_0 < \frac{1}{2}$, then $\frac{vm}{m^2\gamma_0} \leq \frac{|r|}{vm}$ iff $|r| \geq \frac{v^2}{\gamma_0}$, so we require that, $\frac{v^2}{\gamma_0} \leq |r| \leq \frac{v^2m^2}{4}$, $(*)$, $m^2 \geq \frac{4}{\gamma_0}$. With the assumption that v is odd and m^2 is prime, we have that $v^2m^2(1+2t)$ is odd, so that $\frac{v^2m^2(1+2t)}{4r} \notin \mathcal{Z}$, so if $\theta_{v^2,r}(x_0) = 0$, then $mx_0 \notin \mathcal{Z}$, ($\dagger$).

We claim that $Re(f)$ has finitely many zeroes at $\{x_1,\ldots,x_{a(n)}\}$, with $-n \leq x_1 \leq \ldots \leq x_i \leq \ldots \leq x_{a(n)} \leq n$, $(**)$. Suppose not, then, choose an infinite set of zeroes $\{x_i : i \geq 1\} \subset [-n,n]$, with a convergent subsequence $\{x_{i(j)} : j \geq 1\} \subset [-n,n]$. As $[-n,n]$ is closed, the limit $b \in [-n,n]$, and, as $Re(f)$ is continuous, $f(b) = 0$. As $Re(f)$ is analytic, it is identically zero on an open neighborhood $[b-\epsilon, b+\epsilon]$, then, using the fact that $[-n,n]$ is connected, and, repeating the argument, $Re(f) = 0$ on $[-n,n]$, and, similarly, $f \equiv 0$ on $\mathcal{R}$. Let $x_0 = -n$ and $x_{a(n)+1} = n$. Since, $Re(f)|_{[x_i,x_{i+1}]}$ is differentiable, for $0 \leq i \leq a(n)$, as above, it has finitely many maxima and minima, $\{x_{i,1},\ldots,x_{i,j},\ldots,x_{i,b(i)}\}$, with $x_i \leq x_{i,1} \leq x_{i,j} \leq \ldots \leq x_{i,b(i)} \leq x_{i+1}$, $0 \leq i \leq a(n)$, $1 \leq j \leq b(i)-1$. Let $x_{i,0} = x_i$, for $0 \leq i \leq a(n)+1$, then it follows that $Re(f)|_{[x_{i,j},x_{i,j+1}]}$ is monotone for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i)-1$ and $Re(f)|_{[x_{i,b(i)},x_{i+1,0}]}$ is monotone for $0 \leq i \leq a(n)$.

Without loss of generality, there are four cases to consider, $Re(f) \geq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) < Re(f)(x_{i,j+1})$, $Re(f) \geq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) > Re(f)(x_{i,j+1})$, $Re(f) \leq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) < Re(f)(x_{i,j+1})$, $Re(f) \leq 0$ on $[x_{i,j}, x_{i,j+1}]$, with $Re(f)(x_{i,j}) > Re(f)(x_{i,j+1})$, $0 \leq i \leq i \leq a(n)$, $0 \leq j \leq b(i)-1$, and, similarly, for $[x_{i,b(i)}, x_{i+1,0}]$, $0 \leq i \leq a(n)$.

If $x_0 \in \mathcal{R}$, with $mx_0 \in \mathcal{Z}$, then $m^2x_0^2 \in \mathcal{Z}$. If $\{m_1^2, m_2^2\}$ are prime, $\{n_1,n_2\} \subset \mathcal{Z}$, with $x_0^2 = \frac{n_1}{m_1^2} = \frac{n_2}{m_2^2}$, then $n_1m_2^2 = n_2m_1^2$, $n_1 = m_1^2$, $n_2 = m_2^2$, and $|x_0| = 1$. As $Card(\{x_{i,j} : 1 \leq i \leq a(n)-1, 1 \leq j \leq b(i)-1\})$ and $Card(\{x_i : 1 \leq i \leq a(n)-1\})$ are finite, we have that for sufficiently large, and $m \notin \mathcal{Z}$, $mx_i \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$ and $mx_{i,j} \notin \mathcal{Z}$, for $1 \leq i \leq a(n)-1$, $1 \leq j \leq b(i)-1$, $(\dagger\dagger)$.

