

# RIEMANN SUMS FOR RETURNING POINTS

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ABSTRACT.

## 1. PRELIMINARIES

**Lemma 1.1.** *Let  $s \in \mathcal{C}$ , with  $0 < \operatorname{Re}(s) < 1$ , then, if  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  converges, we have that;*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{e^{-i\ln(n)b}}{n^a}$$

where  $a = \operatorname{Re}(s)$ ,  $b = \operatorname{Im}(s)$ .

*Proof.* This follows simply from the fact that;

$$n^{-s} = e^{-s\ln(n)} = e^{-a\ln(n)-ib\ln(n)} = \frac{e^{-ib\ln(n)}}{n^a}, \text{ for } n \in \mathcal{Z}_{\geq 1}.$$

□

**Definition 1.2.** *We let;*

$$\Theta_b = \{\theta \in [-\pi, \pi) : \exists n \in \mathcal{Z}_{\geq 1} (-b\ln(n) \equiv \theta \pmod{2\pi})\}$$

and, if  $\theta \in [-\pi, \pi)$ ;

$$R_{\theta,b} = \{n \in \mathcal{Z}_{\geq 1} : -b\ln(n) \equiv \theta \pmod{2\pi}\}$$

We say that  $\theta$  is finite if  $\sum_{n \in R_{\theta,b}} \frac{1}{n^s}$  converges and, we then let;

$$S_{\theta,b} = \sum_{n \in R_{\theta,b}} \frac{1}{n^s}$$

**Lemma 1.3.** *If  $\theta$  is finite, we have that;*

$$S_{\theta,b} = e^{i\theta} \sum_{n \in R_{\theta,b}} \frac{1}{n^a}$$

and, more generally, without the finiteness assumption,  $n \in R_{\theta,b}$  iff  $K_{\theta,b}(n) = 1$ , where

$$K_{\theta,b}(x) = \cos(\text{bln}(x) + \theta)$$

*Proof.* The proof is clear from Definition 1.2. □

**Lemma 1.4.** *Suppose that  $\theta_0 \in [-\pi, \pi)$ ,  $b \neq 0$  and  $R_{\theta_0,b} \neq \emptyset$ , then, if  $n_0 \in \mathcal{Z}_{\geq 1}$  is minimal with the property that  $n_0 \in R_{\theta_0,b}$ , we have that  $n \in R_{\theta_0,b}$  iff  $n = n_0 C_b^l$ , for some  $l \in \mathcal{Z}_{\geq 0}$ , where  $C_b = e^{\frac{2\pi}{|b|}}$ .*

*Proof.* Assume, first, that  $b > 0$ . We have that  $\theta_0 = -\text{bln}(n_0) + 2k_0\pi$ , for some  $k_0 \in \mathcal{Z}_{\geq 0}$ . Then;

$$\begin{aligned} n \in R_{\theta_0,b} &\text{ iff } -\text{bln}(n) - \theta_0 = -2k\pi \text{ for some } k \in \mathcal{Z}_{\geq k_0} \\ &\text{ iff } -\text{bln}(n_0 + r) - (-\text{bln}(n_0) + 2k_0\pi) = -2k\pi, \text{ for some } k \in \mathcal{Z}_{\geq k_0} \\ &\text{ iff } -\text{bln}\left(\frac{n_0+r}{n_0}\right) = 2(k_0 - k)\pi \\ &\text{ iff } \text{bln}\left(\frac{n_0+r}{n_0}\right) = 2(k - k_0)\pi \\ &\text{ iff } \text{bln}\left(1 + \frac{r}{n_0}\right) = 2l\pi, \text{ for some } l \in \mathcal{Z}_{\geq 0} \\ &\text{ iff } r = n_0(C_b^l - 1) \\ &\text{ iff } n = n_0 C_b^l \end{aligned}$$

where  $n = n_0 + r$ . A similar proof holds when  $b < 0$ , obtaining  $C_b = e^{\frac{-2\pi}{b}}$ . □

**Definition 1.5.** *We say that  $\theta_0 \in \Theta_b$  is unique if  $|R_{\theta_0,b}| = 1$ , and returning if  $R_{\theta_0,b}$  is infinite.*

**Lemma 1.6.**  *$\theta_0 \in \Theta_b$  is not unique, iff there exists  $l_0 \in \mathcal{Z}_{\geq 1}$ , with  $n_0 C_b^{l_0} \in \mathcal{Z}_{\geq 1}$ , where  $n_0$  and  $C_b$  are given in the previous lemma.*

*Proof.* Clear by Lemma 1.4 and Definition 1.5. □

**Lemma 1.7.** *If  $\theta_0 \in \Theta_b$  is not unique, and  $b \neq 0$ , then  $\theta_0$  is finite.*

*Proof.* Choose  $l_0 \in \mathcal{Z}_{\geq 1}$  minimal, with  $C_b^{l_0} \in \mathcal{Q}$ , this is possible as  $\theta_0$  is not unique. Then clearly  $C_b^{kl_0} \in \mathcal{Q}$ , for all  $k \in \mathcal{Z}_{\geq 0}$ , (\*). Suppose that  $n_0 C_b^l \in \mathcal{Z}_{\geq 1}$ . Let  $l = kl_0 + l'$ , where  $0 \leq l' < l_0$  and  $k \in \mathcal{Z}_{\geq 0}$ , then, we have that  $n_0 C_b^{kl_0} C_b^{l'} \in \mathcal{Q}_{\geq 1}$ , hence, by (\*),  $C_b^{l'} \in \mathcal{Q}$ . It follows that  $l' = 0$  and  $l = kl_0$ . Then;

$$\begin{aligned}
& \sum_{n \in R_{\theta,b}} \frac{1}{n^a} \\
& \leq \sum_{q=n_0 C_b^{kl_0}, k \in \mathcal{Z}_{\geq 0}} \frac{1}{n^a} \\
& = \sum_{k \in \mathcal{Z}_{\geq 0}} \frac{1}{(n_0 C_b^{kl_0})^a} \\
& = \frac{1}{n_0^a} \sum_{k \in \mathcal{Z}_{\geq 0}} \left( \frac{1}{C_b^{l_0 a}} \right)^k \\
& = \frac{1}{n_0^a} \frac{1}{1 - C_b^{-l_0 a}} \\
& \text{as } C_b^{l_0 a} > 1, \text{ because } \frac{2\pi l_0 a}{|b|} > 0.
\end{aligned}$$

