SOLVING SCHRODINGER'S EQUATION USING NONSTANDARD ANALYSIS

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Abstract. We propose a new method for solving Schrodinger's equation on a surface, with applications to quantum physics.

Definition 0.1. *We let $S^1(1)$ denote the circle of radius 1, which we identify naturally with the closed interval $[-\pi,\pi]$, via $\mu:[-\pi,\pi]\rightarrow S^1(1)$, $\mu(\theta)=e^{i\theta}$. We let $C^\infty([-\pi,\pi])=\{\mu^*(f):f\in C^\infty(S^1)\}$. We let T denote the closed domain $[-\pi,\pi]\times[-\pi,\pi]$, and let T° denote its interior $(-\pi,\pi)\times(-\pi,\pi)$. If $f\in C(T)$, then if $\{x,y\}\subset[-\pi,\pi]$, we let $\{f_x,f_y\}\subset C([-\pi,\pi])$ be defined by $f^x(y)=f(x,y)$, and $f^y(x)=f(x,y)$. We let;*

$C^\infty(T)=\{f\in C(T):f|_{T^\circ}\in C^\infty(T^\circ), f(-\pi,y)=f(\pi,y), f(x,-\pi)=f(x,\pi), f^x\in C^\infty([-\pi,\pi]), f^y\in C^\infty([-\pi,\pi]), -\pi\leq x\leq\pi, -\pi\leq y\leq\pi\}$

We let;

$V^{\Delta,\lambda}(T)=\{f\in C^\infty(T):(f|_{T^\circ})_{xx}+(f|_{T^\circ})_{yy}=\lambda(f|_{T^\circ})\}$. *If $f\in C^\infty(T)$, we define $(\mathcal{F})(f)\in C([-\pi,\pi]\times\mathcal{Z})$ by;*

$\mathcal{F}(f)(x,m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x,y)e^{-imy}dy$

Lemma 0.2. *If $f\in C^\infty(T)$, and $m\in\mathcal{Z}$, we have that $\mathcal{F}(f_{xx})=(\mathcal{F}(f))_{xx}$ and $\mathcal{F}(f_{yy})=-m^2(\mathcal{F}(f))$. Moreover;*

$f(x,y)=\sum_{m\in\mathcal{Z}}\mathcal{F}(f)(x,m)e^{imy}$

Proof. We have, differentiating under the integral sign, that;

$(\mathcal{F}(f_{xx}))^m(x')$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_{xx})(x',y)e^{-imy}dy$

$= \frac{1}{2\pi}(\int_{-\pi}^{\pi} f(x',y)e^{-imy}dy)_{xx}$

$= ((\mathcal{F}(f))^m)_{xx}(x')$.

and integrating by parts, using the fact that, for $x \in [-\pi,\pi]$, $\{f_y^x, f_{yy}^x\} \subset C([-\pi,\pi])$;

$(\mathcal{F}(f_{yy}))^m(x')$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{yy}(x',y')e^{-imy}dy'$

$= \frac{1}{2\pi}im\int_{-\pi}^{\pi} f_y(x',y')e^{-imy}dy'$

$= -\frac{1}{2\pi}m^2\int_{-\pi}^{\pi} f(x',y')e^{-imy}dy'$

$= -m^2(\mathcal{F}(f))$

the last part follows from the fact that, for $x \in [-\pi,\pi]$, $f^x \in C^\infty[-\pi,\pi]$, and the result of [2].

□

Lemma 0.3. *Let $Z = \{\lambda \in \mathcal{Z}_{\leq 0} : -\lambda = m^2 + n^2, for\ some\ m,n \in \mathcal{Z}\}$, then $V^{\Delta,\lambda}(T) = \emptyset$ iff $\lambda \notin Z$, and if $\lambda \in Z$, $dim_{\mathcal{C}}[V^{\Delta,\lambda}(T)] = 2Card(X_\lambda)$, where $X_\lambda = \{m \in \mathcal{Z} : \lambda + m^2 = -n^2, for\ some\ n \in \mathcal{Z}\}$.*

Proof. Let $f \in V^{\Delta,\lambda}(T)$, and let $g = f|_{T^\circ}$, so $g_{xx} + g_{yy} = \lambda g$, on T°. Applying $\mathcal{F}$, and using Lemma 0.2, we have, for $m \in \mathcal{Z}$, that $(\mathcal{F}(g))_{xx}^m - m^2(\mathcal{F}(g))^m = \lambda(\mathcal{F}(g))^m$, on $(-\pi,\pi)$. Let $h^m = (\mathcal{F}(g))^m$, defined on $[-\pi,\pi]$, then $h^m(-\pi) = h^m(\pi) = c_m$, and $h_{xx}^m = (m^2+\lambda)h^m$, (†). The general solution to $h_{xx}^m = \mu^2 h^m$, for $\mu \in \mathcal{C}$, on $(-\pi,\pi)$ is given by;

