A NONSTANDARD VERSION OF THE FOKKER-PLANCK EQUATION

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Abstract. We derive a nonstandard version of the Fokker-Planck equation.

Definition 0.1. *We let $v \in {}^*N \setminus \mathcal{N}$, and $\eta = 2^\nu$.*

We let $\overline{\Omega_\eta} = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$

and $\overline{\mathcal{T}_\nu} = \{t \in {}^*\mathcal{R} : 0 \leq x \leq 1\}$

We let $\mathcal{C}_\eta$ consist of internal unions of the intervals $[\frac{i}{\eta}, \frac{i+1}{\eta})$, for $0 \leq i \leq \eta - 1$, and let $\mathcal{D}_\nu$ consist of internal unions of $[\frac{i}{\nu}, \frac{i+1}{\nu})$ and $\{1\}$, for $0 \leq i \leq \nu - 1$.

We define counting measures μ_η and λ_ν on $\mathcal{C}_\eta$ and $\mathcal{D}_\nu$ respectively, by setting $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$, for $0 \leq i \leq \eta - 1$, $0 \leq i \leq \nu - 1$ respectively, and $\lambda_\nu(\{1\}) = 0$.

We let $(\overline{\Omega_\eta}, \mathcal{C}_\eta, \mu_\eta)$ and $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$ be the resulting $*$-finite measure spaces, in the sense of [2].

Definition 0.2. *We let $V(\mathcal{C}_\eta) = \{f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}, f(x) = f(\frac{[\eta x]}{\eta})\}$ and $W(\mathcal{C}_\eta) \subset V(\mathcal{C}_\eta)$ be the set of measurable functions $f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, with respect to $\mathcal{C}_\eta$, in the sense of* [2].

Lemma 0.3. *$W(\mathcal{C}_\eta)$ is a $*$-finite vector space over ${}^*\mathcal{C}$, of dimension η, ([1]).*

[1] By a $*$-vector space, one means an internal set V, for which the operations $+ : V \times V \rightarrow V$ of addition and scalar multiplication $. : {}^*\mathcal{C} \times V \rightarrow V$ are internal. Such spaces have the property that $*$-finite linear combinations ${}^*\Sigma_{i \in I} \lambda_i . v_i$, $(*)$, for a $*$-finite index set I, belong to V, by transfer of the corresponding standard result for vector spaces. We say that V is a $*$-finite vector space, if there exists a $*$-finite index set I and elements $\{v_i : i \in I\}$ such that every $v \in V$ can be written as a combination $(*)$, and the elements $\{v_i : i \in I\}$ are independent, in the sense that if $(*) = 0$, then each $\lambda_i = 0$. It is clear, by transfer of the corresponding result for

Proof. This follows straightforwardly, by transfer, from the corresponding result for $\mathcal{C}$-valued functions on finite sets. Namely, any $\mathcal{C}$-valued $f:\mathcal{N}\cap[1,n]\rightarrow\mathcal{C}$ can be written as $\sum_{i=1}^{n}\lambda_i e_i$, where $e_i(j)=\delta_{ij}$, for $1\leq j\leq n$, and $\lambda_i=f(i)$, for $1\leq i\leq n$. □

Definition 0.4. *Given $n\in\mathcal{N}_{>0}$, we let $\Omega_n=\{m\in\mathcal{N}:0\leq m<2^n\}$, and let C_n be the set of sequences of length n, consisting of 1's and -1's. We let $\theta_n:\Omega_n\rightarrow\mathcal{N}^n$ be the map which associates $m\in\Omega_n$ with its binary representation, and let $\phi_n:\Omega_n\rightarrow C_n$ be the composition $\phi_n=(\gamma\circ\theta_n)$, where, for $\bar{m}\in\mathcal{N}^n$, $\gamma(\bar{m})=2.\bar{m}-\bar{1}$. For $\nu\in{}^*\mathcal{N}\setminus\mathcal{N}$, we let $\phi_\nu:\Omega_\nu\rightarrow C_\nu$ be the map, obtained by transfer of ϕ_n, which associates $i\in{}^*\mathcal{N}$, $0\leq i<2^\nu$, with an internal sequence of length ν, consisting of 1's and -1's. Similarly, for $\eta=2^\nu$, we let $\psi_\eta:\overline{\Omega_\eta}\rightarrow C_\nu$ be defined by $\psi_\eta(x)=\phi_\nu([\eta x])$. For $1\leq j\leq\nu$, we let $\omega_j:C_\nu\rightarrow\{1,-1\}$ be the internal projection map onto the j'th coordinate, and let $\omega_j:\overline{\Omega_\eta}\rightarrow\{1,-1\}$ also denote the composition $(\omega_j\circ\psi_\eta)$, so that $\omega_j\in W(\overline{\Omega_\eta})$. By convention, we set $\omega_0=1$. For an internal sequence $\bar{t}\in C_\nu$, we let $\omega_{\bar{t}}:\overline{\Omega_\eta}\rightarrow\{1,-1\}$ be the internal function defined by;*

$$\omega_{\bar{t}}=\prod_{1\leq j\leq\nu}\omega_j^{\frac{\bar{t}(j)+1}{2}}$$

Again, it is clear that $\omega_{\bar{t}}\in W(\overline{\Omega_\eta})$.

Lemma 0.5. *The functions $\{\omega_j:1\leq j\leq\nu\}$ are $*$-independent in the sense of* [1]*, (Definition 19), in particular they are orthogonal with respect to the measure μ_η. Moreover, the functions $\{\omega_{\bar{t}}:\bar{t}\in C_\nu\}$ form an orthogonal basis of $V(\overline{\Omega_\eta})$, and, if $\bar{t}\neq\overline{-1}$, $E_\eta(\omega_{\bar{t}})=0$, and $Var_\eta(\omega_{\bar{t}})=1$, where, E_η and Var_η are the expectation and variance corresponding to the measure μ_η.*

Proof. According to the definition, we need to verify that for an internal index set $J=\{j_1,\ldots,j_s\}\subseteq\{1,\ldots,\nu\}$, and an internal tuple $(\alpha_1,\ldots,\alpha_s)\subset{}^*\mathcal{C}^s$, where $s=|J|$;

$$\mu_\eta(x:\omega_{j_1}(x)<\alpha_1,\ldots,\omega_{j_k}(x)<\alpha_k,\ldots,\omega_{j_s}(x)<\alpha_s)$$

$$=\prod_{k=1}^{s}\mu_\eta(x:\omega_{j_k}(x)<\alpha_k)\ (*)$$

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finite dimensional vector space over $\mathcal{C}$, that V has a well defined dimension given by $Card(I)$, see [3], even though V may be infinite dimensional, considered as a standard vector space.