□

**Lemma 1.8.**  $\theta_0 \in \Theta_b$  is returning iff there exists  $l \in \mathcal{Z}_{\geq 1}$  such that  $C_b^l \in \mathcal{Z}$ . In particular, if there exists  $\theta_0 \in \Theta_b$  which is returning, then every  $\theta \in \Theta_b$  is returning, and if  $\theta_0 \in \Theta_b$  and  $|R_{\theta_0,b}| \geq 3$ , then  $\theta_0$  is returning. If  $\theta_0 \in \Theta_b$  is returning then  $R_{\theta_0,b} = \{n_0 C_b^{ll_0} : l \in \mathcal{Z}_{\geq 0}\}$ , where  $l_0 \in \mathcal{Z}_{\geq 1}$  is minimal with the property that  $C_b^{l_0} \in \mathcal{Z}$ . If  $C_b^l \notin \mathcal{Q}$ , for  $l \in \mathcal{Z}_{\geq 1}$ , then  $|R_{\theta_0,b}| = 1$ , for every  $\theta_0 \in \Theta_b$ . If  $C_b^l \notin \mathcal{Z}$ , for  $l \in \mathcal{Z}_{\geq 1}$ , and there exists  $l \in \mathcal{Z}_{\geq 1}$  with  $C_b^l \in \mathcal{Q}$ , (\*), then  $D_\theta = \{\theta \in \Theta_b : |R_{\theta,b}| = 2\} \neq \emptyset$ , and moreover,  $D_\theta = \{-kbln(n_0) \pmod{2\pi} : k \in \mathcal{Z}_{\geq 1}\}$ , where  $C_b^{l_0} = \frac{n_1}{n_0}$ ,  $(n_1, n_0) = 1$ , and  $l_0 \in \mathcal{Z}_{\geq 1}$  is minimal with the property that  $C_b^{l_0} \in \mathcal{Q}$ .

*Proof.* Suppose that there exists  $l \in \mathcal{Z}_{\geq 1}$  such that  $C_b^l \in \mathcal{Z}$ , then, if  $k \in \mathcal{Z}_{\geq 0}$ , choosing  $\theta_0 \in \Theta_b$ , we have, if  $n_k = n_0 C_b^{kl}$ , that, by Lemma 1.4,  $n_k \in R_{\theta_0,b}$ . Moreover, if  $k_1 \neq k_2$ , then  $n_{k_1} \neq n_{k_2}$ , as  $C_b > 1$ , and  $l \geq 1$ . Hence,  $\theta_0 \in \Theta_b$  is returning. Conversely, suppose that  $\theta_0 \in \Theta_b$  is returning. Let  $n_1$  correspond to the first return, then, there exists  $l_0 \in \mathcal{Z}_{\geq 1}$  such that  $C_b^{l_0} = \frac{n_1}{n_0}$ . If  $l \in \mathcal{Z}_{\geq 1}$ , we have that  $n_0 C_b^l \in \mathcal{Z}$  iff  $n_0 \left(\frac{n_1}{n_0}\right)^{\frac{l}{l_0}} \in \mathcal{Z}$  which implies that  $n_0^{l_0} \left(\frac{n_1}{n_0}\right)^l \in \mathcal{Z}$ , therefore,  $\frac{n_1^l}{n_0^{l-l_0}} \in \mathcal{Z}$ , and  $n_0^{l-l_0} |n_1^l$ . Taking  $l > l_0$ , and considering prime factors, we must have that  $n_0 | n_1$  and  $C_b^{l_0} \in \mathcal{Z}$ , as required. The second part is clear from the above proof. The third part follows from the first part and Lemma 1.4. The fourth claim is clear from the proof. Finally, if (\*) holds, and  $\{l_0, n_0, n_1\} \subset \mathcal{Z}_{\geq 1}$  are chosen, as in the hypotheses, then  $|R_{\theta,b}| = 2$  iff there exist  $\{l, n, n'\} \subset \mathcal{Z}_{\geq 1}$ , with  $C_b^l = \frac{n'}{n}$ , and  $\theta = -bln(n) \pmod{2\pi}$ . By the choice of  $l_0$ , we have that  $l = kl_0$ , for some  $k \in \mathcal{Z}_{\geq 1}$ ,  $n = n_0^k$ ,  $\theta = -kbln(n_0) \pmod{2\pi}$ .

□

**Definition 1.9.** *Until Lemma ?, we assume that every  $\theta_0 \in \Theta_b$  is returning, so that  $R_{\theta_0,b} = \{q \in \mathcal{Q} : n = n_0 C_b^{kl_0}, k \in \mathcal{Z}_{\geq 0}\}$*

We let  $\phi_b : (1 - \epsilon, \infty) \rightarrow \mathcal{C}$  be defined by;

$$\phi_{\theta_0,b}(x) = n_0 C_b^{l_0(x-1)}$$

and let  $Q_{\theta_0,a,b} : (1 - \epsilon, \infty) \rightarrow \mathcal{C}$  be defined by;

$$Q_{\theta_0,a,b}(x) = \phi_{\theta_0,b}(x)^{-a}.$$

**Remarks 1.10.** *Observe that  $\phi_{\theta_0,b}|_{\mathcal{Z}_{\geq 1}}$  defines a bijection between  $\mathcal{Z}_{\geq 1}$  and  $R_{\theta_0,b}$ .*

**Lemma 1.11.**  *$Q_{\theta_0,a,b} \in C^2(1 - \epsilon, \infty)$ , and there exists a constant  $D$ , such that  $\sup_{x \in (1-\epsilon, \infty)} \{x^4 |Q_{\theta_0,a,b}^{(k)}| : 0 \leq k \leq 2\} \leq D$ .*

*Proof.* This is a simple calculation. We have that;

$$Q_{\theta_0,a,b}(x) = \phi_{\theta_0,b}(x)^{-a} = n_0^{-a} C_b^{-al_0(x-1)} = n_0^{-a} C_b^{al_0} C_b^{-al_0 x} = n_0^{-a} C_b^{al_0} e^{\frac{-2\pi al_0 x}{|b|}}$$