Let $\{r_{ijs} : 1 \leq s \leq e(i,j)\}$ enumerate the zeroes of $\theta_{v^2,r}$ on $[x_{i,j}, x_{i,j+1}]$, $r_{ij0} = x_{i,j}$, $r_{ij(e(i,j)+1))} = x_{i,j+1}$, for $0 \leq i \leq a(n)$, $0 \leq j \leq b(i)-1$, and let $\{r_{is} : 1 \leq s \leq e(i)\}$ enumerate the zeroes of $\theta_{v^2,r}$ on $[x_{i,b(i)}, x_{i+1,0}]$, $r_{i0} = x_{i,b(i)}$, $r_{i(e(i)+1)} = x_{i+1}$, $0 \leq i \leq a(n)$. Then, using $(\dagger),(\dagger\dagger)$, we have that;

$$\frac{1}{mv}{}^*\sum_{|l|<nm} Re(f)(\tfrac{l}{m})\theta_{v^2,r}(\tfrac{l}{m})$$

$$= \frac{1}{mv}{}^*\sum_{i=0}^{a(n)}{}^*\sum_{l=[mx_i]+1}^{[mx_{i+1}]-1} Re(f)(\tfrac{l}{m})\theta_{v^2,r}(\tfrac{l}{m})$$

$$= \frac{1}{mv}{}^*\sum_{i=0}^{a(n)}{}^*\sum_{j=0}^{b(i)-1}{}^*\sum_{l=[mx_{i,j}]+1}^{[mx_{i,j+1}]-1} Re(f)(\tfrac{l}{m})\theta_{v^2,r}(\tfrac{l}{m})$$

$+\frac{1}{mv}{}^*\sum_{i=0}^{a(n)*}\sum_{l=[mx_{i,b(i)}]+1}^{[mx_{i+1,0}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$=\frac{1}{mv}{}^*\sum_{i=0}^{a(n)*}\sum_{j=0}^{b(i)-1*}\sum_{s=0}^{e(i,j)*}\sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

$+\frac{1}{mv}{}^*\sum_{i=0}^{a(n)*}\sum_{s=0}^{e(i)}{}^*\sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

We compute $\frac{1}{mv}{}^*\sum_{s=1}^{e(i,j)-1*}\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

Let $\theta_{i,j}(s)=\frac{1}{mv}{}^*\sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}-1]}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})$

We consider Case 2, $Re(f)\geq 0$ on $[x_{i,j},x_{i,(j+1)}]$, with $Re(f)(x_{i,j})>Re(f)(x_{i,(j+1)})$. Assume, without loss of generality, that $\theta_{v^2,r}|_{[r_{ij1},r_{ij2}]}>0$, (†††), then, as $Re(f)|_{[x_{i,j},x_{i,(j+1)}]}\geq 0$, and;

$\frac{\theta_{v^2,r}|_{[r_{ijs},r_{ij(s+1)}]}}{|\theta_{v^2,r}|_{[r_{ijs},r_{ij(s+1)}]}|}=-\frac{\theta_{v^2,r}|_{[r_{ij(s+1)},r_{ij(s+2)}]}}{|\theta_{v^2,r}|_{[r_{ij(s+1)},r_{ij(s+2)}]}|}$, $1\leq s\leq e(i,j)-2$

the sequence $\{\theta_{i,j}(s):1\leq s\leq e(i,j)-1\}$ is alternating. As $Re(f)|_{[x_{i,j},x_{i,(j+1)}]}$ is decreasing, we have that;

$|\theta_{i,j}(s)|\geq|\theta_{i,j}(s+1)|$, for $1\leq s\leq e(i,j)-2$

so that sequence $\{\theta_{i,j}(s):1\leq s\leq e(i,j)-1\}$ is decreasing in magnitude.

We show that $0\leq\theta_{i,j}(1)\leq l_{i,j}$, $l_{i,j}\in\mathcal{R}$, where $l_{i,j}=\theta_{i,j}(1)>0$.

Assume that $e(i,j)$ is odd. We claim that for all sequences $\{\theta_{i,j}(s):1\leq s\leq e(i,j)-1\}$, decreasing in magnitude, with $l_{i,j}>0$, and $e(i,j)$ odd, that ${}^*\sum_{s=1}^{e(i,j)}-1\theta_{i,j}(s)\leq l_{i,j}$. We can prove this by induction, the base case is trivial. Assume true for $e(i,j)$, and consider the sequence $\{\theta_{i,j}(s):1\leq s\leq e(i,j)+1\}$. We have that $\theta_{i,j}(3)>0$, and the sequence $\{\theta_{i,j}(s):3\leq s\leq e(i,j)+1\}$ is alternating and decreasing in magnitude. By the induction hypothesis, we have that;

${}^*\sum_{s=3}^{e(i,j)}+1\theta_{i,j}(s)\leq\theta_{i,j}(3)$

Hence, ${}^*\sum_{s=1}^{e(i,j)}+1\theta_{i,j}(s)$

$=\theta_{i,j}(1)+\theta_{i,j}(2)+{}^*\sum_{s=3}^{e(i,j)}+1\theta_{i,j}(s)$

$\leq \theta_{i,j}(1) + \theta_{i,j}(2) + \theta_{i,j}(3)$

$\leq \theta_{i,j}(1)$

as $\theta_{i,j}(2) + \theta_{i,j}(3) \leq 0$

We have that $0 \leq \theta_{i,j}(1) \leq l_{i,j}$, $l_{i,j} \in \mathcal{R}$.