$h^m(x) = Acos(\mu(x+\pi)) + Bsin(\mu(x+\pi))$

Setting $x = -\pi$, we obtain $A = c_m$ and, setting $x = \pi$, we obtain $c_m = c_m cos(2\mu\pi) + Bsin(2\mu\pi)$. If $sin(2\mu\pi) \neq 0$, (♯), we have $B = \frac{c_m(1-cos(2\mu\pi))}{sin(2\mu\pi)}$. Then;

$h^m(x) = c_m[cos(\mu(x+\pi)) + \frac{(1-cos(2\mu\pi))}{sin(2\mu\pi)}sin(\mu(x+\pi))]$

$(h^m)'(x) = c_m[-\mu sin(\mu(x+\pi)) + \mu\frac{(1-cos(2\mu\pi))}{sin(2\mu\pi)}cos(\mu(x+\pi))]$ (∗)

We have that $h^m \in C^\infty([-\pi,\pi])$, as for $y \in [-\pi,\pi]$, $f^y \in C^\infty([-\pi,\pi])$. Hence, $(h^m)'(-\pi) = (h^m)'(\pi)$ gives that, from $(*)$;

$$\mu\frac{(1-cos(2\mu\pi))}{sin(2\mu\pi)} = \mu cos(2\mu\pi)\frac{(1-cos(2\mu\pi))}{sin(2\mu\pi)} - \mu sin(2\mu\pi)$$

$$(1-cos(2\mu\pi))^2 = -sin^2(2\mu\pi)$$

$$cos(2\mu\pi) = 1$$

hence, $\mu \in \mathcal{Z}$, and $sin(2\mu\pi) = 0$, contradicting $(\sharp)$. It follows that $sin(2\mu\pi) = 0$, giving $c_m = c_m cos(2\mu\pi)$. If $c_m \neq 0$, we obtain that $cos(2\mu\pi) = 1$, and $\mu \in \mathcal{Z}$, then $g^m(x) = c_m cos(n(x+\pi))$, for $n \in \mathcal{Z}$. If $c_m = 0$, we obtain $A = 0$, and $g_m(x) = Bsin(\mu(x+\pi))$, then $g_m(-\pi) = 0 = g_m(\pi) = Bsin(2\mu\pi) = 0$, giving again that $\mu \in \mathcal{Z}$, and $g_m(x) = d_m sin(n(x+\pi))$, for $n \in \mathcal{Z}$. Simplifying this, we obtain that $g_m(x) = c_m cos(n(x+\pi))$, or $g_m(x) = d_m sin(n(x+\pi))$, $n \in \mathcal{Z}_{\geq 0}$, $(\dagger\dagger)$. From $(\dagger), (\dagger\dagger)$, we obtain that $m^2 + \lambda = -n^2$, hence, $\lambda \in \mathcal{Z}_{\leq 0}$, and we obtain a non-trivial solution to $(\dagger)$ iff $\lambda \in Z$. For a given $\lambda \in Z$, observing that $Card(X_\lambda$ is finite, using $(\dagger\dagger)$ and Lemma 0.2, we obtain that;

$$(\mathcal{F}(f))(x,m)$$

$$= c_m cos((-(m^2+\lambda))^{\frac{1}{2}}(x+\pi)) + d_m sin((-(m^2+\lambda))^{\frac{1}{2}}(x+\pi))$$

$$f(x,y) = \sum_{m \in X_\lambda} c_m cos((-(m^2+\lambda))^{\frac{1}{2}}(x+\pi))e^{imy}$$

$$+ d_m sin((-(m^2+\lambda))^{\frac{1}{2}}(x+\pi))e^{imy} \ (\dagger\dagger\dagger)$$

It is easily checked that $f(x,y)$, of the form given in $(\dagger\dagger\dagger)$, belongs to $V^{\Delta,\lambda}(T)$, as required.

□

Remarks 0.4. *The Hodge Theorem for compact manifolds implies that, if $f \in L^2(T)$;*

$$f = \sum_{\lambda \in Z} \sum_{1 \leq i \leq 2Card(X_\lambda)} < f, e_{\lambda,i} > e_{\lambda,i} \ (\dagger)$$

where, for $\lambda \in Z$, $\{e_{\lambda,i} : 1 \leq i \leq 2Card(X_\lambda)\}$ is an orthonormal basis of $V^{\Delta,\lambda}(T)$