Observing that  $(x^4 e^{\frac{-2\pi al_0 x}{|b|}})' = 0$  iff  $x = 0$  or  $x = \frac{2|b|}{\pi al_0}$ , we obtain that;

$$\sup_{x \in (1-\epsilon, \infty)} x^4 e^{\frac{-2\pi al_0 x}{|b|}} = x^4 e^{\frac{-2\pi al_0 x}{|b|}} \Big|_{\frac{2|b|}{\pi al_0}}$$

$$= \frac{16|b|^4}{\pi^4 a^4 l_0^4 e^4}, \text{ if } |b| > \frac{(1-\epsilon)\pi al_0}{2}$$

$$\sup_{x \in (1-\epsilon, \infty)} x^4 e^{\frac{-2\pi al_0 x}{|b|}} = x^4 e^{\frac{-2\pi al_0 x}{|b|}} \Big|_{1-\epsilon}$$

$$= (1 - \epsilon)^4 e^{\frac{-2\pi al_0(1-\epsilon)}{|b|}}, \text{ if } 0 < |b| \leq \frac{(1-\epsilon)\pi al_0}{2}$$

Letting  $D_{1,b} = \max(\frac{16|b|^4}{\pi^4 a^4 l_0^4 e^4}, (1 - \epsilon)^4 e^{\frac{-2\pi al_0(1-\epsilon)}{|b|}})$ , we have that;

$$\sup_{x \in (1-\epsilon, \infty)} (x^4 |Q_{\theta_0,a,b}| \leq D_{1,b} n_0^{-a} C_b^{al_0}$$

$$\sup_{x \in (1-\epsilon, \infty)} (x^4 |Q'_{\theta_0,a,b}| \leq D_{1,b} n_0^{-a} C_b^{al_0} \frac{2\pi al_0}{|b|}$$

$$\sup_{x \in (1-\epsilon, \infty)} (x^4 |Q''_{\theta_0,a,b}| \leq D_{1,b} n_0^{-a} C_b^{al_0} (\frac{2\pi al_0}{|b|})^2, \quad (1)$$

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<sup>1</sup>In fact,  $Q_{\theta_0,a,b} \in \mathcal{S}(1 - \epsilon, \infty)$

Letting  $D = \max(D_{1,b}n_0^{-a}C_b^{al_0}, D_{1,b}n_0^{-a}C_b^{al_0\frac{2\pi al_0}{|b|}}, D_{1,b}n_0^{-a}C_b^{al_0}(\frac{2\pi al_0}{|b|})^2)$ , we obtain the result. □

**Lemma 1.12.** *If  $\theta \in \Theta_b$ , we have that;*

$$S_{\theta,b} = e^{i\theta} \sum_{n \in R_{\theta,b}} \frac{1}{n^a} = e^{i\theta} \sum_{n \in \mathcal{Z}_{\geq 1}} Q_{\theta,a,b}(n)$$

*Proof.* The result follows immediately from Remark . □

**Definition 1.13.** *Let  $r \in \mathcal{Z}_{\geq 1}$ , and  $\theta_0 \in \Theta_b$ , then we define  $F_{\theta_0,a,b,\delta,r} : \mathcal{R} \rightarrow \mathcal{C}$  by letting;*

$$F_{\theta_0,a,b,r}(x) = Q_{\theta_0,a,b}(x), \quad (x \geq r)$$

$$F_{\theta_0,a,b,r}(x) = Q_{\theta_0,a,b}(-x), \quad (x \leq -r)$$

$$F_{\theta_0,a,b,r}(x) = p_{\theta_0,a,b,r}(x), \quad (|x| \leq r)$$

where  $p_{\theta_0,a,b,r}$  is a symmetric polynomial with  $p_{\theta_0,a,b,r}^{(k)}(r) = Q_{\theta_0,a,b}^{(k)}(r)$ , for  $0 \leq k \leq 2$ , and  $p_{\theta_0,a,b,r}(n) = Q_{\theta_0,a,b,r}(n)$ , for  $1 \leq n \leq r$ .

**Lemma 1.14.**  *$F_{\theta_0,a,b,r} \in C^2(\mathcal{R})$ , and there exists a constant  $F$ , such that  $\sup_{x \in \mathcal{R}} \{x^4 |F_{\theta_0,a,b,r}^{(k)}| : 0 \leq k \leq 2\} \leq F$ .*

*Proof.* The proof is clear by Lemma 1.11 and the construction in Definition 1.13. □

**Lemma 1.15.** *For  $\theta_0 \in \Theta_b$  and  $r \in \mathcal{Z}_{\geq 1}$ ;*

$$\begin{aligned} & \sum_{m \in R_{\theta_0,b}} m^{-a} \\ &= \sum_{j=0}^{r+1} (\bar{a}_{\theta_0,a,b,r})_{j+1} \left( \frac{B_{2j+1}(r)}{2j+1} \right) + \frac{n_0^{-a} e^{\frac{2\pi al_0(1-r)}{|b|}}}{2} + \frac{|b| n_0^{-a} e^{\frac{2\pi al_0(1-r)}{|b|}}}{2\pi al_0} + R_{\theta_0,a,b,r} \end{aligned}$$

where  $\lim_{r \rightarrow \infty} R_{\theta_0,a,b,r} = 0$ , and;

$$|R_{\theta_0,a,b,r}| \leq \frac{D}{9r^3}$$

$$p_{\theta_0,a,b,r} = \sum_{j=0}^{r+1} (\bar{a}_{\theta_0,a,b,r})_{j+1} x^{2j}$$

*Proof.* The first part follows the fact that, using Lemma 1.14,  $F_{\theta_0,a,b,r}$  satisfies the conditions of Lemma 1.16, and Lemma 1.20 in [5], gives;

$$\begin{aligned}
& \sum_{m \in R_{\theta_0,b}, m \geq \phi_{\theta,b}(r)} m^{-a} \\
&= \sum_{n=r}^{\infty} Q_{\theta_0,a,b}(n) \\
&= \int_r^{\infty} Q_{\theta_0,a,b}(x) dx + \frac{p_{\theta_0,a,b,r}(r)}{2} + \frac{R_{\theta_0,a,b,r}}{2} \\
& \sum_{m \in R_{\theta_0,b}, m < \phi_{\theta,b}(r)} m^{-a} \\
&= \sum_{n=1}^{r-1} Q_{\theta_0,a,b}(n) \\
&= \sum_{j=0}^{r+1} (\bar{a}_{\theta_0,a,b,r})_{j+1} \binom{B_{2j+1}(r)}{2j+1}
\end{aligned}$$

where;

$$\begin{aligned}
R_{\theta_0,a,b,r} &= R_{\theta_0,a,b,r,1} + R_{\theta_0,a,b,r,2} \\
&= \sum_{n \in \mathbb{Z} \neq 0} \left( \int_r^{\infty} e^{2\pi i n x} Q_{\theta_0,a,b}(x) dx \right) + \sum_{n \in \mathbb{Z} \neq 0} \left( \int_r^{\infty} e^{-2\pi i n x} Q_{\theta_0,a,b}(x) dx \right)
\end{aligned}$$