As;

$\theta_{i,j}(2p-1) + \theta_{i,j}(2p) \geq 0$, for $1 \leq p \leq \frac{e(i,j)-1}{2}$

$0 \leq {}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)$

therefore, $0 \leq {}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$, $(***)$

Assume that $e(i,j)$ is even, then as;

$\theta_{i,j}(2p) + \theta_{i,j}(2p+1) \leq 0$, for $1 \leq p \leq \frac{e(i,j)-2}{2}$

so that;

${}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$

and, as;

$\theta_{i,j}(2p-1) + \theta_{i,j}(2p) \geq 0$, for $1 \leq p \leq \frac{e(i,j)-2}{2}$

$\theta_{i,j}(e(i,j)-1) \geq 0$

therefore;

$0 \leq {}^*\sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s) \leq l_{i,j}$, $(****)$

We compute;

$l_{i,j} = \frac{1}{mv} {}^*\sum_{l=[mr_{ij1}]}^{[mr_{ij2}]} Re(f)(\frac{l}{m}) \theta_{v^2,r}(\frac{l}{m})$

$\leq \frac{1}{mv} {}^*\sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1} D$, where $|Re(f)| \leq D$, and $0 \leq \theta_{v^2,r}|_{[r_{ij1},r_{ij2}]} \leq 1$

$\leq \frac{D}{mv}(([mr_{ij2}]-1) - ([mr_{ij1}]+1) + 1)$

$= \frac{D}{mv}([mr_{ij2}] - [mr_{ij1}] - 1)$

$\leq \frac{D}{mv}((mr_{ij2} + 1) - (mr_{ij1} - 1) - 1)$

$= \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

so that, using $(***), (****)$;

$0 \leq \frac{1}{mv} {}^*\sum_{s=1}^{e(i,j)-1} {}^*\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}) \leq l_{i,j}$

$\leq \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

$= \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$, using $(\sharp)$

We can remove the assumption $(\dagger\dagger\dagger)$, to obtain;

$\frac{1}{mv}|{}^*\sum_{s=1}^{e(i,j)-1} {}^*\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})| \leq \frac{D}{v}((r_{ij2} - r_{ij1}) + \frac{1}{m})$

Cases 4 is similar, with the same bound. For Cases 1 and 3, reversing the sequences, we obtain;

$\frac{1}{mv}|{}^*\sum_{s=1}^{e(i,j)-1} {}^*\sum_{l=[mx_{ijs}]+1}^{[mx_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{D}{v}((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{m})$

$\leq \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

Similarly, considering all 4 cases,we obtain the same bound;

$\frac{1}{mv}|{}^*\sum_{s=1}^{e(i)-1} {}^*\sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})| \leq \frac{D}{v}(\frac{v^2 m}{2r} + \frac{1}{m})$

We have that;

$\frac{1}{mv}|{}^*\sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv} {}^*\sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} D$

$\leq \frac{D}{mv}(([mr_{ij1}] - 1) - ([mr_{ij0}] + 1) + 1)$

$= \frac{D}{mv}([mr_{ij1}] - [mr_{ij0}] - 1)$

$\leq \frac{D}{mv}((mr_{ij1}+1)-(mr_{ij0}-1)-1)$

$= \frac{D}{v}((r_{ij1}-r_{ij0})+\frac{1}{m})$

$= \frac{D}{v}((r_{ij1}-x_{ij})+\frac{1}{m})$

$\leq \frac{D}{v}((r_{ij1}-r_{i(j-1)b(j-1)})+\frac{1}{m})$

$= \frac{D}{v}(\frac{v^2m}{2r}+\frac{1}{m})$

and, similarly;

$max(A_{i,j}, B_{i,j}, C_i, D_i)$

$\leq \frac{D}{v}(\frac{v^2m}{2r}+\frac{1}{m})$

where;

$A_{i,j} = \frac{1}{mv}|^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$B_{i,j} = \frac{1}{mv}|^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$C_i = \frac{1}{mv}|^* \sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$D_i = \frac{1}{mv}|^* \sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|)$

It follows that;

$\frac{1}{mv}|^* \sum_{|l|<nm} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv}|^* \sum_{i=0}^{a(n)} {}^* \sum_{j=0}^{b(i)-1} {}^* \sum_{s=0}^{e(i,j)} {}^* \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)} {}^* \sum_{s=0}^{e(i)} {}^* \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$\leq \frac{1}{mv}|^* \sum_{i=0}^{a(n)} {}^* \sum_{j=0}^{b(i)-1} {}^* \sum_{s=1}^{e(i,j)-1} {}^* \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)} {}^* \sum_{j=0}^{b(i)-1} ({}^* \sum_{l=[mr_{ij0}]+1}^{[mr_{ij1}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}) + {}^* \sum_{l=[mr_{ije(i,j)}]+1}^{[mr_{ij(e(i,j)+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}))|$