and convergence is with respect to the L^2 *norm. The proof can be found in* [7]*, and uses the existence of a heat kernel* $e^{t\Delta}$*. One of the aims of this paper is to improve this result to uniform convergence, in the case when* $f \in C^\infty(T)$*, using a nonstandard argument. Alternatively, observe that, defining* $Y_\lambda = \{(m,n) \in \mathcal{Z}^2 : m^2 + n^2 = -\lambda\}$*,* $\lambda \in Z$*, we have that, if* $f \in V^{\Delta,\lambda}(T)$*, by* $(\dagger\dagger\dagger)$ *in Lemma 0.3;*

$$f = \sum_{(m,n)\in Y_\lambda} c_{(m,n)} e^{imx} e^{iny}$$

Observe that the family $\{e^{imx}e^{iny} : (m,n) \in \mathcal{Z}^2\}$ *is orthonormal in* $L^2(T)$*, with respect to the inner product;*

$$< f, g >= \frac{1}{4\pi^2} \int_T f\bar{g} d\mu$$

where μ *is Lebesgue measure on* T*. Moreover, the family* $W = \{e^{imx}, e^{iny} : (m,n) \in \mathcal{Z}^2\}$ *separates points, hence, the* $*$*-subalgebra generated by* W *is dense in* $C(T)$*. By the usual arguments, see Lemma 3.2, Chapter 2, of* [8]*, adapted to the* 2*-variable case,* W *is dense in* $L^2(T)$*. It follows that, given* $f \in L^2(T)$*, we can obtain explicitly;*

$$f = \sum_{\lambda \in Z} \sum_{(m,n)\in Y_\lambda} < f, e^{imx}e^{iny} > e^{imx}e^{iny} \quad (\dagger\dagger)$$

with convergence in $L^2(T)$*, and, without invoking the Hodge Theorem. If* $f \in C^\infty(T)$*, letting* $f_{m,n} =< f, e^{imx}e^{iny} >$*, we have, using integration by parts, that, for any* $r \in \mathcal{Z}_{\geq 1}$*,* $|m| + |n| \neq 0$*;*

$$|f_{m,n}| \leq \frac{||\Delta^r f||}{(m^2+n^2)^r}$$

Rearranging $(\dagger\dagger)$*, we have that;*

$$|\sum_{(m,n)\mathcal{Z}^2} < f, e^{imx}e^{iny} > e^{imx}e^{iny}|$$

$$\leq |f_{0,0}| + 2\sum_{n\in\mathcal{Z}_{\geq 1}} \frac{||\Delta^{r+1}f||}{n^{2(r+1)}} + 2\sum_{m\in\mathcal{Z}_{\geq 1}} \sum_{n\in\mathcal{Z}_{\geq 0}} \frac{||\Delta^{r+1}f||}{(m^2+n^2)^{r+1}}$$

$$\leq |f_{0,0}| + 2(||\Delta^{r+1}f|| + \frac{||\Delta^{r+1}f||}{2r+1}) + 2\sum_{m\in\mathcal{Z}_{\geq 1}} \frac{||\Delta^{r+1}f||}{rm^{2r}}$$

$$\leq |f_{0,0}| + 2(||\Delta^{r+1}f|| + \frac{||\Delta^{r+1}f||}{2r+1}) + \frac{2}{r}(||\Delta^{r+1}f|| + \frac{||\Delta^{r+1}f||}{(2r-1)})$$

Taking $r = 1$*, guarantees that the series defined by* $(\dagger\dagger)$*, converges uniformly to define a continuous function* $h \in C(T)$*, such that all the Fourier cofficients* $(f-h)_{m,n} = 0$*,* $(*)$*. We claim that* $f - h = 0$*. Suppose that* $f - h$ *is real valued, and, without loss of generality,*

$(f-h)(0,0) > 0$. *As* $(f-h)$ *is continuous, we have that* $|(f-h)(x,y)| > \frac{(f-h)(0,0)}{2}$, *for* $(x,y) \in S_\delta$, *where* $S_\delta = \{(x,y) : |(x,y)| < \delta\}$. *Choose* $\epsilon > 0$, *so that* $V_1 = \{(x,y) : (\epsilon + cosx)(\epsilon + cosy)) > 1\} \subset S_\delta$, $(**)$. *Letting* $p_k(x,y) = (\epsilon + cosx)^k(\epsilon + cosy)^k$, *we have, by* $(*)$,$(**)$ *that;*

$\int_T (f-h)p_k dxdy = 0$, *for* $k \in \mathcal{Z}_{\geq 0}$

$lim_{k\rightarrow\infty} \int_{\overline{V_1}^c} (f-h)p_k dxdy = 0$

$lim_{k\rightarrow\infty} \int_{V_1} (f-h)p_k dxdy = \infty$

a contradiction, hence, $f - h = 0$. *If* $(f-h)$ *is not real valued, we can repeat the same argument, noting that* $Re(f-h) = \frac{(f-h)+\overline{(f-h)}}{2}$, *and the Fourier coefficients of* $\overline{(f-h)}$ *also vanish. It follows that the series defined by* $(\dagger\dagger)$ *converges uniformly to* f *on* T, *not just in* $L^2(T)$.

