

# A NONSTANDARD APPROACH TO EQUIDISTRIBUTION

TRISTRAM DE PIRO

ABSTRACT. Using nonstandard analysis, we generalise a classical result on equidistributions to integrable functions, and give an application of the Weil conjectures for algebraic curves, to equidistribution in characteristic zero.

**Lemma 0.1.** *Let  $\{x_n : n \in \mathcal{N}\}$  be equidistributed on  $[0, 1)$ , then, if  $f \in L^1([0, 1))$ , we have that;*

$$\int_0^1 f d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(x_j)$$

*Proof.* We give a nonstandard proof of this result. Choose  $\eta \in {}^*\mathcal{N}$  infinite. By transfer, we can find an internal sequence  $\{s_i : i \in ({}^*\mathcal{N} \cap [1, \eta])\} \subset {}^*[0, 1)$ , with  $s_i = x_i$ , for  $i \in \mathcal{N}$ . We adopt the notation  $(\overline{S}_\eta, \mathcal{C}_\eta)$  of Definition 0.4 in [4], and define  $\delta_\eta : \mathcal{C}_\eta \rightarrow {}^*\mathcal{R}$  by setting;

$$\delta_\eta\left(\left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\right) = \frac{1}{\eta} {}^*\text{Card}\left(\left\{i \in {}^*\mathcal{N} \cap [1, \eta] : s_i \in \left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\right\}\right)$$

$$\delta_\eta(V) = {}^*\sum_{\frac{j}{\eta} \in V} \delta_\eta\left(\left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\right) \quad (*)$$

for  $0 \leq j \leq \eta - 1$  and  $V \in \mathcal{C}_\eta$ . It is easily verified that  $\delta_\eta$  is finitely additive, hence  $*$ -finitely additive. Following [3], we let  $(L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta))$  denote the associated Loeb space. We claim that the standard part mapping;

$$st : (L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta)) \rightarrow ([0, 1], \mathcal{B}, \mu)$$

is measurable and measure preserving, (\*\*), where  $\mathcal{B}$  denotes the completion of the Borel field on  $[0, 1]$  and  $\mu$  is Lebesgue measure. Observe, for  $\{a, b\} \subset ([0, 1) \cap \mathcal{Q})$ , using (\*) and the fact that  $[a, b) = \bigcup_{a\eta \leq j < b\eta} \left[\frac{j}{\eta}, \frac{j+1}{\eta}\right)$ ;

$$\delta_\eta({}^*[a, b)) = \frac{1}{\eta} {}^*\text{Card}\{1 \leq i \leq \eta : s_i \in [a, b)\}$$

The internal sequence  $\{s_{a,b}^i\}_{1 \leq i \leq \eta}$ , defined by  $s_{a,b}^i = \frac{1}{i} * \text{Card}(\{k \in {}^* \mathcal{N} \cap [1, i] : s_k \in {}^* [a, b]\})$ , has the property that  $s_{a,b}^\eta \simeq b - a$ , using Theorem 2.22(i) of [3]. Hence  $L(\delta_\eta)({}^* [a, b]) = b - a$ . Now, let  $\{c, d\} \subset ([0, 1] \cap \mathcal{R})$ , and assume that  $c \neq 0$ , <sup>(1)</sup>. Choose sequences  $\{c_{l,n}, c_{u,n}, d_{l,n}, d_{u,n} : n \in \mathcal{Z}_{\geq 1}\} \subset ({}^* [0, 1] \cap \mathcal{Q})$  such that  $c_{u,n} < c < c_{l,n} < d_{l,n} < d < d_{u,n}$ ,  $\lim_{n \rightarrow \infty} c_{u,n} = \lim_{n \rightarrow \infty} c_{l,n} = c$  and  $\lim_{n \rightarrow \infty} d_{u,n} = \lim_{n \rightarrow \infty} d_{l,n} = d$ . We have that  $[c_{l,n}, d_{l,n}] \subset [\frac{[n]c}{n}, \frac{[n]d}{n}] \subset [c_{u,n}, d_{u,n}]$ , for  $n \in \mathcal{Z}_{\geq 1}$ . Then, using elementary properties of measures, we have that;

$$\begin{aligned} L(\delta_\eta)\left(\left[\frac{[n]c}{n}, \frac{[n]d}{n}\right]\right) &= \lim_{n \rightarrow \infty} L(\delta_\eta)([c_{l,n}, d_{l,n}]) \\ &= \lim_{n \rightarrow \infty} L(\delta_\eta)([c_{u,n}, d_{u,n}]) \\ &= \lim_{n \rightarrow \infty} (d_{l,n} - c_{l,n}) \\ &= \lim_{n \rightarrow \infty} (d_{u,n} - c_{u,n}) = d - c \end{aligned}$$