We have that;

$$\begin{aligned}
R_{\theta_0,a,b,r,1} &= \sum_{n \in \mathbb{Z} \neq 0} \frac{-Q_{\theta_0,a,b}(r)}{2\pi i n} - \sum_{n \in \mathbb{Z} \neq 0} \frac{1}{2\pi i n} \int_r^{\infty} Q'_{\theta_0,a,b}(x) e^{2\pi i n x} dx \\
&= \sum_{n \in \mathbb{Z} \neq 0} \frac{Q'_{\theta_0,a,b}(r)}{(2\pi i n)^2} + \sum_{n \in \mathbb{Z} \neq 0} \frac{1}{(2\pi i n)^2} \int_r^{\infty} Q''_{\theta_0,a,b}(x) e^{2\pi i n x} dx \\
&= \frac{-Q'_{\theta_0,a,b}(r)}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} + D_{\theta_0,a,b,r,1} \\
&= \frac{-Q'_{\theta_0,a,b}(r)}{24} + D_{\theta_0,a,b,r,1}
\end{aligned}$$

where;

$$|D_{\theta_0,a,b,r,1}| \leq \frac{C_{\theta_0,a,b,r,1}}{4\pi^2} \sum_{n \in \mathbb{Z} \neq 0} \frac{1}{n^2} = \frac{C_{\theta_0,a,b,r,1}}{24}$$

$$\text{and } C_{\theta_0,a,b,r,1} \leq \int_r^{\infty} |Q''_{\theta_0,a,b}(x)| dx$$

$$\leq D \int_r^{\infty} \frac{dx}{x^4}$$

$$= \frac{D}{3r^3}$$

It follows that;

$$\begin{aligned} |R_{\theta_0, a, b, r, 1}| &\leq \frac{|E'_{\theta_0, a, b}(r)|}{24} + \frac{D}{72r^3} \\ &\leq \frac{D}{24r^4} + \frac{D}{72r^3} \\ &\leq \frac{D}{18r^3} \end{aligned}$$

Similarly,  $|R_{\theta_0, a, b, r, 2}| \leq \frac{D}{18r^3}$ , so that  $|R_{\theta_0, a, b, r}| \leq \frac{D}{9r^3}$ .

We have that;

$$\int_r^\infty Q_{\theta_0, a, b}(x) dx = \frac{bn_0^{-a} C_b^{al_0}}{2\pi al_0} e^{-\frac{2\pi al_0 r}{|b|}}$$

Therefore, for  $r \in \mathcal{Z}_{\geq 1}$ ;

$$\begin{aligned} &\sum_{m \in R_{\theta_0, b}} m^{-a} \\ &= \sum_{j=0}^{r+1} (\bar{a}_{\theta_0, a, b, r})_{j+1} \left( \frac{B_{2j+1}(r)}{2j+1} \right) + \frac{n_0^{-a} e^{-\frac{2\pi al_0(1-r)}{|b|}}}{2} + \frac{|b| n_0^{-a} e^{-\frac{2\pi al_0(1-r)}{|b|}}}{2\pi al_0} + R_{\theta_0, a, b, r} \end{aligned}$$

where  $\lim_{r \rightarrow \infty} R_{\theta_0, a, b, r} = 0$ , as required.

□

We now show a series of results about equidistributed sequences.

**Lemma 1.16.** *Let  $\{x_n : n \in \mathcal{Z}_{\geq 1}\}$  be equidistributed on  $[0, 1)$ , then, if  $f \in L^1([0, 1))$ , we have that;*

$$\int_0^1 f dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(x_j)$$

*Proof.* We give a nonstandard proof of this result. Choose  $\eta \in {}^*\mathcal{N}$  infinite. By transfer, we can find an internal sequence  $\{s_i : i \in ({}^*\mathcal{Z} \cap [1, \eta])\} \subset {}^*[0, 1]$ , with  $s_i = x_i$ , for  $i \in \mathcal{Z}_{\geq 1}$ . We adopt the notation  $(\bar{S}_\eta, \mathcal{C}_\eta)$  of Definition 0.4 in [7], and define  $\delta_\eta : \mathcal{C}_\eta \rightarrow {}^*\mathcal{R}$  by setting;

$$\delta_\eta\left(\left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\right) = \frac{1}{\eta} {}^* \text{Card}\{1 \leq i \leq \eta : s_i \in \left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\}$$

$$\delta_\eta(V) = {}^* \sum_{\frac{j}{\eta} \in V} \nu\left(\left[\frac{j}{\eta}, \frac{j+1}{\eta}\right]\right) (*)$$

for  $V \in \mathcal{C}_\eta$ . It is easily verified that  $\delta_\eta$  is finitely additive, hence  $*$ -finitely additive. We let  $(L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta))$  denote the associated Loeb space. We claim that the standard part mapping;

$$st : (L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta)) \rightarrow ([0, 1], \mathcal{B}, \mu)$$

is measurable and measure preserving, ( $\dagger$ ). First, observe that, for  $\{a, b\} \subset ([0, 1] \cap \mathcal{Q})$ , we have that;

$$\delta_\eta(*[a, b]) = \frac{1}{\eta} * \text{Card}\{1 \leq i \leq \eta : s_i \in [a, b]\}$$