Without loss of generality, we can assume that each $\alpha_{j_k} > -1$, as if some $\alpha_{j_k} \leq -1$, both sides of $(*)$ are equal to zero. Let $J' = \{j' \in J : -1 < \alpha_{j'} \leq 1\}$ and $J'' = \{j'' \in J : 1 < \alpha_{j''}\}$, so $J = J' \cup J''$. Then;

$\mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \ldots, \omega_{j_s}(x) < \alpha_s)$

$= \frac{1}{\eta} Card(z \in C_\nu : z(j') = -1 \; for \; j' \in J', z(j'') \in \{-1, 1\} \; for \; j'' \in J'')$

$= \frac{1}{2^\nu} Card(z \in C_\nu : z(j') = -1 \; for \; j' \in J') = \frac{2^{\nu - s'}}{2^\nu} = 2^{-s'}$

where $s' = Card(J')$. Moreover;

$\prod_{k=1}^{s} \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) = \prod_{j' \in J'} \mu_\eta(x : \omega_{j'}(x) = -1) = 2^{-s'}$

as $\mu_\eta(x : \omega_j(x) = -1) = \frac{1}{2}$, for $1 \leq j \leq \nu$. Hence, $(*)$ is shown. That $*$-independence implies orthogonality follows easily by transfer, from the corresponding fact, for finite measure spaces, that $E(X_{j_1} X_{j_2}) = E(X_{j_1}) E(X_{j_2})$, for the standard expectation E and independent random variables $\{X_{j_1}, X_{j_2}\}$, $(**)$. Hence, by $(**)$;

$E_\eta(\omega_{j_1} \omega_{j_2}) = E_\eta(\omega_{j_1}) E_\eta(\omega_{j_2}) = 0, \; (j_1 \neq j_2) \; (***)$

as clearly $E_\eta(\omega_j) = 0$, for $1 \leq j \leq \nu$. If $\bar{t} \neq \overline{-1}$, let $J' = \{j' : 1 \leq j' \leq \nu, \bar{t}(j') = 1\}$, then;

$E_\eta(\omega_{\bar{t}}) = E_\eta(\prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}) = E_\eta(\prod_{j' \in J'} \omega_{j'}) = \prod_{j' \in J'} E_\eta(\omega_{j'}) = 0 \; (\sharp)$

where, in $(\sharp)$, we have used the facts that $J' \neq \emptyset$ and internal, and a simple generalisation of $(***)$, by transfer from the corresponding fact for finite measure spaces. Hence, $1 = \omega_{\overline{-1}}$ is orthogonal to $\omega_{\bar{t}}$, for $\bar{t} \neq \overline{-1}$. If $\overline{t_1} \neq \overline{t_2}$ are both distinct from $\overline{-1}$, then, if $J_1 = \{j : 1 \leq j \leq \nu, \overline{t_1}(j) = 1\}$ and $J_2 = \{j : 1 \leq j \leq \nu, \overline{t_2}(j) = 1\}$, so $J_1 \neq J_2$ and $J_1, J_2 \neq \emptyset$, we have;

$E_\eta(\omega_{\overline{t_1}} \omega_{\overline{t_2}})$

$= E_\eta(\prod_{j \in J_1} \omega_j . \prod_{j \in J_2} \omega_j) \; (\sharp\sharp)$

$= E_\eta(\prod_{j \in (J_1 \setminus J_2)} \omega_j . \prod_{j \in (J_2 \setminus J_1)} \omega_j) \; (\sharp\sharp\sharp)$

$= E_\eta(\prod_{j\in(J_1\setminus J_2)}\omega_j)E_\eta(\prod_{j\in(J_2\setminus J_1)}\omega_j) = 0$ $(\sharp\sharp\sharp\sharp)$

In $(\sharp\sharp)$, we have used the definition of J_1 and J_2, and in $(\sharp\sharp\sharp)$, we have used the fact that $(J_1 \cup J_2) = (J_1 \cap J_2) \sqcup (J_1 \setminus J_2) \sqcup (J_2 \setminus J_1)$, and $\omega_j^2 = 1$, for $1 \leq j \leq \nu$. Finally, in $(\sharp\sharp\sharp\sharp)$, we have used the facts that $(J_1 \setminus J_2)$ and $(J_2 \setminus J_1)$ are disjoint, and at least one of these sets is nonempty, the result of $(\sharp)$ and a similar generalisation of $(***)$. This shows that the functions $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$ are orthogonal, $(****)$. That they form a basis for $V(\overline{\Omega_\eta})$ follows immediately, by transfer, from $(****)$ and the corresponding fact for finite dimensional vector spaces. The final calculation is left to the reader. □

We require the following;

Definition 0.6. *For $0 \leq l \leq \nu$, we define $\sim'_l$, on C_ν, to be the internal equivalence relation given by;*

$\bar{t}_1 \sim'_l \bar{t}_2$ *iff* $\bar{t}_1(j) = \bar{t}_2(j)$ $(\forall j \leq l)$

We extend this to an internal equivalence relation on $\overline{\Omega}_\eta$, which we denote by $\sim_l$;

$x_1 \sim_l x_2$ *iff* $\psi_\eta(x_1) \sim_l \psi_\eta(x_2)$ $(*)$

We let $\mathcal{C}_\eta^l$ be the $*$-finite algebra generated by the partition of $\overline{\Omega}_\eta$ into the 2^l equivalence classes with respect to $\sim_l$, $(*)$. As is easily verifed, we have $\mathcal{C}_\eta^{l_1} \subseteq \mathcal{C}_\eta^{l_2}$, if $l_1 \leq l_2$, $\mathcal{C}_\eta^0 = \{\emptyset, \overline{\Omega}_\eta\}$ and $\mathcal{C}_\eta = \mathcal{C}_\eta^\nu$. For $0 \leq l \leq \nu$, we let $W(\mathcal{C}_\eta^l) \subseteq W(\mathcal{C}_\eta)$ be the set of measurable functions $f : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$, with respect to $\mathcal{C}_\eta^l$. We will refer to the collection $\{\mathcal{C}_\eta^l : 0 \leq l \leq \nu\}$ of $*$-finite algebras, as the nonstandard filtration associated to $\overline{\Omega}_\eta$. We also require a slight modification of the construction of nonstandard Brownian motion in [1]. *Namely, we define $\chi : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ by;*

$\chi(t, x) = \frac{1}{\sqrt{\nu}}({}^*\sum_{i=1}^{[\nu t]}\omega_i)$, $[\nu t] \geq 1$

$\chi(t, x) = 0$, $0 \leq [\nu t] < 1$

If $\lambda : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ is measurable, we say that λ is progressively measurable if;

(i). For $t \in \overline{\mathcal{T}_\nu}$, $Y_{\frac{[\nu t]}{\nu}}$ is measurable with respect to $\mathcal{C}_\eta^{[\nu t]}$.