We can now follow Theorem 14 in [1], to obtain that  $L(\delta_\eta)(st^{-1}([c, d])) = d - c$ , and then (\*\*) is shown, using the same proof. For  $g \in V(\overline{S}_\eta)$ , and  $A \in \mathcal{C}_\eta$ , we let  $\int_A g d\delta_\eta$  be as in Definition 3.9 of [3], and define  $S$ -integrability, as in Definition 3.17 of [3]. Then, we have, by Theorem 3.20 of [3], that, for  $g$   $S$ -integrable;

$$\circ \int_{\overline{S}_\eta} g d\delta_\eta = \int_{\overline{S}_\eta} \circ g dL(\delta_\eta), (***)$$

If  $f \in L^1([0, 1], \mathcal{B}, \mu)$ , using the result (\*\*), we must have that  $st^*(f) \in L^1(L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta))$ . We claim that there exists  $g \in SL^1(\overline{S}_\eta)$ , <sup>(2)</sup>, with the property that  $g(x_i) = f(x_i)$ , for  $1 \leq i \leq \eta$  and  $\circ g = st^*(f)$  a.e  $d(L(\delta_\eta))$ , (\*\*\*\*). We follow the case by case proof of Theorem 3.31 in [3]. The case when  $st^*(f)$  is bounded follows by choosing the initial sequence of  $\mathcal{C}_\eta$ -measurable functions  $\{f_n\}_{n \in \mathcal{N}_{>0}}$  to have the property that  $f_n(x_i) = f(x_i)$ , for  $1 \leq i \leq n$ . After extending the sequence  $\{f_n\}_{n \in \mathcal{N}_{>0}}$  to an internal sequence  $\{f_n\}_{1 \leq n \leq \omega'}$ , for some infinite  $\omega'$ , this property continues to hold by overflow, quantifying over the internal sequence  $\{f(s_i)\}_{1 \leq i \leq \min(\omega', \eta)}$ . Choosing  $\omega \leq \omega'$ , as in the proof of Theorem 3.13, we obtain that  $f_\omega(x_i) = f(x_i)$ , for  $i \in \mathcal{N}$ , (\*\*\*\*). For the general case, we can follow the proof, requiring, using (\*\*\*\*), and replacing  $\overline{S}_\eta$  by  $A_n$ , that the sequence  $\{f_n\}_{n \in \mathcal{N}_{>0}}$ ,

<sup>1</sup>The case  $c = 0$  can be dealt with, by observing that  $[n]0 = 0$ , and taking  $c_{u,n} = 0$ .

<sup>2</sup>Using the notation in [1] for  $S$ -integrable functions.

has the property that  $f_n(x_i) = f(x_i)$ , for any  $s_i \in A_n$ . Hence, (\*\*\*) is shown. Then, using (\*\*), (\*\*\*), (\*\*\*);

$$\begin{aligned}
& \circ(\frac{1}{\eta}^* \sum_{1 \leq j \leq \eta} f(s_j)) \\
&= \circ(\frac{1}{\eta}^* \sum_{1 \leq j \leq \eta} g(s_j)) \\
&= \circ \int_{\bar{S}_\eta} g d\delta_\eta \\
&= \int_{\bar{S}_\eta} \circ g dL(\delta_\eta) \\
&= \int_{\bar{S}_\eta} st^*(f) dL(\delta_\eta) = \int_0^1 f d\mu
\end{aligned}$$

The lemma then follows, this time using Theorem 2.22(ii) of [3].  $\square$

**Definition 0.2.** If  $\eta \in {}^*\mathcal{N}$  is infinite, we say that an internal sequence  $\{s_i\}_{1 \leq i \leq \eta} \subset {}^*[0, 1)$  is equidistributed if it corresponds, by transfer, to a standard equidistributed sequence  $\{x_i\}_{i \in \mathbb{Z}_{\geq 1}} \subset [0, 1)$ . An internal sequence  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed if, for the associated measure  $L(\delta_\eta)$ ,  $L(\delta_\eta)(a, b) = b - a$ , for  $\{a, b\} \subset [0, 1)$ .