using ( $*$ ) and the fact that  $[a, b] = \bigcup_{a\eta \leq j < b\eta} [\frac{j}{\eta}, \frac{j+1}{\eta})$ . The internal sequence  $\{s_{a,b}^i : \frac{1}{i} * \text{Card}\{1 \leq k \leq i : s_k \in *[a, b]\}\}$ , has the property that  $s_{a,b}^\eta \simeq b - a$ , using Theorem 2.22(i) of [8]. Hence  $L(\delta_\eta)(*[a, b]) = b - a$ . Now, let  $\{c, d\} \subset ([0, 1] \cap \mathcal{R})$ , and assume that  $c \neq 0$ , ( $^2$ ), and choose sequences  $\{c_{l,n}, c_{u,n}, d_{l,n}, d_{u,n} : n \in \mathcal{Z}_{\geq 1}\} \subset (*[0, 1] \cap \mathcal{Q})$  such that  $c_{u,n} < c < c_{l,n} < d_{l,n} < d < d_{u,n}$ , and  $\lim_{n \rightarrow \infty} c_{u,n} = \lim_{n \rightarrow \infty} c_{l,n} = c$ ,  $\lim_{n \rightarrow \infty} d_{u,n} = \lim_{n \rightarrow \infty} d_{l,n} = d$ . We have that  $[c_{l,n}, d_{l,n}) \subset [\frac{\lfloor \eta c \rfloor}{\eta}, \frac{\lfloor \eta d \rfloor}{\eta}) \subset [c_{u,n}, d_{u,n})$ , for  $n \in \mathcal{Z}_{\geq 1}$ . Then, using elementary properties of measures, we have that;

$$\begin{aligned} L(\delta_\eta)([\frac{\lfloor \eta c \rfloor}{\eta}, \frac{\lfloor \eta d \rfloor}{\eta})) &= \lim_{n \rightarrow \infty} L(\delta_\eta)([c_{l,n}, d_{l,n})) \\ &= \lim_{n \rightarrow \infty} L(\delta_\eta)([c_{u,n}, d_{u,n})) \\ &= \lim_{n \rightarrow \infty} (d_{l,n} - c_{l,n}) \\ &= \lim_{n \rightarrow \infty} (d_{u,n} - c_{u,n}) = d - c \end{aligned}$$

We can now follow Theorem 14 in [1], to obtain that  $L(\delta_\eta)(st^{-1}([c, d])) = d - c$ , and then ( $\dagger$ ) is shown, using the same proof. For  $g \in V(\overline{S}_\eta)$ , and  $A \in \mathcal{C}_\eta$ , we let  $\int_A g d\delta_\eta$  be as in Definition 3.9 of [8], and define  $S$ -integrability, as in Definition 3.17 of [8]. Then, we have, by Theorem 3.20 of [8] and ( $\dagger$ ), that, for  $g$   $S$ -integrable;

$$\circ \int_{\overline{S}_\eta} g d\delta_\eta = \int_{\overline{S}_\eta} \circ g d\delta_\eta, (\dagger\dagger)$$

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<sup>2</sup>The case  $c = 0$  can be dealt with, by observing that  $[\eta 0] = 0$ , and taking  $c_{u,n} = 0$  below.

If  $f \in L^1([0, 1], \mathcal{B}, \mu)$ , we have, by  $(\dagger)$ , that  $st^*(f) \in L^1(L(\overline{S}_\eta), L(\mathcal{C}_\eta), L(\delta_\eta))$ . We claim, using Theorem 3.31 of [8], that there exists  $g \in SL^1(\overline{S}_\eta)$  with the property that  $g(x_i) = f(x_i)$ , for  $1 \leq i \leq \eta$  and  ${}^\circ g = st^*(f)$  a.e  $d(L(\delta_\eta))$ ,  $(\dagger\dagger\dagger)$  (The case when  $st^*(f)$  is bounded follows by choosing the initial sequence of  $\mathcal{C}_\eta$ -measurable functions  $\{f_n\}_{n \in \mathcal{N}_{>0}}$  to have the property that  $f_n(x_i) = f(x_i)$ , for  $1 \leq i \leq n$ . After extending the sequence  $\{f_n\}_{n \in \mathcal{N}_{>0}}$  to an internal sequence  $\{f_n\}_{1 \leq n \leq \omega'}$ , for some infinite  $\omega'$ , this property continues to hold by overflow, quantifying over the internal sequence  $\{{}^*f(s_i)\}_{1 \leq i \leq \min(\omega', \eta)}$ . Choosing  $\omega \leq \omega'$ , as in the proof of Theorem 3.13, we obtain that  $f_\omega$  has the required property that  $f_\omega(x_i) = f(x_i)$ , for  $i \in \mathcal{N}$ ,  $(*)$ . For the general case in Theorem 3.31, we can follow the proof, requiring, using  $(*)$ , and replacing  $\overline{S}_\eta$  by  $A_n$ , that the sequence  $\{f_n\}_{n \in \mathcal{N}_{>0}}$ , has the property that  $f_n(x_i) = f(x_i)$ , for any  $s_i \in A_n$ .) Then, using  $(\dagger)$ ,  $(\dagger\dagger)$ ,  $(\dagger\dagger\dagger)$ ;

$$\begin{aligned}
& {}^\circ(\frac{1}{\eta}{}^* \sum_{1 \leq j \leq \eta} f(s_j)) \\
&= {}^\circ(\frac{1}{\eta}{}^* \sum_{1 \leq j \leq \eta} g(s_j)) \\
&= {}^\circ \int_{\overline{S}_\eta} g d\delta_\eta \\
&= \int_{\overline{S}_\eta} {}^\circ g dL(\delta_\eta) \\
&= \int_{\overline{S}_\eta} st^*(f) dL(\delta_\eta) = \int_0^1 f d\mu
\end{aligned}$$

The lemma then follows, again using Theorem 2.22(ii) of [8].  $\square$

**Definition 1.17.** *An internal sequence  $\{s_i\}_{1 \leq i \leq \eta}$  is equidistributed if, it corresponds to a standard equidistributed sequence  $\{x_i\}_{i \in \mathbb{Z}_{\geq 1}}$ . An internal sequence  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed if, for the associated measure  $L(\delta_\eta)$ ,  $L(\delta_\eta)(a, b) = b - a$ , for  $\{a, b\} \subset ({}^*[0, 1] \cap \mathcal{R})$ .*

**Remarks 1.18.** *Observe, from the proof of Lemma 1.16, that equidistributed implies weakly equidistributed, and, if  $\{s_i\}_{1 \leq i \leq \eta}$  is equidistributed or weakly equidistributed, then for any standard  $f \in L^1([0, 1])$ ,*  
 $(\frac{1}{\eta}{}^* \sum_{1 \leq j \leq \eta} f(s_j)) \simeq \int_0^1 f dx$