If λ is progressively measurable, we define the stochastic integral $\mu(t,x) = \int_0^t \lambda(t',x)d\chi(t',x)$ by;

$^*\sum_{j=1}^{[\nu t]}\lambda(\frac{j-1}{\nu},x)(\chi(\frac{j}{\nu},x)-\chi(\frac{j-1}{\nu},x))$, *for* $[\nu t] \geq 1$

$\mu(t,x) = 0$, *for* $0 \leq [\nu t] < 1$

Remarks 0.7. *Observe that, for $t \in \overline{T}_\nu$, $E_\eta(\mu(t,x) = 0$. This follows as;*

$$E_\eta(^*\sum_{j=1}^{[\nu t]}\lambda(\tfrac{j-1}{\nu},x)(\chi(\tfrac{j}{\nu},x)-\chi(\tfrac{j-1}{\nu},x)))$$

$$= {}^*\sum_{j=1}^{[\nu t]}E_\eta(\lambda(\tfrac{j-1}{\nu},x)(\chi(\tfrac{j}{\nu},x)-\chi(\tfrac{j-1}{\nu},x)))$$

$$= {}^*\sum_{j=1}^{[\nu t]}E_\eta(\lambda(\tfrac{j-1}{\nu},x))E_\eta(\chi(\tfrac{j}{\nu},x)-\chi(\tfrac{j-1}{\nu},x)) = 0$$

Lemma 0.8. *For $0 \leq l \leq \nu$, a basis of the $*$-finite vector space $W(\mathcal{C}_\eta^l)$ is given by $D_l = \bigcup_{0\leq m\leq l} B_m$, where, for $1 \leq m \leq \nu$, $B_m = \{\omega_{\overline{t}} : \overline{t}(m) = 1, \overline{t}(m') = -1, m < m' \leq \nu\}$, and $B_0 = \{\omega_{\overline{-1}}\}$.*

Proof. The case when $l = 0$ is clear as $\omega_{\overline{-1}} = 1$, and using the description of $\mathcal{C}_\eta^0$ in Definition 0.6. Using the observation $(*)$ there, we have, for $1 \leq l \leq \nu$, that $W(\mathcal{C}_\eta^l)$ is a $*$-finite vector space of dimension 2^l. Using Lemma 0.5, and the fact that $Card(D_l) = 2^l$, it is sufficient to show each $\omega_{\overline{t}} \in D_l$ is measurable with respect to $\mathcal{C}_\eta^l$. We have that, for $1 \leq j \leq l$, ω_j is measurable with respect to $\mathcal{C}_\eta^j \subseteq \mathcal{C}_\eta^l$. Hence, the result follows easily, by transfer of the result for finite measure spaces, that the product $X_{j_1}X_{j_2}$, of two measurable random variables X_{j_1} and X_{j_2} is measurable.

□

Definition 0.9. *We define a nonstandard martingale to be a $\mathcal{D}_\nu \times \mathcal{C}_\eta$-measurable function $Y : \overline{\mathcal{T}_\nu} \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$, such that;*

(i). For $t \in \overline{\mathcal{T}_\nu}$, $Y_{\frac{[\nu t]}{\nu}}$ is measurable with respect to $\mathcal{C}_\eta^{[\nu t]}$.

(ii). $E_\eta(Y_{\frac{[\nu t]}{\nu}}|\mathcal{C}_\eta^{[\nu s]}) = Y_{\frac{[\nu s]}{\nu}}$, for $(0 \leq s \leq t \leq 1)$.

Lemma 0.10. *Let* $Y : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ *be a nonstandard martingale, then;*

$Y_t(x) = \sum_{j=0}^{[\nu t]} c_j(t,x)\omega_j(x)$

where, if $s = \frac{j}{\nu}$, $j \geq 1$, $c_j : [s,1] \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ *is* $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$ *measurable,* $c_0 : [0,1] \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ *is* $\mathcal{D}_\nu \times \mathcal{C}_\eta^0$, *measurable,* $c_j(s,x) = c_j(t,x)$, *for* $s \leq t \leq 1$.

Proof. Using (ii) of Definition 0.9, we have that $E_\eta(Y_t) = E_\eta(Y_t|\mathcal{C}_\eta^0) = Y_0$. Replacing Y_t by $Y_t - Y_0$, we can, without loss of generality, assume that $E_\eta(Y_t) = 0$, for $t \in [0,1]$. By (i) of Definition 0.9 and Lemma 0.8;

$Y_t = \sum_{j=1}^{[\nu t]} c_j(t,x)\omega_j(x)$

where;

$c_j(t,x) = \sum_{a=1}^{j-1} \sum_{i_1 < \ldots < i_a;1}^{j-1} p_j^{(i_1,\ldots,i_a)}(t)\omega_{i_1}\ldots\omega_{i_a}(x)$

Using Lemma 0.8, we obtain the first claim for $s = \frac{j}{\nu}$. Again, using (ii) of Definition 0.9, and the fact that $c_{[\nu t']}(t,x)\omega_{\nu t'}$ is orthogonal to $c_{[\nu s']}(t,x)\omega_{[\nu s']}$, for $[\nu s'] \leq [\nu s] < [\nu t'] \leq [\nu t]$, $(*)$, we have;

$\sum_{j=1}^{[\nu s]} c_j(t,x)\omega_j(x) = \sum_{j=1}^{[\nu s]} c_j(s,x)\omega_j(x)$

Equating coefficients, and using the fact that $D_{[\nu t]}$ is a basis for $W(\mathcal{C}_\eta^{[\nu t]})$, we obtain $c_j(s,x) = c_j(t,x)$, for all $0 \leq s \leq t \leq 1$.

□

Lemma 0.11. *Martingale Representation Theorem*

Let $Y : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ *be a nonstandard martingale, with* $E_\eta(Y_t) = 0$, *then there exists a progressively measurable* $Z : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ *such that;*

$Y_t = \int_0^t Z(t',x)d\chi(t',x)$

Proof. By Lemma 0.10, we can find $c_j : [\frac{j}{\nu}, 1] \rightarrow {}^*\mathcal{C}$, $j \geq 1$, such that;

$Y_t(x) = \sum_{j=1}^{[\nu t]} c_j(t,x)\omega_j(x)$, for $[\nu t] \geq 1$

Set $Z(t,x)=\sqrt{\nu}c_{[\nu t]+1}(1,x)$, $0\leq[\nu t]<1$, and $Z(1,x)=0$, ,then Z is progressively measurable and;

$Y_t=\int_0^t Z(t',x)d\chi(t',x)$

as required. □

Definition 0.12. *If $\eta'\in {}^*\mathcal{N}\setminus\mathcal{N}$, we define $\overline{\mathcal{R}_{\eta'}}=[-\eta',\eta')\cap {}^*\mathcal{R}$. We let $\mathcal{F}_{\eta'}$ consist of internal unions of intervals of the form $[\frac{i}{\eta'},\frac{i+1}{\eta'})$, for $-\eta'^2\leq i\leq\eta'^2-1$. We define a measure $\phi_{\eta'}$ on $\mathcal{F}_{\eta'}$, by setting $\phi_{\eta'}([\frac{i}{\eta'},\frac{i+1}{\eta'}))=\frac{1}{\eta'}$, for $-\eta'^2\leq i\leq\eta'^2-1$. We let $W(\overline{\mathcal{R}_{\eta'}})$ consist of the set of measurable functions on $\overline{\mathcal{R}_{\eta'}}$. If $f\in W(\overline{\mathcal{R}_{\eta'}})$, we define f' by;*

$f'(\frac{i}{\eta'})=\eta'(f(\frac{i+1}{\eta'})-f(\frac{i}{\eta'}))$, *for* $-\eta'^2\leq i\leq\eta'^2-2$