$+\frac{1}{mv}|^* \sum_{i=0}^{a(n)} {}^* \sum_{s=1}^{e(i)-1} {}^* \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})|$

$+\frac{1}{mv}|^*\sum_{i=0}^{a(n)}(^*\sum_{l=[mr_{i0}]+1}^{[mr_{i1}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m})+^*\sum_{l=[mr_{ie(i)}]+1}^{[mr_{i(e(i)+1)}]-1}Re(f)(\frac{l}{m})\theta_{v^2,r}(\frac{l}{m}))|$

$\leq {}^*\sum_{i=0}^{a(n)*}\sum_{j=0}^{b(i)-1}(\frac{D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$+^*\sum_{i=0}^{a(n)*}\sum_{j=0}^{b(i)-1}(\frac{2D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$+^*\sum_{i=0}^{a(n)}(\frac{D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$+^*\sum_{i=0}^{a(n)}(\frac{2D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$= {}^*\sum_{i=0}^{a(n)*}\sum_{j=0}^{b(i)-1}(\frac{3D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$+^*\sum_{i=0}^{a(n)}(\frac{3D}{v}(\frac{v^2m}{2r}+\frac{1}{m}))$

$(w(n)+a(n)+1)\frac{3D}{v}(\frac{v^2m}{2r}+\frac{1}{m})$

where $w(n)=Card(Re(f)'|_{[-n,n]}=0)$, $a(n)=Card(Re(f)|_{[-n,n]}=0)$

It follows, by transfer, that;

$|\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})}Re(f^{\sqrt{c}})cos_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$=\frac{1}{\sqrt{\eta c}}|^*\sum_{|l|<n\sqrt{\eta}}Re(f^*)(\frac{l}{\sqrt{\eta}})\theta_{c,k}(\frac{l}{\sqrt{\eta}}))|$

$\leq (w(n)+a(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

and, similarly;

Similarly;

$B_n)\leq (w(n)+a(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

and;

$max(A_n,C_n)\leq (w'(n)+a'(n)+1)\frac{3D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

where;

$w'(n) = Card(Im(f)'|_{[-n,n]} = 0)$ and $a'(n) = Card(Im(f)|_{[-n,n]} = 0)$

$A_n = |i\int\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}})cos_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$B_n = |-i\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Re(f^{\sqrt{c}})sin_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

$C_n = |\int_{(\frac{-n\sqrt{\eta}}{\sqrt{\eta c}},\frac{n\sqrt{\eta}}{\sqrt{\eta c}})} Im(f^{\sqrt{c}})sin_{\eta c}(2\pi xy)d\mu_{\eta c}(x)|$

It follows that;

$|\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2k}+\frac{1}{\sqrt{\eta}})$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{c\sqrt{\eta}}{2}\frac{1}{c\eta^{1-\gamma}}+\frac{1}{\sqrt{\eta}})$

$= (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{\sqrt{\eta}}{2\eta^{1-\gamma}}+\frac{1}{\sqrt{\eta}})$

as;

$|\frac{k}{\sqrt{\eta c}}| \geq \sqrt{c}(\eta^{\frac{1}{2}-\gamma})$

$|k| \geq c\eta^{1-\gamma}$

$\frac{1}{|k|} \leq \frac{1}{c\eta^{1-\gamma}}$

so that;

$|\int_{\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq |\int_{(|x|<\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|+|\int_{(|x|>\frac{[n\sqrt{\eta}]}{\sqrt{\eta c}})\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{\sqrt{\eta}}{2\eta^{1-\gamma}}+\frac{1}{\sqrt{\eta}})+\frac{E}{(n-1)\sqrt{c}}$

$= (w(n)+a(n)+w'(n)+a'(n)+2)\frac{6D}{\sqrt{c}}(\frac{\eta^{\gamma-\frac{1}{2}}}{2}+\frac{1}{\sqrt{\eta}})+\frac{E}{(n-1)\sqrt{c}}$

□

Lemma 0.85. *If $f\in S(\mathcal{R})$, then for $|y|\geq 1$;*

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq \frac{16L}{\sqrt{c}|y|^2}$

In particular, for $|y| \geq \frac{\sqrt{\eta c}}{4}$*;*

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq \frac{64L}{\sqrt{\eta c}}$

where $L \in \mathcal{R}$*.*

Proof. We have, for $\frac{-(\eta c-1)}{2} \leq i \leq \frac{(\eta c-5)}{2}$, that;

$(f^{\sqrt{c}})^{D^2}(\frac{i}{\sqrt{\eta c}})$

$= \eta c(f^{\sqrt{c}}(\frac{i+2}{\sqrt{\eta c}}) - 2f^{\sqrt{c}}(\frac{i+1}{\sqrt{\eta c}}) + f^{\sqrt{c}}(\frac{i}{\sqrt{\eta c}}))$