**Lemma 1.19.** *An internal sequence  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed iff  $\frac{1}{\eta}{}^* \sum_{1 \leq i \leq \eta} \exp_\eta(2\pi i k s_i) \simeq 0$ , for finite  $k \in \mathcal{Z}_{\eta, \neq 0}$ , and some (any)  $\eta \in {}^*\mathcal{N}$  infinite.*

*Proof.* Suppose that  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed, then, using the above proof, we have that, for finite  $k \in \mathcal{Z}_{\eta, \neq 0}$ , as  $\exp_\eta(2\pi i k x)$  is

$S$ -integrable;

$$\begin{aligned}
&= \circ(\frac{1}{\eta}^* \sum_{1 \leq j \leq \eta} \exp_{\eta}(2\pi i k s_j)) \\
&= \circ \int_{\bar{S}_{\eta}} \exp_{\eta}(2\pi i k x) d\delta_{\eta} \\
&= \int_{\bar{S}_{\eta}} \circ \exp_{\eta}(2\pi i k x) dL(\delta_{\eta}) \\
&= \int_{\bar{S}_{\eta}} st^*(\exp(2\pi i k x)) dL(\delta_{\eta}) = \int_0^1 \exp(2\pi i k x) d\mu = 0
\end{aligned}$$

Conversely, suppose that  $\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \exp_{\eta}(2\pi i k s_i) \simeq 0$ , (\*), for finite  $k \in \mathcal{Z}_{\eta, \neq 0}$ . Let  $\{a, b\} \subset [0, 1]$ , and choose  $f \in C^{\infty}([0, 1])$ , with  $f(0) = f(1)$ , such that  $\|f - \chi_{[a, b]}\|_{C([0, 1])} < \epsilon$ . Let  $f = g + r$ , where  $r = \int_0^1 f dx$ ,  $g \in C^{\infty}([0, 1])$ , with  $g(0) = g(1)$ . We have that;

$$g_{\eta}(x) = \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_{\eta}(k) \exp_{\eta}(2\pi i k x)$$

Hence;

$$\begin{aligned}
\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} g_{\eta}(s_i) &= \frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_{\eta}(k) \exp_{\eta}(2\pi i k s_i) \\
&= \sum_{k \in \mathcal{Z}_{\eta, \neq 0}} \hat{g}_{\eta}(k) \frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \exp_{\eta}(2\pi i k s_i) \simeq 0
\end{aligned}$$

Using the decay rate on the coefficients  $\hat{g}_{\eta}(k)$ , see [7], (\*), and the fact that  $|\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \exp_{\eta}(2\pi i k s_i)| \leq 1$ , for  $k \in \mathcal{Z}_{\eta}$ . Hence;

$$\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} f_{\eta}(s_i) \simeq r$$

It follows that;

$$|\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \chi_{[a, b], \eta}(s_i) - \frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} f_{\eta}(s_i)| \leq \frac{1}{\eta} \eta \epsilon = \epsilon$$

Hence;

$$|\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \chi_{(a, b), \eta}(s_i) - r| < 2\epsilon$$

$$|r - (b - a)| < \epsilon$$

$$|\frac{1}{\eta}^* \sum_{1 \leq i \leq \eta} \chi_{(a, b), \eta}(s_i) - (b - a)| < 3\epsilon$$

and, as  $\epsilon$  was arbitrary;

$$\frac{1}{\eta} \sum_{1 \leq i \leq \eta} \chi_{(a,b),\eta}(s_i) \simeq (b-a)$$

It follows that  $\{s_i\}_{1 \leq i \leq \eta}$  is weakly equidistributed.  $\square$

**Lemma 1.20.** *Let  $p \in \mathcal{R}[x]$  be a standard polynomial,  $p(x) = \sum_{l=0}^d a_l x^l$ , with  $0 \leq a_l < 1$ , then;*

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{[a_l q]}{q} = a_l$$

and, if  $0 < a < b < 1$  and  $f \in L^1([0, 1])$ ;

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} |\{i : 0 \leq i \leq q-1, \sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1} \in (a, b)\}| = (b-a)$$

$$\int_0^1 f dx = \lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} \sum_{i=0}^{q-1} f\left(\sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1}\right)$$

*Proof.* The first claim follows easily from the fact that, for infinite  $\eta$ ,  $\frac{[a_l \eta]}{\eta} \simeq a_l$ , and Theorem 2.2(i) of [7]. We claim that, for infinite  $\eta$  prime, and infinite  $\nu \in {}^* \mathcal{N}$ , the sequence  $\{\sum_{l=0}^d \frac{[a_l \eta]}{\eta} i^l : 1 \leq i \leq \nu \pmod{1}\}$  is weakly equidistributed, ( $\dagger$ ). By the previous lemma, it is sufficient to show that there exists an infinite  $\eta \in {}^* \mathcal{N}$ , with  $\frac{1}{\eta} \sum_{1 \leq j \leq \eta} \exp_{\eta}(2\pi i k \frac{p_{\eta}}{\eta}(j)) \simeq 0$ , for  $k \in \mathcal{Z}_{\eta}$ ,  $k \neq 0$ .

Suppose that  $p(x) = \sum_{l=0}^d a_l x^l$ , and that  $0 \leq |a_l| < 1$ , for  $0 \leq l \leq d$ . For  $q$  prime in  ${}^* \mathcal{N}$ , Let  $t_{l,q} = [qa_l]$ , so that  $a_l \simeq \frac{[qa_l]}{q}$ , and  $0 < t_{l,q} < q$ , for  $\eta$  infinite, therefore  $0 < t_{l,q} < q$ , for sufficiently large  $q \in \mathcal{N}$ , (\*). Let  $p_q = \sum_{l=0}^d t_{l,q} x^l$ , and let  $F_q \cong \mathcal{Z}/q\mathcal{Z}$  denote a finite field with  $q$  elements. Using Lemma 0.5 of [9], we have that, for  $q$  prime sufficiently large, see (\*), coprime to  $d$ ;

$$|\sum_{0 \leq j \leq q-1} e^{2\pi i \frac{k}{q} p_q(j)}| \leq (d-1)q^{\frac{1}{2}} + 1$$

for  $0 < k \leq q-1$ .