$f'(\eta'-\frac{1}{\eta'})=0$

and $\int_{\overline{\mathcal{R}_{\eta'}}}fd\phi_{\eta'}=\frac{1}{\eta'}{}^*\sum_{i=-\eta'^2}^{\eta'^2-1}f(\frac{i}{\eta'})$

If $f\in C^\infty(\mathcal{R})$, we let $f_{\eta'}\in W(\overline{\mathcal{R}_{\eta'}})$ be defined by;

$f(\frac{i}{\eta'})={}^*f(\frac{i}{\eta'})$, *for* $-\eta'^2\leq i\leq\eta'^2-1$

Remarks 0.13. *Observe that if $f\in W(\overline{\mathcal{R}_{\eta'}})$;*

$f''(\frac{i}{\eta'})=\eta'^2(f(\frac{i+2}{\eta'})-2f(\frac{i+1}{\eta'})+f(\frac{i}{\eta'}))$, *for* $-\eta'^2\leq i\leq\eta'^2-3$

$f''(\eta'-\frac{2}{\eta'})=-\eta'^2(f(\eta'-\frac{1}{\eta'})-f(\eta'-\frac{2}{\eta'}))$

$f''(\eta'-\frac{1}{\eta'})=0$

We now introduce distribution functions;

Definition 0.14. *Let $Y:\overline{\mathcal{T}_\nu}\times\overline{\Omega_\eta}\rightarrow {}^*\mathcal{R}$ be measurable, and suppose that $|Y|\leq\eta''$, with $\eta''\in {}^*\mathcal{N}\setminus\mathcal{N}$, and $\eta'\geq\eta''+1$. Let $t\in\overline{\mathcal{T}_\nu}$. Define $R_t:\overline{\mathcal{R}_{\eta'}}\rightarrow {}^*[0,1]$ and $r_t:\overline{\mathcal{R}_{\eta'}}\rightarrow {}^*\mathcal{R}$ by;*

$R_t(x)=\mu_\eta(Y_{\frac{[\nu t]}{\nu}}\leq\frac{[x\eta]}{\eta})$, $r_t(x)=R'_t(x)$

Lemma 0.15. *We have that;*

$R_t(\eta'-\frac{1}{\eta'})=\int_{\overline{\mathcal{R}_{\eta'}}}r_t(x)d\phi_{\eta'}=1$

$R_t(-\eta')=0$

Proof. The facts about R_t follow immediately from the definition and the property that $|Y|\leq\eta'-1$. We have that;

$\int_{\overline{\mathcal{R}_{\eta'}}}r_t(x)d\phi_{\eta'}$

$=\frac{1}{\eta'}{}^*\sum_{i=-\eta'^2}^{\eta'^2-1}R_t'(\frac{i}{\eta'})$

$=\frac{1}{\eta'}{}^*\sum_{i=-\eta'^2}^{\eta'^2-2}\eta(R_t(\frac{i+1}{\eta'})-R_t(\frac{i}{\eta'}))$

$=R_t(\eta'-\frac{1}{\eta'})-R_t(-\eta)=1$

□

Lemma 0.16. *Let* $W\in C^3(\mathcal{R})$, *and let* $max(W,\frac{dW}{dx},\frac{d^2W}{dx^2},\frac{d^3W}{dx^3})\leq K$, *then* $(W_{\eta'})'\simeq(\frac{dW}{dx})_{\eta'}$ *on* $\overline{\mathcal{R}_{\eta'}}\setminus[\eta'-\frac{1}{\eta'},\eta')$ *and* $(W_{\eta'})''\simeq(\frac{d^2W}{dx^2})_{\eta'}$ *on* $\overline{\mathcal{R}_{\eta'}}\setminus[\eta'-\frac{2}{\eta'},\eta')$

Proof. We have, by the mean value theorem, that, for $x_0\in\mathcal{R}$, $n\in\mathcal{N}$;

$W(x_0+\frac{1}{n})-W(x_0)=\frac{dW}{dx}|_{x_0+c}\frac{1}{n}$, where $c\in(0,\frac{1}{n})$

Therefore;

$|\frac{dW}{dx}|_{x_0}-n(W(x_0+\frac{1}{n})-W(x_0))$

$=|\frac{dW}{dx}|_{x_0}-\frac{dW}{dx}|_{x_0+c}|=c\frac{d^2W}{dx^2}|_{x_0+d}$, where $d\in(0,c)$

Therefore;

$|\frac{dW}{dx}|_{x_0}-n(W(x_0+\frac{1}{n})-W(x_0))|\leq\frac{K}{n}$

We have that;

$n^2(W(x_0+\frac{2}{n})-2W(x_0+\frac{1}{n})+W(x_0))$

$= n(\frac{dW}{dx}|_{x_0+c+\frac{1}{n}} - \frac{dW}{dx}|_{x_0+c})$

$= \frac{d^2W}{dx^2}|_{x_0+c+e}$ where $e \in (0, \frac{1}{n})$

Hence;

$|\frac{d^2W}{dx^2}|_{x_0} - n^2(W(x_0 + \frac{2}{n}) - 2W(x_0 + \frac{1}{n}) + W(x_0))|$

$= |\frac{d^2W}{dx^2}|_{x_0} - \frac{d^2W}{dx^2}|_{x_0+c+e}$

$= (c + e)\frac{d^3W}{dx^3}|_{x_0+f}$, where $f \in (0, c + e)$

$\leq \frac{2K}{n}$

Then;

$\mathcal{R} \models (\forall x \in \mathcal{R})(\forall n \in \mathcal{N})max(|\frac{dW}{dx} - n(W(x + \frac{1}{n}) - W(x))|,$

$|\frac{d^2W}{dx^2} - n^2(W(x + \frac{2}{n}) - 2W(x + \frac{1}{n}) + W(x))|) \leq \frac{2K}{n}$

Hence, ${}^*\mathcal{R}$ satisfies the transferred statement. In particular, for any infinite $\eta' \in {}^*\mathcal{N}$, as $\frac{2K}{\eta'} \simeq 0$, we obtain the result from Definition 0.12 and Remark 0.13. □

Lemma 0.17. *Let hypotheses be as in the previous lemma. Then, there exist* $\{f, g\} \subset W(\overline{\mathcal{R}}_{\eta'})$, *with* $|f| \leq \frac{K}{\eta'}$, $|g| \leq \frac{2K}{\eta'}$, *and* $C \in \mathcal{R}$ *such that;*

$(\frac{dW}{dx})_{\eta'} = (W_{\eta'})' + f + \delta_{\eta'-\frac{1}{\eta'}}(\frac{dW}{dx})_{\eta'}(\eta' - \frac{1}{\eta'})$

$(\frac{d^2W}{dx^2})_{\eta'} = (W_{\eta'})'' + g + \delta_{\eta'-\frac{2}{\eta'}}((\frac{d^2W}{dx^2})_{\eta'}(\eta' - \frac{2}{\eta'}) + \eta'(\frac{dW}{dx})_{\eta'}(\eta' - \frac{2}{\eta'}) + C) + \delta_{\eta'-\frac{1}{\eta'}}((\frac{d^2W}{dx^2})_{\eta'}(\eta' - \frac{1}{\eta'}))$

Proof. The proof just requires the observation that;

$W'_{\eta'}(\eta' - \frac{1}{\eta'}) = W''_{\eta'}(\eta' - \frac{1}{\eta'}) = 0$

and

$W''_{\eta'}(\eta' - \frac{2}{\eta'})$

$= -\eta'(W_{\eta'}(\eta' - \frac{2}{\eta'}))$

$= -\eta'((\frac{dW}{dx})_{\eta'}(\eta' - \frac{2}{\eta'}) + \frac{C}{\eta'})$

$= -\eta'((\frac{dW}{dx})_{\eta'}(\eta' - \frac{2}{\eta'}) - C$

where $C \in \mathcal{R}$.