$= \eta c(f^*)(\frac{i+2}{\sqrt{\eta}}) - 2(f^*(\frac{i+1}{\sqrt{\eta}}) + f^*(\frac{i}{\sqrt{\eta}})$

$= cf_{\eta,c}^{D^2}(\frac{i}{\sqrt{\eta}})$

where, for $\frac{(-\eta c-1)}{2} \leq i \leq \frac{(\eta c-1)}{2}$, $f_{\eta,c} \in V(R_{\eta,c})$ is defined by;

$f_{\eta,c}(\frac{i}{\sqrt{\eta}}) = f^*(\frac{i}{\sqrt{\eta}})$

A straightforward adaptation of Lemma 0.21 shows that there exists $G \in \mathcal{R}_{>0}$, with $|(f_{\eta,c})^{D^2}| \leq \frac{G}{|x_{\eta,c}|^2}$, for $x \in R_{\eta,c}$, $|x| > 1$, where;

$x_{\eta,c}(\frac{i}{\sqrt{\eta}}) = x^*(\frac{i}{\sqrt{\eta}})$

It follows that;

$|(f^{\sqrt{c}})^{D^2}(\frac{i}{\sqrt{\eta c}})|$

$= c|f_{\eta,c}^{D^2}(\frac{i}{\sqrt{\eta}})|$

$\leq \frac{Gc}{|\frac{i}{\sqrt{\eta}}|^2} = \frac{Gc\eta}{i^2} = \frac{G}{|x_{\eta c}|^2}$, $(*)$

for $|i| > [\sqrt{\eta}]$, as $|x_{\eta c}|^2(\frac{i}{\sqrt{\eta c}}) = \frac{i^2}{\eta c}$. Using Lemma 0.19, we have that;

$\mathcal{F}_{\eta c}((f^{\sqrt{c}})^{D^2})(y) = \chi_\eta^2(y)V_\eta^2(y)\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)$

It follows for $|y| \geq 1$, that;

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq \frac{|\mathcal{F}_{\eta c}(f^{\sqrt{c}})^{D^2}(y)|}{|\chi_\eta^2(y)V_\eta^2(y)|}$, $(**)$

We have, using $(*)$, that;

$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})^{D^2}(y)|$

$= |\int_{\overline{\mathcal{R}_{\eta c}}}(f^{\sqrt{c}})^{D^2} exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)|$

$\leq \int_{|x|\leq\frac{[\sqrt{\eta}]}{\sqrt{\eta c}}\cap\overline{\mathcal{R}_{\eta c}}}|(f^{\sqrt{c}})^{D^2}|d\mu_{\eta c}(x) + \int_{|x|>\frac{[\sqrt{\eta}]}{\sqrt{\eta c}}\cap\overline{\mathcal{R}_{\eta c}}}|(f^{\sqrt{c}})^{D^2}|d\mu_{\eta c}(x)$

$\leq \int_{|x|\leq\frac{[\sqrt{\eta}]}{\sqrt{\eta c}}\cap\overline{\mathcal{R}_{\eta c}}}|(f^{\sqrt{c}})^{D^2}|d\mu_{\eta c}(x) + \int_{|x|>\frac{[\sqrt{\eta}]}{\sqrt{\eta c}}\cap\overline{\mathcal{R}_{\eta c}}}\frac{G}{|x_{\eta c}|^2}d\mu_{\eta c}(x)$

$= \frac{1}{\sqrt{c}}\int_{(|x|\leq 1)\cap\overline{\mathcal{R}_{\eta,c}}}c|(f_{\eta,c}^{D^2})|d\mu_{\eta,c}(x) + \frac{1}{\sqrt{\eta c}}{}^*\sum_{|i|=[\sqrt{\eta}]+1}^{\frac{(\eta c-1)}{2}}\frac{G}{|x_{\eta c}|^2}$

$\leq \frac{M}{\sqrt{c}} + \frac{2}{\sqrt{\eta c}}\int_{[\sqrt{\eta}]}^{\frac{(\eta c-1)}{2}}\frac{G}{x^2}$ (by transfer, and, using the fact that $(f_{\eta,c})^{D^2} \leq M$, see Lemma 0.17.)

$= \frac{M}{\sqrt{c}} + \frac{2}{\sqrt{\eta c}}[\frac{-G}{x}]_{[\sqrt{\eta}]}^{\frac{(\eta c-1)}{2}}$

$= \frac{M}{\sqrt{c}} + \frac{2}{\sqrt{\eta c}}(\frac{G}{[\sqrt{\eta}]} - \frac{2G}{(\eta c-1)})$

$\leq \frac{M}{\sqrt{c}} + \frac{2G}{\sqrt{\eta c}(\frac{\sqrt{\eta}}{2})}$

$= \frac{M}{\sqrt{c}} + \frac{4G}{\eta\sqrt{c}}$

$\leq \frac{M+4G}{\sqrt{c}}$, $(***)$

Using $(**), (***)$, we have
$|\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq \frac{M+4G}{\sqrt{c}|\chi_\eta^2(y)V_\eta^2(y)|}$

$= \frac{M+4G}{\sqrt{c}|\chi_\eta^2(y)|}$

$\leq \frac{16(M+4G)}{\sqrt{c}|y|^2}$

using Lemma 0.20. The result follows, taking $L = M + 4G$.