By transfer, for infinite  $\eta$  prime, which must be coprime to  $d$ , for  $0 < k \leq \eta-1$ ;

$$\frac{1}{\eta} \left| \sum_{0 \leq j \leq \eta-1} \ast \exp(2\pi i \frac{k}{\eta} \ast p_{\eta}(j)) \right| \leq \frac{(d-1)}{\eta^{\frac{1}{2}}} + \frac{1}{\eta} \simeq 0$$

The characters  $\{e^{2\pi i \frac{k}{q}} : -(q-1) \leq k \leq -1\}$  are just a re-enumeration of the characters  $\{e^{2\pi i \frac{k}{q}} : 1 \leq k \leq q-1\}$  on  $F_q$  for  $q$  prime, and, therefore, by the same argument;

$$\frac{1}{\eta} \left| \sum_{0 \leq j \leq \eta-1} \ast \exp(2\pi i \frac{k}{\eta} \ast p_{\eta}(j)) \right| \simeq 0, \text{ for } k \in \mathcal{Z}_{\eta} \setminus \{-\eta, 0\}$$

As  $\exp(2\pi i kx)$  is continuous on  $[0, 1)$ , for  $k \in \mathcal{Z}$ , we have that;

$$\frac{1}{\eta} \left| \sum_{0 \leq j \leq \eta-1} \ast \exp(2\pi i \frac{k}{\eta} \ast p_{\eta}(j)) \right| \simeq 0, \text{ for finite } k \in \mathcal{Z}_{\eta, \neq 0}$$

It follows, using the previous lemma, that the sequence  $\{\frac{\ast p_{\eta}}{\eta}(j)\}_{0 \leq j \leq \nu-1}$  is weakly distributed, for any infinite  $\nu$ , hence (†) is shown.

We have that, for any given  $\epsilon > 0$  standard,  $\nu$  infinite,  $q$  infinite prime ;

$$(b-a) - \epsilon < \frac{1}{\nu} \left| \{i : 0 \leq i \leq \nu-1, \sum_{l=0}^d \frac{[a_l \eta]}{\eta} i^l \pmod{1} \in (a, b)\} \right| < (b-a) + \epsilon$$

By underflow, there exists a standard  $N(\epsilon)$  prime, such that, for all standard primes  $q \geq N(\epsilon)$ ;

$$(b-a) - \epsilon < \frac{1}{q} \left| \{i : 0 \leq i \leq q-1, \sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1} \in (a, b)\} \right| < (b-a) + \epsilon$$

hence, the second claim is shown. Using Remarks 1.18, we have that, for any given  $f \in L^1([0, 1))$ , standard  $\epsilon > 0$ ,  $\nu$  infinite and  $q$  infinite prime;

$$\int_0^1 f dx - \epsilon < \frac{1}{q} \sum_{i=0}^{\nu-1} f(\sum_{l=0}^d \frac{[a_l \eta]}{\eta} i^l \pmod{1}) < \int_0^1 f dx + \epsilon$$

Again, by underflow, there exists a standard  $M(\epsilon, f)$  prime, such that, for all standard primes  $q \geq M(\epsilon, f)$ ;

$$\int_0^1 f dx - \epsilon < \frac{1}{q} \sum_{i=0}^{q-1} f(\sum_{l=0}^d \frac{[a_l q]}{q} i^l \pmod{1}) < \int_0^1 f dx + \epsilon$$

Hence, the final claim is shown. □

**Lemma 1.21.** *Let  $\{r, s\} \subset \mathcal{R}[x]$  be standard polynomials,  $r(x) = \sum_{l=0}^d a_l x^l$ ,  $s(x) = \sum_{m=0}^e b_m x^m$ , with  $0 \leq a_l < 1$  and  $0 \leq b_m < 1$ , then;*

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{[a_l q]}{q} = a_l, \lim_{q \rightarrow \infty, q \text{ prime}} \frac{[b_m q]}{q} = b_m$$

for  $0 \leq l \leq d$ ,  $0 \leq m \leq e$ . For  $q$  prime, letting;

$$W_q = \{i : 0 \leq i \leq q-1, \sum_{m=0}^d \frac{[a_l q]}{q} i^l \pmod{1} \neq 0\}$$

we have, if  $0 < a < b < 1$ ,  $f \in L^1([0, 1])$ ;

$$\lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} |\{i : 0 \leq i \leq q-1, i \notin W_q, \frac{\sum_{l=0}^d \frac{[a_l q]}{q} i^l}{\sum_{m=0}^e \frac{[b_m q]}{q} i^m} \pmod{1} \in (a, b)\}| = (b-a)$$

$$\int_0^1 f dx = \lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} \sum_{i \notin W_q} f\left(\frac{\sum_{l=0}^d \frac{[a_l q]}{q} i^l}{\sum_{m=0}^e \frac{[b_m q]}{q} i^m} \pmod{1}\right)$$

*Proof.* The first claim follows easily from the fact that, for infinite  $\eta$ ,  $\frac{[a_l \eta]}{\eta} \simeq a_l$ ,  $\frac{[b_m \eta]}{\eta} \simeq b_m$ , for  $0 \leq l \leq d$ ,  $0 \leq m \leq e$ , and Theorem 2.2(i) of [7].

Sketch proof; use case  $q = \eta^2$  to get correct scaling, see [9] for the quotient. Need to get rid of trace term.

□

**Definition 1.22.** *For  $b \neq 0$ , if  $\theta \in \Theta_b$ , we let  $n_\theta = \mu n (n \in R_{\theta, b})$ , and;*

$$\begin{aligned} N_b &= \bigcup_{\theta \in \Theta_b} n_\theta \\ &= \{n \in \mathcal{Z} : (\exists \theta \in \Theta_b)(n \in R_{\theta, b} \wedge (\forall m \in R_{\theta, b})(n \leq m))\} \end{aligned}$$