$\square$

Definition 0.18. *If $V \in C^3(\mathcal{R})$, and $Y \in W(\overline{\Omega}_\eta)$, we let $(V \circ Y)_\eta$ be defined by;*

$(V \circ Y)_\eta(\frac{i}{\eta}) = {}^*V(Y(\frac{i}{\eta}))$, *for* $0 \leq i \leq \eta - 1$

Suppose that $|Y| \leq \eta''$, and $\eta' = \eta'' + 1$, let $V_{\eta'}$ be defined by;

$V_{\eta'}(\frac{i}{\eta'}) = {}^*V(\frac{i}{\eta'})$, *for* $-\eta'^2 \leq i \leq \eta'^2 - 1$

so that $(V \circ Y)_\eta \in W(\overline{\Omega}_\eta)$ and $V_{\eta'} \in W(\overline{\mathcal{R}_{\eta'}})$.

Lemma 0.19. *Let $Y \in W(\overline{\Omega}_\eta)$, with $|Y| \leq \eta''$, and $\eta' \geq \eta'' + 1$. Let $V \in C^3(\mathcal{R})$, with $max(V, \frac{dV}{dx}) \leq K$. Then;*

$E_\eta((V \circ Y)_\eta) \simeq \int_{\overline{\mathcal{R}_{\eta'}}} V_{\eta'} r_Y d\phi_{\eta'}$

Proof. We have that;

$\gamma_1 = \int_{\overline{\mathcal{R}_{\eta'}}} V_{\eta'} r_Y d\phi_{\eta'}$

$= \frac{1}{\eta'} {}^*\sum_{-\eta'^2 \leq i \leq \eta'^2 - 1} V_{\eta'}(\frac{i}{\eta'}) r_Y(\frac{i}{\eta'})$

$= {}^*\sum_{-\eta'^2 \leq i \leq \eta'^2 - 2} V_{\eta'}(\frac{i}{\eta'})(R_Y(\frac{i+1}{\eta'}) - R_Y(\frac{i}{\eta'}))$

$= \frac{1}{\eta} {}^*\sum_{-\eta'^2 \leq i \leq \eta'^2 - 2} {}^*\sum_{j \in J_i} V_{\eta'}(\frac{i}{\eta'})$ $(*)$

where $J_i = \{j \in {}^*\mathcal{Z} \cap [0, \eta - 1] : Y(\frac{j}{\eta}) \in (\frac{i}{\eta'}, \frac{i+1}{\eta'}]\}$

$\gamma_2 = E_\eta((V \circ Y)_\eta)$

$= \frac{1}{\eta} {}^*\sum_{0 \leq j \leq \eta - 1} {}^*V(Y(\frac{j}{\eta}))$

$= \frac{1}{\eta}{}^*\sum_{-\eta'^2\leq i\leq \eta'^2-2}{}^*\sum_{j\in J_i}{}^*V(Y(\frac{j}{\eta}))$ $(**)$

$|\gamma_1-\gamma_2|\leq \frac{1}{\eta}{}^*\sum_{-\eta'^2\leq i\leq \eta'^2-2}{}^*\sum_{j\in J_i}|V_{\eta'}(\frac{i}{\eta'})-{}^*V(Y(\frac{j}{\eta}))|$

$= \frac{1}{\eta}{}^*\sum_{-\eta'^2\leq i\leq \eta'^2-2}{}^*\sum_{j\in J_i}|V_{\eta'}(\frac{i}{\eta'})-{}^*V(\frac{i+c(j)}{\eta'})|$, $0\leq c(j)<1$

$\leq \frac{1}{\eta}{}^*\sum_{-\eta'^2\leq i\leq \eta'^2-2}{}^*\sum_{j\in J_i}\frac{K}{\eta'}$

$= \frac{K}{\eta'}\simeq 0$

where, in the penultimate step, we have used the proof of Lemma 0.16.

□

Definition 0.20. *If $t\in\mathcal{R}$ and $V\in C^3([0,t_0]\times\mathcal{R}$, and $Y\in W(\overline{\mathcal{T}_\nu}\times\overline{\Omega_\eta})$, we define $(V\circ(t,Y))_{\nu,\eta}$ by;*

$(V\circ(t,Y))_{\nu,\eta}(t,x)={}^*V(\frac{[\nu t]}{\nu},Y(t,x))$ *for* $(t,x)\in(\overline{\mathcal{T}_\nu}\cap{}^*[0,t_0])\times\overline{\Omega_\eta}$

so that $(V\circ(t,Y))_{\nu,\eta}\in W((\overline{\mathcal{T}_\nu}\cap{}^*[0,t_0])\times\overline{\Omega_\eta})$.

Lemma 0.21. *Let $Y:\overline{\mathcal{T}_\nu}\times\overline{\Omega_\eta}\rightarrow{}^*\mathcal{R}$ be a nonstandard martingale with associated $Z:\overline{\mathcal{T}_\nu}\times\overline{\Omega_\eta}\rightarrow{}^*\mathcal{R}$, and let $D=\frac{Z^2}{2}$. For $t_0\in[0,1]$, let V_{t_0} and W be as given. Let $W(t,x)=V_{t_0}(t)W(x)$, and let $S_{t_0,\eta}$ be defined by;*

$S_{t_0,\nu,\eta}(t,x)={}^*W(\frac{[\nu t]}{\nu},Y(t,x))$

$(\frac{\partial V_{t_0}}{\partial t}\circ(t,Y))_{\nu,\eta}(t,x)={}^*\frac{\partial V_{t_0}}{\partial t}(\frac{[\nu t]}{\nu},Y(t,x))$

Then, for $t'\in\mu(t_0)\cap{}^*[0,t_0]$;

$S_{t_0,\nu,\eta}(t',x)$

$\simeq\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,Y))_{\nu,\eta}+D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,Y))_{\nu,\eta}d\lambda_\nu+\int_0^{t'}Z(\frac{\partial W_{t_0}}{\partial x}\circ(t,Y))_{\nu,\eta}d\chi$

Proof. We have that;

$S_{t_0,\nu,\eta}(t',x)$

$= {}^*W(\frac{[\nu t']}{\nu}, Y(t', x)) - {}^*W(0, Y(0, x))$

$= {}^* \sum_{j=1}^{[\nu t']} ({}^*W(\frac{j}{\nu}, Y(\frac{j}{\nu}, x)) - {}^*W(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))$

$= {}^* \sum_{j=1}^{[\nu t']} {}^*\frac{\partial W}{\partial t}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))\frac{1}{\nu}$