□

Lemma 0.86. *For $\kappa, \lambda \subset \overline{R}_\eta$, with $1 \leq \kappa \leq \lambda$, and $f_\eta \in V(\overline{\mathcal{R}_\eta})$, with $|f_\eta| \leq \frac{C}{|y_\eta}|^2$, for $|y| \geq \kappa$, and $C \in {}^*\mathcal{R}_{>0}$, then we have that;*

$|\int_{\kappa\leq|y|\leq\lambda} f_\eta d\mu_\eta(y)|$

$\leq \frac{4C}{\kappa}$

Proof. We have that;

$|\int_{\kappa\leq|y|\leq\lambda} f_\eta d\mu_\eta(y)|$

$\leq \int_{\kappa\leq|y|\leq\lambda} |f_\eta| d\mu_\eta$

$\leq \int_{\kappa\leq|y|\leq\lambda} \frac{C}{|y_\eta|^2} d\mu_\eta$

$\leq C\int_{|y|\geq\kappa} \frac{1}{|y_\eta|^2} d\mu_\eta$

$\leq \frac{2C}{\sqrt{\eta}} * \sum_{i\geq[\kappa]\sqrt{\eta}]} \frac{1}{(\frac{i}{\sqrt{\eta}})^2}$

$= \frac{2C\eta}{\sqrt{\eta}} * \sum_{i\geq[\kappa\sqrt{\eta}]} \frac{1}{i^2}$

$\leq 2C\sqrt{\eta}\int_{[\kappa\sqrt{\eta}]-1}^{\eta-1} \frac{dx}{x^2}$

$= \frac{2\sqrt{\eta}}{[\kappa\sqrt{\eta}]-1}$

$\leq \frac{2C\sqrt{\eta}}{\kappa\sqrt{\eta}-2}$

$\leq \frac{2C\sqrt{\eta}}{(\kappa\sqrt{\eta}-\frac{\kappa\sqrt{\eta}}{2})}$

$= \frac{4C}{\kappa}$

□

Lemma 0.87. *let $f \in S(\mathcal{R})$, and, let $\{h_c, g_{\sqrt{c}}\}$ be as in Definition 0.71 above, then, for finite t, for $|y| \geq \frac{\sqrt{\eta c}}{4}$, $x \in \overline{\mathcal{R}}_{\eta c}$, we have;*

$|(h_c(t,y) - g_{\sqrt{c}}(t,y))| \leq 2;$

$|(h_c - g_{\sqrt{c}})(t,y)\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)| \leq \frac{32L}{\sqrt{c}|y|^2}$

and $|\int_{|y|\geq\frac{\sqrt{\eta c}}{4}}((h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)|$

$\leq \frac{512L}{\sqrt{\eta}c}$

for $1 \leq |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c}$, $t \in \overline{\mathcal{T}_\nu}$;

$|(h_c - g_{\sqrt{c}}(t,y))|$

$\leq 2t(\frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma})$;

$|(h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}})|(t,y)|$

$\leq \frac{16L}{\sqrt{\eta}c|y|^2}(2t(\frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma}))$

and $|\int_{1\leq|y|\leq\eta^{\frac{1}{2}-\gamma}\sqrt{c}}((h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)|$

$\leq \frac{64L}{\sqrt{\eta}c}(2t(\frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma}))$

and for $\eta^{\frac{1}{2}-\gamma}\sqrt{c} \leq |y| \leq \frac{\sqrt{\eta c}}{4}$

$|(h_c - g_{\sqrt{c}}(y))| \leq 2$;

$|(h_c - g_{\sqrt{c}}\mathcal{F}_{\eta c}(f^{\sqrt{c}})|(y)| \leq \frac{32L}{\sqrt{c}|y|^2}$

and $|\int_{\eta^{\frac{1}{2}-\gamma}\sqrt{c}\leq|y|\leq\frac{\sqrt{\eta c}}{4}}((h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)| \leq \frac{128L}{\eta^{\frac{1}{2}-\gamma}c}$

Proof. Using Definition 0.71, for finite $t \in \overline{T}_\nu$, $|y| \geq \frac{\sqrt{\eta c}}{4}$, we have that;

$|(h_c - g_{\sqrt{c}}(t,y))|$

$= |{}^*exp(\frac{-4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2}) - {}^*exp(\frac{-4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta})| \leq 2$

It follows, using Lemma 0.85, that, for $|y| \geq \frac{\sqrt{\eta c}}{4} \geq 1$;