**Remarks 0.3.** Observe, from the proof of Lemma 0.1, that equidistributed implies weakly equidistributed, and, if  $\{s_i\}_{1 \leq i \leq \eta}$  is equidistributed or weakly equidistributed, then for any standard  $f \in L^1([0, 1))$ ,  $(\frac{1}{\eta}^* \sum_{1 \leq j \leq \eta} f(s_j)) \simeq \int_0^1 f d\mu$ .

**Lemma 0.4.** If  $\eta \in {}^*\mathcal{N}$  is infinite, an internal sequence  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed iff  $\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \exp_\eta(2\pi i k s_i) \simeq 0$ , for finite  $k \in \bar{\mathcal{Z}}_{\eta, \neq 0}$ , (<sup>3</sup>).

*Proof.* Suppose that  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed, then, using the proof of Lemma 0.1 and Remark 0.2, we have that, for finite  $k \in \mathcal{Z}_{\eta, \neq 0}$ , as  $\exp_\eta(2\pi i k x)$  is  $S$ -integrable;

$$\begin{aligned}
& \circ(\frac{1}{\eta}^* \sum_{1 \leq j \leq \eta} \exp_\eta(2\pi i k s_j)) \\
&= \circ \int_{\bar{S}_\eta} \exp_\eta(2\pi i k x) d\delta_\eta
\end{aligned}$$

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<sup>3</sup>We adopt the notation of Definition 0.8 in [4], letting  $\exp_\eta(2\pi i k x)$  denote the  $\mathcal{C}_\eta$ -measurable counterpart of  ${}^*\exp(2\pi i k x)$  on  ${}^*[0, 1)$ , and  $\bar{\mathcal{Z}}_{\eta, \neq 0} = \{k \in {}^*\mathcal{Z} : -\eta \leq k \leq \eta - 1\}$ .

$$\begin{aligned}
&= \int_{\overline{S}_\eta} \circ \exp_\eta(2\pi i k x) dL(\delta_\eta) \\
&= \int_{\overline{S}_\eta} st^*(\exp(2\pi i k x)) dL(\delta_\eta) = \int_0^1 \exp(2\pi i k x) d\mu = 0
\end{aligned}$$

Conversely, suppose that  $\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \exp_\eta(2\pi i k s_i) \simeq 0$ , (\*), for finite  $k \in \mathcal{Z}_{\eta, \neq 0}$ . Let  $\{a, b\} \subset [0, 1)$ ,  $\epsilon > 0$  and choose  $f \in C^\infty([0, 1])$ , (<sup>4</sup>), such that  $\|f - \chi_{[a, b]}\|_{C([0, 1])} < \epsilon$ , (\*\*). Suppose that  $f = g + r$ , where  $r = \int_0^1 f d\mu$ , so that  $g \in C^\infty([0, 1])$  and  $\int_0^1 g d\mu = 0$ . Using Lemma 0.9 of [4], we have that;

$$g_\eta(x) = \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_\eta(k) \exp_\eta(2\pi i k x), \quad (5)$$

Hence, using (\*), the fact that  $|\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \exp_\eta(2\pi i k s_i)| \leq 1$ , and  $|\hat{g}_\eta(k)| \leq \frac{H}{k^2}$ , (\*\* \*), for  $k \in \overline{\mathcal{Z}}_\eta$ , where  $H \in \mathcal{R}$ , (<sup>6</sup>);

$$\begin{aligned}
\frac{1}{\eta} \sum_{1 \leq i \leq \eta} g_\eta(s_i) &= \frac{1}{\eta} \sum_{1 \leq i \leq \eta} \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_\eta(k) \exp_\eta(2\pi i k s_i) \\
&= \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_\eta(k) \frac{1}{\eta} \sum_{1 \leq i \leq \eta} \exp_\eta(2\pi i k s_i) \simeq 0
\end{aligned}$$

Hence;

$$\frac{1}{\eta} \sum_{1 \leq i \leq \eta} f_\eta(s_i) \simeq r$$

and, using (\*\*);

$$|\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \chi_{[a, b], \eta}(s_i) - \frac{1}{\eta} \sum_{1 \leq i \leq \eta} f_\eta(s_i)| \leq \frac{1}{\eta} \eta \epsilon = \epsilon$$

Hence, as  $|\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \chi_{(a, b), \eta}(s_i) - r| < 2\epsilon$  and  $|r - (b - a)| < \epsilon$ , we have that;

$$|\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \chi_{(a, b), \eta}(s_i) - (b - a)| < 3\epsilon$$

and, as  $\epsilon$  was arbitrary;

$$\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \chi_{(a, b), \eta}(s_i) \simeq (b - a)$$

It follows that  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed.