$+{}^* \sum_{j=1}^{[\nu t']} {}^*\frac{\partial W}{\partial x}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))$

$+{}^* \sum_{j=1}^{[\nu t']} (\frac{1}{2}{}^*\frac{\partial^2 W}{\partial t^2}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)) + \epsilon_{tt})\frac{1}{\nu^2}$

$+{}^* \sum_{j=1}^{[\nu t']} (\frac{1}{2}{}^*\frac{\partial^2 W}{\partial x^2}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)) + \epsilon_{xx})(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))^2$

$+{}^* \sum_{j=1}^{[\nu t']} ({}^*\frac{\partial^2 W}{\partial t \partial x}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)) + \epsilon_{tx})\frac{1}{\nu}(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))$

$W(t, x) = e^{-x^2} sin(\frac{\pi t}{t_0})$

$max(W_{xxx}, W_{txx}, W_{ttx}, W_{ttt}) \leq K$

$|W_{xx}(t + h_1, x + h_2) - W_{xx}(t, x)|$

$\leq |W_{xx}(t + h_1, x + h_2) - W_{xx}(t + h_1, x)| + |W_{xx}(t + h_1, x) - W_{xx}(t, x)|$

$\leq h_2 K + h_1 K$

Similarly, for W_{tx}, W_{tt}.

$\mathcal{R} \models (\forall (t, x) \forall ||(t_1, t_2)||_{\mathcal{R}^2} < \sqrt{2}\nu^{\frac{-1}{3}})$

$max(W_{xx}(\overline{x}+\overline{h}) - W_{xx}(\overline{x}), W_{tx}(\overline{x}+\overline{h}) - W_{tx}(\overline{x}), W_{tt}(\overline{x}+\overline{h}) - W_{tt}(\overline{x})) \leq 2K\sqrt{2}\nu^{\frac{-1}{3}} = \epsilon$

Assume that $|Z| < min\{\nu^{\frac{1}{6}}, \epsilon^{\frac{-1}{3}}\} = min(\nu^{\frac{1}{6}}, (2K\sqrt{2})^{\frac{-1}{3}}\nu^{\frac{1}{9}}) = (2K\sqrt{2})^{\frac{-1}{3}}\nu^{\frac{1}{9}})$

$|(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))| \leq |Z(\frac{j}{\nu}, x)||\frac{e_j}{\sqrt{\nu}}| < min\{\nu^{\frac{-1}{3}}, \epsilon^{\frac{-1}{3}}\nu^{\frac{-1}{2}}\}$

$|(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))|^2 \leq min\{\nu^{\frac{-2}{3}}, \epsilon^{\frac{-2}{3}}\nu^{-1}\}$

It follows that;

$|{}^* \sum_{j=1}^{[\nu t']} \epsilon_{xx}(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))^2|$

$\leq \nu|\epsilon_{xx}|\epsilon^{\frac{-2}{3}}\nu^{-1}$

$\leq |\epsilon_{xx}|\epsilon^{\frac{-2}{3}}$

where $\epsilon_{xx} = sup_{||\overline{\delta}||_2 \leq ||\overline{u}-\overline{s}||_2}|(W_{xx}(\overline{s}+\overline{\delta}) - W_{xx}(\overline{s}))$, for $W(\overline{u})$

$= \epsilon\epsilon^{\frac{-2}{3}} = \epsilon^{\frac{1}{3}} \simeq 0$, as $||(\frac{j}{\nu}, Y(\frac{j}{\nu})) - (\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}))||_2 \leq \sqrt{2}\nu^{\frac{-1}{3}}$

Similarly,

$|^*\sum_{j=1}^{[\nu t']}\epsilon_{tt}\frac{1}{\nu^2}|$

$\leq \nu|\epsilon_{tt}|\frac{1}{\nu^2} = \frac{\epsilon}{\nu} \simeq 0$

Finally;

$|^*\sum_{j=1}^{[\nu t']}\epsilon_{tx}\frac{1}{\nu}(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))|$

$\leq \nu|\epsilon_{tx}|\frac{1}{\nu}\epsilon^{\frac{-1}{3}}\nu^{\frac{-1}{2}}$

$= |\epsilon_{tx}|\epsilon^{\frac{-1}{3}}\nu^{\frac{-1}{2}}$

$= 2\epsilon\epsilon^{\frac{-1}{3}}\nu^{\frac{-1}{2}}$

$= 2\epsilon^{\frac{2}{3}}\nu^{\frac{-1}{2}} \simeq 0$

Then;

$(i).^*\sum_{j=1}^{[\nu t']}{}^*\frac{\partial W}{\partial t}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))\frac{1}{\nu} = \int_0^{t'}(\frac{\partial W_{t_0}}{\partial t} \circ (t, Y))_{\nu,\eta} d\lambda_\nu$

$(ii).^*\sum_{j=1}^{[\nu t']}{}^*\frac{\partial W}{\partial x}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))(Y(\frac{j}{\nu}, x) - Y(\frac{j-1}{\nu}, x))$

$= {}^*\sum_{j=1}^{[\nu t']}{}^*\frac{\partial W}{\partial x}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x)))Z(\frac{j}{\nu}, x)\frac{e_j}{\sqrt{\nu}}$

$= \int_0^{t'} Z(\frac{\partial W_{t_0}}{\partial x} \circ (t, Y))_{\nu,\eta} d\chi$

$(iii).|^*\sum_{j=1}^{[\nu t']}\frac{1}{2}{}^*\frac{\partial^2 W}{\partial t^2}(\frac{j-1}{\nu}, Y(\frac{j-1}{\nu}, x))\frac{1}{\nu^2}| \leq \frac{\nu K}{\nu^2} \simeq 0$

where $K = max(\frac{\partial^2 W}{\partial t^2}, \frac{\partial^2 W}{\partial t \partial x})$.

$(iv).{}^*\sum_{j=1}^{[\nu t']}\frac{1}{2}{}^*\frac{\partial^2 W}{\partial x^2}(\frac{j-1}{\nu},Y(\frac{j-1}{\nu},x))(Y(\frac{j}{\nu},x)-Y(\frac{j-1}{\nu},x))^2$

$={}^*\sum_{j=1}^{[\nu t']}\frac{1}{2}{}^*\frac{\partial^2 W}{\partial x^2}(\frac{j-1}{\nu},Y(\frac{j-1}{\nu},x))Z^2(\frac{j}{\nu},x)\frac{1}{\nu}$

$=\int_0^{t'}D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,Y))_{\nu,\eta}d\lambda_\nu$

$(v).|{}^*\sum_{j=1}^{[\nu t']}{}^*\frac{\partial^2 W}{\partial t\partial x}(\frac{j-1}{\nu},Y(\frac{j-1}{\nu},x))\frac{1}{\nu}(Y(\frac{j}{\nu},x)-Y(\frac{j-1}{\nu},x))|$

$\leq\frac{\nu K\nu^{\frac{1}{6}}}{\nu\nu^{\frac{1}{2}}}\simeq 0$

giving the result. □

Lemma 0.22. *With assumptions as in the previous lemma, we have that;*

$E_\eta(\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,Y))_{\nu,\eta}+D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,Y))_{\nu,\eta}d\lambda_\nu)$