$|(h_c - g_{\sqrt{c}}\mathcal{F}_{\eta c}(f^{\sqrt{c}})|(y)|$

$\leq 2.\frac{16L}{\sqrt{c}|y|^2}$

$= \frac{32L}{\sqrt{c}|y|^2}$

Applying Lemma 0.86, and using the fact that $|((h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)| \leq \frac{32L}{\sqrt{c}|y|^2}$, for $y \geq \frac{\sqrt{\eta c}}{4}$, $x \in \overline{\mathcal{R}}_\eta$, we have that;

$|\int_{|y|\geq\frac{\sqrt{\eta c}}{4}}((h_c - g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)|$

$\leq \frac{4.32L}{\sqrt{c}}\frac{4}{\sqrt{\eta c}}$

$= \frac{512}{\sqrt{\eta}c}$

For the second part, using Lemma 0.70, we have for $1 \leq |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c}$, that;

$|\mathcal{F}_{\eta c}(K_c) - \frac{-4\pi^2 i[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}|$

$\leq \frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma}$

It follows, for $t \in \overline{\mathcal{T}_\nu}$ finite, $1 \leq |y| \leq \eta^{\frac{1}{2}-\gamma}\sqrt{c}$, that;

$| - \frac{4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2} - \frac{4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}$

$\leq t(\frac{16\pi^2}{\eta c} + 4\pi^2 + \frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}} + \frac{4\pi^2}{\eta} + \frac{16\pi^2}{\eta\sqrt{c}} + \frac{8\pi^2}{\eta^\gamma})$

For $\{z, w\} \subset \mathcal{C}$, with $|z - w| < \frac{1}{2}$, and w imaginary, we have that;

$|e^z - e^w|$

$= |e^w(e^{z-w} - 1)|$

$= |\sum_{n=0}^{\infty}\frac{(z-w)^n}{n!} - 1|$

$= \sum_{n=1}^{\infty}\frac{|z-w|^n}{n!}|$

$\leq \sum_{n=1}^{\infty}|z - w|^n|$

$= \frac{|z-w|}{(1-|z-w|)}$

$\leq 2|z - w|$

It follows, by transfer, that for $\{z, w\} \subset {}^*\mathcal{C}$, with w imaginary, and $z \simeq w$, that;

$|{}^*exp(z) - {}^*exp(w)| \leq 2|z - w|$

taking $z = -\frac{4\pi^2 it[\sqrt{\eta c}y]^2}{\eta c^2}$, $w = \frac{4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}$, we obtain that;

$|h_c(t,y)-g_{\sqrt{c}}|$

$=|{}^*exp(-\frac{4\pi^2 it[\sqrt{\eta}cy]^2}{\eta c^2})-{}^*exp(\frac{4\pi^2 it[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta})|$

$\leq 2t(\frac{16\pi^2}{\eta c}+4\pi^2+\frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma})$

It follows, using Lemma 0.85 that, for $1\leq|y|\leq\eta^{\frac{1}{2}-\gamma}\sqrt{c}$, $t\in\overline{\mathcal{T}_\nu}$ finite;

$|(h_c(t,y)-g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}})(y)|$

$\leq\frac{16L}{\sqrt{\eta}c|y|^2}(2t(\frac{16\pi^2}{\eta c}+4\pi^2+\frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma}))$

We then have, applying Lemma 0.86 that;

$|\int_{1\leq|y|\leq\eta^{\frac{1}{2}-\gamma}\sqrt{c}}((h_c-g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)|$

$\leq\frac{64L}{\sqrt{\eta}c}(2t(\frac{16\pi^2}{\eta c}+4\pi^2+\frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma}))$

For the last part, for $\eta^{\frac{1}{2}-\gamma}\sqrt{c}\leq|y|\leq\frac{\sqrt{\eta c}}{4}$, as above;

$|(h_c(t,y)-g_{\sqrt{c}}(t,y))|\leq 2;$

$|(h_c-g_{\sqrt{c}}\mathcal{F}_{\eta c}(f^{\sqrt{c}})|(y)|\leq\frac{32L}{\sqrt{c}|y|^2}$

Applying Lemma 0.86 again, and using the fact that;

$|((h_c-g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)|\leq\frac{32L}{\sqrt{c}|y|^2}$

for $y\geq\eta^{\frac{1}{2}-\gamma}\sqrt{c}\geq 1$, $x\in\overline{\mathcal{R}}_\eta$ we have that;

$|\int_{\eta^{\frac{1}{2}-\gamma}\sqrt{c}\leq|y|\leq\frac{\sqrt{\eta c}}{4}}((h_c-g_{\sqrt{c}})\mathcal{F}_{\eta c}(f^{\sqrt{c}}))exp_{\eta c}(2\pi ixy)d\mu_{\eta c}(y)|\leq\frac{128L}{\eta^{\frac{1}{2}-\gamma}c}$