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<sup>4</sup>We let  $C^\infty([0, 1]) = \{f \in C[0, 1] : \exists g \in C^\infty(S^1), \text{ang}^* g = f\}$ , where  $\text{ang}(\theta) = e^{2\pi i \theta}$ , for  $\theta \in [0, 1]$ .

<sup>5</sup>We adopt the notation, in Definition 0.8, for  $\{g_\eta, \hat{g}_\eta, \overline{\mathcal{Z}}_\eta\}$ .

<sup>6</sup>For (\*\* \*), see Lemma 0.16 and Theorem 0.19 of [4].

□

**Lemma 0.5.** *Let  $p \in \mathcal{R}[x]$  be a standard polynomial of degree  $d$ ,  $p(x) = \sum_{l=0}^d a_l x^l$ , with  $0 \leq a_l < 1$ , then;*

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{[a_l q]}{q} = a_l$$

and, if  $0 \leq a < b < 1$  and  $f \in L^1([0, 1])$ ;

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} |\{i : 1 \leq i \leq q, \sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1} \in (a, b)\}| = (b - a)$$

$$\int_0^1 f d\mu = \lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} \sum_{i=1}^q f(\sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1})$$

*Proof.* The first claim follows easily from the fact that, for infinite  $\eta \in {}^*\mathcal{N}$  prime,  $\frac{[a_l \eta]}{\eta} \simeq a_l$ , and Theorem 2.2(i) of [4]. For  $q$  prime in  ${}^*\mathcal{N}$ , let  $t_{l,q} = [qa_l]$ . We have  $a_l \simeq \frac{[qa_l]}{q}$ , and, therefore,  $0 \leq t_{l,q} < q$ , for  $\eta \in {}^*\mathcal{N}$  infinite prime. It follows, using underflow, that  $0 \leq t_{l,q} < q$ , for sufficiently large  $q \in \mathcal{N}$  prime,  $q \geq N(p)$ , (\*). For  $q \in {}^*\mathcal{N}$  prime, let  $p_q = \sum_{l=0}^d t_{l,q} x^l$ . We now claim that, for infinite  $\eta \in {}^*\mathcal{N}$  prime, the sequence  $\{\frac{p_\eta}{\eta}(j)\}_{1 \leq j \leq \eta}$  is weakly equidistributed, (\*\*). By Lemma 0.4, it is sufficient to show that there exists an infinite  $\eta \in {}^*\mathcal{N}$ , with  $\frac{1}{\eta} \sum_{1 \leq j \leq \eta} \exp_\eta(2\pi i k \frac{p_\eta}{\eta}(j)) \simeq 0$ , for  $k \in \overline{\mathcal{Z}}_\eta$ ,  $k$  finite, (\*\*\*) .

Let  $F_q \cong \mathcal{Z}/q\mathcal{Z}$  denote a finite field with  $q$  elements. Using Lemma 0.5 of [2], we have that, for  $q \geq N(p)$ ,  $(q, d) = 1$ , and for  $0 < k \leq \eta - 1$ ;

$$|\sum_{0 \leq j \leq q-1} e^{2\pi i \frac{k}{q} p_q(j)}| \leq (d-1)q^{\frac{1}{2}} + 1$$

If  $\eta \in {}^*\mathcal{N}$  is prime, then  $(\eta, d) = 1$ ,  $\eta \geq N(p)$ , and, by transfer, for  $0 < k \leq \eta - 1$ ;

$$\frac{1}{\eta} |\sum_{0 \leq j \leq \eta-1} {}^* \exp(2\pi i \frac{k}{\eta} p_\eta(j))| \leq \frac{(d-1)}{\eta^{\frac{1}{2}}} + \frac{1}{\eta} \simeq 0$$