$\simeq\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}+D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,x))_{\nu,\eta'}p(t,x)d\lambda_\nu d\phi_{\eta'}\simeq 0$

Proof. As $V_{t_0}(t')\simeq 0$, we have that $S_{t_0,\nu,\eta}(t',x)\simeq 0$ on $\overline{\Omega}_\eta$, hence $E_\eta(S_{t_0,\nu,\eta}(t',x))\simeq 0$. We have $E_\eta(\int_0^{t'}Z(\frac{\partial W_{t_0}}{\partial x}\circ(t,Y))_{\nu,\eta}d\chi)=0$, by Remark 0.13. This gives the first part of the claim. For the second part, observe that, using Lemma 0.19 and the fact that $\int_0^{t'}fd\lambda_\nu\simeq 0$, if $f\simeq 0$;

$E_\eta(\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,Y))_{\nu,\eta}$

$\int_0^{t'}\int_{\overline{\Omega}_\eta}(\frac{\partial W_{t_0}}{\partial t}\circ(t,Y))_{\nu,\eta}d\mu_\eta d\lambda_\nu$

$\simeq\int_0^{t'}\int_{\overline{\mathcal{R}_{\eta'}}}(\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}p(t,x)d\phi_{\eta'}d\lambda_\nu$

$=\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}p(t,x)d\lambda_\nu d\phi_{\eta'}$

and a similar calculation for $E_\eta(\int_0^{t'}D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,Y))_{\nu,\eta}d\lambda_\nu)$. □

Lemma 0.23. *We have that;*

$\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(\frac{\partial(W_{t_0})_{\nu,\eta'}}{\partial t}+D\frac{\partial^2(W_{t_0})_{\nu,\eta'}}{\partial x^2})p(t,x))d\lambda_\nu d\phi_{\eta'}\simeq 0$

Proof. By Lemma 0.16, we have that $\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}=\frac{\partial(W_{t_0})_{\nu,\eta'}}{\partial t}+g+\epsilon$, where $|g|\leq\frac{K}{\eta'}$ and ϵ is an error term supported at the endpoint, and

$\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,x))_{\nu,\eta'}=\frac{\partial^2 (W_{t_0})_{\nu,\eta'}}{\partial x^2}+h+\delta$, where $h|\leq\frac{2K}{\eta'}$ and δ is an error term supported at 2 endpoints. Hence;

$$|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}p(x,t)d\lambda_\nu d\phi_{\eta'}-\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(\frac{\partial (W_{t_0})_{\nu,\eta'}}{\partial t}p(t,x)d\lambda_\nu d\phi_{\eta'}|$$

$$\leq|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(g+\epsilon)p(t,x)d\lambda_\nu d\phi_{\eta'}|$$

$$\leq\frac{K}{\eta'}\int_0^{t'}\int_{\overline{\mathcal{R}_{\eta'}}}|p(t,x)|d\lambda_\nu d\phi_{\eta'}$$

$$+\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}|\delta_{[\nu t']-\frac{1}{\nu}}\frac{\partial W_{t_0}}{\partial t}\circ(t,x))_{\nu,\eta'}([\nu t']-\tfrac{1}{\nu})p(t,x)|d\lambda_\nu d\phi_{\eta'}$$

$$\leq\frac{Kt'}{\eta'}+\frac{D}{\nu}\int_{\overline{\mathcal{R}_{\eta'}}}|p([\nu t_0]-\tfrac{1}{\nu},x)|d\phi_{\eta'}$$

$$=\frac{Kt'}{\eta'}+\frac{D}{\nu}\simeq 0$$

where we have used the fact that $p(t,x)$ is a probability distribution, and $|\frac{\partial W_{t_0}}{\partial t}|\leq D$.

Similarly;

$$|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}D(\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,x))_{\nu,\eta'}p(t,x)d\lambda_\nu d\phi_{\eta'}-\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}D\frac{\partial^2 (W_{t_0})_{\nu,\eta'}}{\partial x^2}p(t,x))d\lambda_\nu d\phi_{\eta'}|$$

$$\leq|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}D(h+\delta)p(t,x)d\lambda_\nu d\phi_{\eta'}|$$

$$\leq\frac{2KC\nu^{\frac{1}{9}}}{\eta'}\int_0^{t'}\int_{\overline{\mathcal{R}_{\eta'}}}|p(t,x)|d\lambda_\nu d\phi_{\eta'}$$

$$+\int_0^{t'}\int_{\overline{\mathcal{R}_{\eta'}}}|(\delta_{\eta'-\frac{2}{\eta'}}((\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,x))_{\nu,\eta'}(\eta'-\tfrac{2}{\eta'})+\eta'(\frac{\partial W_{t_0}}{\partial x}\circ(t,x))_{\nu,\eta'}(\eta'-\tfrac{2}{\eta'})+C)+\delta_{\eta'-\frac{1}{\eta'}}((\frac{\partial^2 W_{t_0}}{\partial x^2}\circ(t,x))_{\nu,\eta'}(\eta'-\tfrac{1}{\eta'})))Dp(x,t)|d\phi_{\eta'}d\lambda_\nu$$

$$\leq\frac{2Kt'C\nu^{\frac{1}{9}}}{\eta'}\simeq 0$$

for sufficiently large η', where we have used the fact that $p(t,\eta'-\frac{2}{\eta'})=p(t,\eta'-\frac{1}{\eta'})=0$, $p(t,x)\geq 0$, and $\int_{\overline{\mathcal{R}_{\eta'}}}p(t,x)d\phi_{\eta'}=1$, for $t\in\overline{\mathcal{T}_\nu}$

□

Lemma 0.24. *If $f\in W(\overline{R}_{\eta'})$, then;*

$$\int_{\overline{R}_{\eta'}}f'd\phi_{\eta'}=f(\eta'-\tfrac{1}{\eta'})-f(-\eta')$$

$\int_{\overline{R}_{\eta'}} f'gd\phi_{\eta'} = fg(\eta'-\frac{1}{\eta'}) - fg(-\eta') - \int_{\overline{R}_{\eta'}} f^{sh}g'd\phi_{\eta'}$

where $f^{sh}(\frac{i}{\eta'}) = f(\frac{i+1}{\eta'})$ *for* $-\eta'^2 \leq i \leq \eta'^2-2,\ f^{sh}(\eta'-\frac{1}{\eta'}) = 0.$

$\int_{\overline{R}_{\eta'}} f''gd\phi_{\eta'} = (f'g(\eta'-\frac{1}{\eta'}) - f'g(-\eta')) - (f^{sh}g'(\eta'-\frac{1}{\eta'}) - f^{sh}g'(-\eta')) - (f(\eta'-\frac{1}{\eta'})g'(\eta'-\frac{2}{\eta'})) + \int_{\overline{R}_{\eta'}} f^{sh^2}g''d\phi_{\eta'}$

Proof. For the first part, we have that;

$\int_{\overline{R}_{\eta'}} f'd\phi_{\eta'} = \frac{1}{\eta'}{}^*\sum_{0\leq i\leq 2\eta'^2-2} f'(-\eta'+\frac{i}{\eta'})$