$\leq\frac{32L}{\sqrt{c}}\frac{4}{\eta^{\frac{1}{2}-\gamma}\sqrt{c}}$

$=\frac{128L}{\eta^{\frac{1}{2}-\gamma}c}$

□

Lemma 0.88. *Let $f\in S(\mathcal{R})$, and $g_{\sqrt{c}}$ as in Lemma 0.87, then, for $x\in\overline{\mathcal{R}}_{\eta c}$, $\nu\geq(\eta c)^5$, and the convolution equation;*

$\frac{\partial F}{\partial t} - K_c * F = 0$ *on* $\overline{\mathcal{R}_{\eta c}} \times \overline{\mathcal{T}_\nu}$

with initial condition $f \in V(\overline{\mathcal{R}_{\eta c}})$, *we have that, for finite* $t \in \overline{T}_\nu$, $|y| \geq \frac{\sqrt{\eta c}}{4}$;

$|\mathcal{F}_{\eta c}(y,t) - g_{\sqrt{c}}(y,t)| \leq ?$

$|\int_{|y| \geq \frac{\sqrt{\eta c}}{4}} (\mathcal{F}_{\eta c}(y,t) - g_{\sqrt{c}}(y,t)) \mathcal{F}_{\eta c} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(y)| \leq ?$

for finite $t \in \overline{T}_\nu$, $1 \leq |y| \geq \eta^{\frac{1}{2}-\gamma}$, $\gamma > 0$;

$|\mathcal{F}_{\eta c}(y,t) - g_{\sqrt{c}}(y,t)| \leq ?$

$|\int_{1 \leq |y| \leq \eta^{\frac{1}{2}-\gamma}} (\mathcal{F}_{\eta c}(y,t) - g_{\sqrt{c}}(y,t)) \mathcal{F}_{\eta c} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(y)| \leq ?$

Proof. Using Lemmas 0.87, Lemma 0.33, $\nu \geq (\eta c)^5$, and footnote 6 of [7],we have, for finite $t \in \overline{T}_\nu$, $1 \leq |y| \geq \eta^{\frac{1}{2}-\gamma}$ that;

$|(1 - \frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2 \nu})^\nu - exp^*(\frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2})| \leq \frac{1}{\nu^{\frac{1}{2}}}$

$|(1 - \frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2 \nu})^{\nu t} - exp^*(\frac{4\pi^2 it[\sqrt{\eta c} y]^2}{\eta c^2})| \leq 4et \frac{1}{\nu^{\frac{1}{2}} |exp^*(\frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2})|}$

$= \frac{4et}{\nu^{\frac{1}{2}}}$

for ${}^\circ t > 0$, $\frac{1}{\nu^{\frac{1}{2}}} \leq min(\frac{|-exp^*(\frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2})}{2}|, \frac{|-exp^*(\frac{4\pi^2 i[\sqrt{\eta c} y]^2}{\eta c^2})}{4t}|) = min(\frac{1}{2}, \frac{1}{4t})$, which is satisfied with ν infinite. For $\{z, w\} \subset \mathcal{C}$, using the above calculation in Lemma 0.87, we have that;

$|z^w - 1|$

$= |e^{wlog(z)} - 1|$

$\leq 2|wlog(z)|$

$= 2|w||log(1 + (z-1))|$

$\leq 2|w||\sum_{n=1}^\infty \frac{(-1)^{n+1}(z-1)^n}{n}|$

$\leq 2|w||z-1|$

It follows that, using transfer, that;

$|(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{\nu t}-(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}$

$=|(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}((1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{\nu t-[\nu t]}-1)$

$\leq 2|2|\nu t-[\nu t]||(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})-1|$

$\leq 4.1.|\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu}|$

$\leq 4.4\pi^2.\frac{1}{c\nu}(\frac{\sqrt{\eta c}}{2}^2)$

$=\frac{16\pi^2\eta}{4\nu}=\frac{4\pi^2\eta}{\nu}$

Then;

$|\mathcal{F}_{\eta c}(y,t)-g_{\sqrt{c}}(y,t)|$

$=|(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}-g_{\sqrt{c}}(y,t)|$

$\leq|(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}-h_c(t,y)|+|h_c(t,y)-g_{\sqrt{c}}|$

$\leq|(1-\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2\nu})^{[\nu t]}-exp^*(\frac{4\pi^2 i[\sqrt{\eta c}y]^2}{\eta c^2})|$

$+2t(\frac{16\pi^2}{\eta c}+4\pi^2+\frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma})$

$\leq\frac{4\pi^2\eta}{\nu}+2t(\frac{16\pi^2}{\eta c}+4\pi^2+\frac{8\pi^2}{\eta^\gamma}\frac{1}{c^{\frac{1}{2}}}+\frac{4\pi^2}{\eta}+\frac{16\pi^2}{\eta\sqrt{c}}+\frac{8\pi^2}{\eta^\gamma})$

□

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