The characters  $\{e^{2\pi i \frac{k}{q}} : -(q-1) \leq k \leq -1\}$  are just a re-enumeration of the characters  $\{e^{2\pi i \frac{k}{q}} : 1 \leq k \leq q-1\}$  on  $F_q$  for  $q$  prime, and, therefore, by the same argument;

$$\frac{1}{\eta} |\sum_{0 \leq j \leq \eta-1} {}^* \exp(2\pi i \frac{k}{\eta} p_\eta(j))| \simeq 0, \text{ for } k \in \mathcal{Z}_\eta \setminus \{-\eta, 0\}$$

As  $\exp(2\pi i k x)$  is continuous on  $[0, 1)$ , for  $k \in \mathcal{Z}$ , and;

$$\max(\frac{1}{\eta} \exp_{\eta}(2\pi i \frac{k}{\eta} p_{\eta}(0)), \frac{1}{\eta} (\exp_{\eta} 2\pi i \frac{k}{\eta} p_{\eta}(\eta))) \simeq 0$$

we have that;

$$\frac{1}{\eta} | \sum_{1 \leq j \leq \eta} \exp_{\eta}(2\pi i \frac{k}{\eta} p_{\eta}(j)) | \simeq 0, \text{ for finite } k \in \mathcal{Z}_{\eta, \neq 0}$$

It follows that  $(**), (***)$  hold. We have that, for any given  $\epsilon > 0$  standard,  $\eta \in {}^* \mathcal{N}$  infinite prime, that;

$$(b-a) - \epsilon < \frac{1}{\eta} |\{i : 1 \leq i \leq \eta, \sum_{l=0}^d \frac{[a_l \eta]}{\eta} i^l \pmod{1} \in (a, b)\}| < (b-a) + \epsilon$$

By underflow, there exists a standard  $N(\epsilon, p)$  prime, such that, for all standard primes  $q \geq N(\epsilon, p)$ ;

$$(b-a) - \epsilon < \frac{1}{q} |\{i : 1 \leq i \leq q, \sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1} \in (a, b)\}| < (b-a) + \epsilon$$

hence, the second claim is shown. Using Remarks 0.3, for any given  $f \in L^1([0, 1))$ , standard  $\epsilon > 0$ ,  $\eta \in {}^* \mathcal{N}$  infinite prime;

$$\int_0^1 f d\mu - \epsilon < \frac{1}{\eta} \sum_{i=1}^{\eta} f(\sum_{l=0}^d \frac{[a_l \eta]}{\eta} i^l \pmod{1}) < \int_0^1 f d\mu + \epsilon$$

Again, by underflow, there exists a standard  $M(\epsilon, p, f)$  prime, such that, for all standard primes  $q \geq M(\epsilon, p, f)$ ;

$$\int_0^1 f d\mu - \epsilon < \frac{1}{q} \sum_{i=1}^q f(\sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1}) < \int_0^1 f d\mu + \epsilon$$

Hence, the final claim is shown. □

**Definition 0.6.** If  $p \in \mathcal{R}[x]$ , and  $p_q, q$  prime, are as in Lemma 0.5, we define the associated measure;

$$\mu_{p,q} = \frac{1}{q} (\delta_{\frac{p_q}{q}(1) \pmod{1}} + \dots + \delta_{\frac{p_q}{q}(i) \pmod{1}} + \dots + \delta_{\frac{p_q}{q}(q) \pmod{1}})$$

where  $\{\delta_{\frac{p_q}{q}(i) \pmod{1}} : 1 \leq i \leq q\}$  are point measures supported at  $\{\frac{p_q}{q}(i) : 1 \leq i \leq q\}$ .

**Lemma 0.7.** If  $p \in \mathcal{R}[x]$ , then the sequence  $\{\mu_{p,q} : q \in \mathcal{N}, q \text{ prime}\}$  converges weakly to Lebesgue measure on  $[0, 1)$ .

*Proof.* The proof follows immediately from the last part of Lemma 0.5.  $\square$

## REFERENCES

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MATHEMATICS DEPARTMENT, HARRISON BUILDING, STREATHAM CAMPUS,  
UNIVERSITY OF EXETER, NORTH PARK ROAD, EXETER, DEVON, EX4 4QF,  
UNITED KINGDOM

*E-mail address:* `t.depiro@magneticstrix.net`