$= \frac{1}{\eta'}{}^*\sum_{0\leq i\leq 2\eta'^2-2} \eta'(f'(-\eta'+\frac{i+1}{\eta'}) - f(\eta'+\frac{i}{\eta'})$

$= {}^*\sum_{0\leq i\leq 2\eta'^2-2} f'(-\eta'+\frac{i+1}{\eta'}) - f(-\eta'+\frac{i}{\eta'})$

$= f(\eta'-\frac{1}{\eta'}) - f(-\eta')$

For the second part observe, that, for $-\eta'^2 \leq i \leq \eta'^2-2$;

$(fg)'(\frac{i}{\eta'}) = \eta'(fg(\frac{i+1}{\eta'}) - fg(\frac{i}{\eta'}))$

$= \eta'(fg(\frac{i+1}{\eta'}) - g(\frac{i}{\eta'})f(\frac{i+1}{\eta'}) + g(\frac{i}{\eta'})f(\frac{i+1}{\eta'}) - fg(\frac{i}{\eta'}))$

$= g'(\frac{i}{\eta'})f((\frac{i+1}{\eta'})) + g(\frac{i}{\eta'})f'(\frac{i}{\eta'})$

$= (g'f^{sh} + gf')(\frac{i}{\eta'})$

and that $(fg)'(\eta'-\frac{1}{\eta'}) = (g'f^{sh}+gf')(\eta'-\frac{1}{\eta'}) = 0$

to obtain $(fg)' = (f'g + f^{sh}g')$, $(*)$. Then, by the first part and $(*)$;

$fg(\eta'-\frac{1}{\eta'}) - fg(-\eta')$

$= \int_{\overline{R}_{\eta'}} (fg)'d\phi_{\eta'}$

$= \int_{\overline{R}_{\eta'}} (f'g + f^{sh}g')d\phi_{\eta'}$

as required. Using the second part, we have that;

$\int_{\overline{R}_{\eta'}} (f''g)d\phi_{\eta'}$

$= f'g(\eta'-\frac{1}{\eta'}) - f'g(-\eta') - \int_{\overline{R}_{\eta'}}(f'^{sh}g')d\phi_{\eta'}$, $(**)$

For $-\eta'^2 \leq i \leq \eta'^2-3$, we have that;

$f'^{sh}(\frac{i}{\eta'}) = f'(\frac{i+1}{\eta'})$

$= \eta'(f(\frac{i+2}{\eta'}) - f(\frac{i+1}{\eta'}))$

$= \eta'(f^{sh}(\frac{i+1}{\eta'}) - f^{sh}(\frac{i}{\eta'}))$

$= (f^{sh})'(\frac{i}{\eta'})$

$f'^{sh}(\eta'-\frac{1}{\eta'}) = (f^{sh})'(\eta'-\frac{1}{\eta'}) = 0$

$f'^{sh}(\eta'-\frac{2}{\eta'}) = f'(\eta'-\frac{1}{\eta'}) = 0$

$(f^{sh})'(\eta'-\frac{2}{\eta'})$

$= \eta'(f^{sh}(\eta'-\frac{1}{\eta'}) - f^{sh}(\eta'-\frac{2}{\eta'}))$

$= -\eta' f(\eta'-\frac{1}{\eta'})$

Therefore, $f'^{sh} = (f^{sh})' + \delta_{\eta-\frac{2}{\eta'}}\eta' f(\eta'-\frac{1}{\eta'})$

We therefore have that;

$\int_{\overline{R}_{\eta'}}(f'^{sh}g')d\phi_{\eta'}$

$= \int_{\overline{R}_{\eta'}}((f^{sh})'g' + \delta_{\eta-\frac{2}{\eta'}}\eta' f(\eta'-\frac{1}{\eta'})g')d\phi_{\eta'}$

$= f(\eta'-\frac{1}{\eta'})g'(\eta'-\frac{2}{\eta'}) + \int_{\overline{R}_{\eta'}}(f^{sh})'g'd\phi_{\eta'}$

$= f(\eta'-\frac{1}{\eta'})g'(\eta'-\frac{2}{\eta'}) + (f^{sh}g'(\eta'-\frac{1}{\eta'}) - f^{sh}g'(-\eta')) - \int_{\overline{R}_{\eta'}}(f^{sh^2})g''d\phi_{\eta'}$, $(***)$

Combining $(**)$ and $(***)$ gives the result.

□

Lemma 0.25. *We have that;*

$\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})^{sh_t}_{\nu,\eta'}\frac{\partial p}{\partial t} + (W_{t_0})^{sh_x^2}_{\nu,\eta'}\frac{\partial^2 Dp}{\partial x^2}d\lambda_\nu d\phi_{\eta'} \simeq 0$

Lemma 0.26. *If* $\frac{\partial p}{\partial t}$ *and* $\frac{\partial^2(Dp)}{\partial x^2}$ *are* S*-integrable, we have that;*

$\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})_{\nu,\eta'}(\frac{\partial p}{\partial t}+\frac{\partial^2 Dp}{\partial x^2})d\lambda_\nu d\phi_{\eta'}\simeq 0$

Proof. To see this, observe that, as $(W_{t_0})_{\nu,\eta'}$ is S-continuous, $(W_{t_0})_{\nu,\eta'}^{sh_t}=(W_{t_0})_{\nu,\eta'}+\epsilon_1$, $(W_{t_0})_{\nu,\eta'}^{sh_x^2}=(W_{t_0})_{\nu,\eta'}+\epsilon_2$, where ϵ_1,ϵ_2 are infinitesimal functions. We have that;

$|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})_{\nu,\eta'}^{sh_t}\frac{\partial p}{\partial t}d\lambda_\nu d\phi_{\eta'}-\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})_{\nu,\eta'}(\frac{\partial p}{\partial t})d\lambda_\nu d\phi_{\eta'}|$

$\leq\epsilon\int_{\mathcal{R}_{\eta'}}\int_0^{t'}|\frac{\partial p}{\partial t}|d\lambda_\nu d\phi_{\eta'}$

$=D\epsilon\simeq 0$

Similarly;

$|\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})_{\nu,\eta'}^{sh_x^2}\frac{\partial^2 Dp}{\partial x^2}d\lambda_\nu d\phi_{\eta'}-\int_{\overline{\mathcal{R}_{\eta'}}}\int_0^{t'}(W_{t_0})_{\nu,\eta'}(\frac{\partial^2 Dp}{\partial x^2})d\lambda_\nu d\phi_{\eta'}|$

$\leq\epsilon\int_{\mathcal{R}_{\eta'}}\int_0^{t'}|\frac{\partial^2 Dp}{\partial x^2}|d\lambda_\nu d\phi_{\eta'}$

$=C\epsilon\simeq 0$

□

Lemma 0.27. *If* $\frac{\partial p}{\partial t}$ *and* $\frac{\partial^2(Dp)}{\partial x^2}$ *are Lebesgue-integrable, we have that;*

$\frac{\partial p}{\partial t}+\frac{\partial^2(Dp)}{\partial x^2}\simeq 0$

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