Let  $\kappa : \mathcal{N} \rightarrow N_b$  be an order preserving enumeration, let  $\{z_n : n \geq 1\}$  be the sequence defined by  $z_n = |b \ln(\kappa(n))|$ , and let  $\{y_n : n \geq 1\} \subset [0, 1]$  be the sequence defined by  $y_n = \frac{1}{1+z_n}$ , so that  $\exp(\frac{az_n}{|b|}) = \exp(\frac{a(1-y_n)}{|b|y_n})$ . For  $m \in \mathcal{Z}_{\geq 1}$ , we let  $p_{b,m} \in \mathcal{R}[x]$  be a polynomial, defined by;

$$p_{b,m}(n) = y_{n+1}, \text{ for } n \in \mathcal{Z} \cap [0, m-1].$$

We let;

$$f_{1,r,s}(y) = \frac{e^{\frac{2\pi a l_0(1-r)}{|b|}} \exp\left(\frac{a(1-y)}{|b|y}\right)}{2}$$

$$f_{2,r,s}(y) = \frac{|b| \exp\left(\frac{a(1-y)}{|b|y}\right) e^{\frac{2\pi a l_0(1-r)}{|b|}}}{2\pi a l_0}$$

$$f_{3,r,s}(x, y) = C_b^{-al_0(x-1)} \exp\left(\frac{a(1-y)}{|b|y}\right)$$

$f_{4,r,s}(x, y)$  is a symmetric polyomial in  $x$ , with;

$$\frac{\partial f_{4,r,s}}{\partial x^k}(r, y) = \frac{\partial f_{3,r,s}}{\partial x^k}(r, y), \text{ for } 0 \leq k \leq 2$$

$$f_{4,r,s}(n, y) = f_{3,r,s}(n, y), \text{ for } 1 \leq n \leq r$$

$$\bar{f}_{4,r,s,j+1}(y) = \frac{\partial f_{4,r,s}}{\partial x^{2j}}(0, y), \text{ (} 0 \leq j \leq r+1 \text{)}$$

$$f_{5,r,s}(y) = \sum_{j=0}^{r+1} (\bar{f}_{4,r,s,j+1})(y) \binom{B_{2j+1}(r)}{2j+1}$$

$$f_{6,r,s}(y) = f_{5,r,s}(y) + f_{2,r,s}(y) + f_{1,r,s}(y)$$

**Remarks 1.23.** If  $n_0 \in N_b$ ;

$$|b|n_0^{-a} = |b| \exp \ln(n_0^{-a})$$

$$= |b| \exp(-a \ln(n_0))$$

$$= |b| \exp\left(\frac{a}{|b|} (|b| \ln(n_0))\right)$$

$$= |b| \exp\left(\frac{ax_{\kappa^{-1}(n_0)}}{|b|}\right) = |b| \exp\left(\frac{a(1-y_{\kappa^{-1}(n_0)})}{|b|y_{\kappa^{-1}(n_0)}}\right)$$

Then;

$$f_{3,r,s}(x, y_{\kappa^{-1}(n_0)}) = Q_{\theta_0,a,b}(x)$$

$$f_{4,r,s}(x, y_{\kappa^{-1}(n_0)}) = p_{\theta_0,a,b,r}(x)$$

$$\bar{f}_{4,r,s,j+1}(y_{\kappa^{-1}(n_0)}) = (\bar{a}_{\theta_0,a,b,r})_{j+1}, \text{ (} 0 \leq j \leq r+1 \text{)}$$

$$f_{5,r,s}(y_{\kappa^{-1}(n_0)}) = \sum_{j=0}^{r+1} (\bar{a}_{\theta_0,a,b,r})_{j+1} \binom{B_{2j+1}(r)}{2j+1}$$

$$f_{6,r,s}(y_{\kappa^{-1}(n_0)}) + R_{\theta_0,a,b,r} = \sum_{m \in R_{\theta_0,b}} m^{-a}$$

$$\sum_{n \in \mathcal{Z}_{\geq 1}} f_{6,r,s}(y_n) + \sum_{\theta \in \Theta_b} R_{\theta,a,b,r} = \sum_{m \in \mathcal{Z}_{\geq 1}} m^{-s}$$

Moreover, for  $m \in \mathcal{Z}_{\geq 1}$ ;

$$\int_0^1 f_{6,r,s}(y) dy = \lim_{q \rightarrow \infty, q \text{ prime}} \frac{1}{q} \sum_{i=0}^{q-1} f_{6,r,s}\left(\frac{[qp_{b,m}]}{q}(i)(\text{mod } 1)\right)$$

We have that;

$$\begin{aligned} & \left| \sum_{n \in \mathcal{Z}_{\geq 1}} f_{6,r,s}(y_n) - \sum_{n \in \mathcal{Z}_{\geq 0}} f_{6,r,s}(p_{b,m}(n)(\text{mod } 1)) \right| \\ & \leq \sum_{n \in \mathcal{Z}_{\geq m+1}} |f_{6,r,s}(y_n)| + \sum_{n \in \mathcal{Z}_{\geq m}} |f_{6,r,s}(p_{b,m}(n)(\text{mod } 1))| \end{aligned}$$

(Bound first term, decreasing with  $m$ , and second term, using a rational function, (\*\*) replacing the polynomial, using fact that  $\lim_{y \rightarrow 0} f_{6,r,s}(y) = 0$ , exponentially. Modify Lemma ?? to include (\*\*), and consider restriction on coefficients as well.)

## REFERENCES

- [1] A Non-Standard Representation for Brownian Motion and Ito Integration, Israel Journal of Mathematics, Vol. 25, (1976)
- [2] Complex Algebraic Surfaces, LMS Student Texts 34, A. Beauville, (1996).
- [3] La Conjecture de Weil: I, Pierre Deligne, Publications mathematiques de l'I.H.E.S, tome 43, (1974).
- [4] Etale Cohomology and the Weil Conjecture, E. Fretag, R. Kiehl, Springer, (1988).
- [5] An Application of Fourier Analysis to Riemann Sums, Tristram de Piro, available at <http://www.curvalinea.net>, (2015).
- [6] A Simple Proof of the Inversion Theorem using Nonstandard Analysis, Tristram de Piro, available at <http://www.curvalinea.net>, (2013).
- [7] A Simple Proof of the Uniform Convergence of Fourier Series using Non-standard Analysis, Tristram de Piro, available at <http://www.curvalinea.net>, (2013).
- [8] Applications of Nonstandard Analysis to Probability Theory, Tristram de Piro, available at <http://www.curvalinea.net>, (2013). (MOVE TO dep1)
- [9] A Note on the Weil Conjectures for Curves, Tristram de Piro, available at <http://www.curvalinea.net>, (2015).

- [10] Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes, M. Raynaud, Séminaire Bourbaki, 286, 1964-1966.
- [11] The Arithmetic of Elliptic Curves, 2nd Edition, J. Silverman, Springer, (1986)

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