Lemma 0.5. *Let* $[a,b] \subset \mathcal{R}$ *be a bounded interval, and suppose that, for all* $\alpha > 0$, $G \in C^\infty((a-\alpha, b+\alpha) \times \mathcal{C}^n, \mathcal{C}^n)$. *Let* $\overline{x}_0 \in (\mathcal{C})^n$ *be given,* $\eta \in {}^*\mathcal{R}$ *infinite, and suppose that there exists* $K > 0$, $L > 0$, *such that for all* $t \in [a,b]$ *and* $\overline{x} \in \mathcal{C}^n$*;*

$||G(t,\overline{x})||_{\mathcal{C}^n} \leq K||\overline{x}||_{\mathcal{C}^n} + L$

Let $\overline{X} : {}^*[a,b] \rightarrow ({}^*\mathcal{R})^n$ *be defined inductively by;*

$\overline{X}(a) = \overline{x}_0$

$\overline{X}(a+\frac{i+1}{\eta}) - \overline{X}(a+\frac{i}{\eta}) = \frac{1}{\eta}({}^*G)(a+\frac{i}{\eta}, \overline{X}(a+\frac{i}{\eta}))\ 0 \leq i \leq [\eta(b-a)]-2$

$\overline{X}(a+\tau) = \overline{X}(a+\frac{[\eta\tau]}{\eta})\ \tau \in {}^*[0, b-a)$

$\overline{X}(b) = \overline{X}(b-\frac{1}{\eta})\ (*)$

Then $\overline{X}$ *is S-continuous, and letting* $\overline{x} = {}^\circ\overline{X} : [a,b] \rightarrow (\mathcal{C})^n$, *we have that;*

$\overline{x}(t) - \overline{x}_0 = \int_a^t G(s, \overline{x}(s))ds$

In particular, $\overline{x} \in C^{\infty}([a,b])$, ([1]), *and solves the differential equation;*

$\overline{x}'(t) = G(t, \overline{x}(t))$, *for* $t \in (a,b)$, *with initial condition* $\overline{x}(a) = \overline{x}_0$.

Proof. Assume first that $||G(t,\overline{x})||_{\mathcal{C}^n} \leq c$, for all $(t,\overline{x}) \in [a,b] \times \mathcal{C}^n$, with c finite. We have that, for $\{\tau, \tau'\} \subset {}^*[a,b]$, $||\overline{X}(\tau') - \overline{X}(\tau)||_{({}^*\mathcal{C})^n} \leq c|\tau' - \tau|$. This follows, by internal induction, see Lemma 2.12(ii) of [3]. We have that $||\overline{X}(\tau) - \overline{X}(\tau)||_{({}^*\mathcal{C})^n} = 0$, and, if $||\overline{X}(\tau + \frac{i}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{C})^n} \leq \frac{ci}{\eta}$;

$||\overline{X}(\tau + \frac{i+1}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{C})^n}$

$\leq ||\overline{X}(\tau + \frac{i+1}{\eta}) - \overline{X}(\tau + \frac{i}{\eta})||_{({}^*\mathcal{C})^n}$

$+||\overline{X}(\tau + \frac{i}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{C})^n}$

$\leq ||\frac{1}{\eta}({}^*G)(a + \frac{i}{\eta}, \overline{X}(a + \frac{i}{\eta}))||_{({}^*\mathcal{R})^n} + \frac{ci}{\eta} = \frac{c}{\eta} + \frac{ci}{\eta} = \frac{c(i+1)}{\eta}$

Then $\overline{X}$ is S-continuous, in the sense that, if $\{\tau'', \tau'''\} \subset {}^*[a,b]$, with $\tau'' \simeq \tau'''$, then $||\overline{X}(\tau'') - \overline{X}(\tau''')||_{({}^*\mathcal{C})^n} \simeq 0$. It follows that $\overline{x} = {}^{\circ}\overline{X} \in C([a,b])$. Now let $F : {}^*[a,b] \rightarrow ({}^*\mathcal{R})^n$ be defined by;

$F_1(\tau) = {}^*G(\tau, \overline{X}(\tau))$

$F(a+\tau) = F_1(a + \frac{[\eta\tau]}{\eta})$, $\tau \in {}^*[0, b-a)$

$F(b) = F(b - \frac{1}{\eta})$

Then, for all $\tau \in {}^*[0, b-a]$, we have, using the fact that $\overline{X}$ and ${}^*G$ are S-continuous;

$F(a+\tau) = F_1(a + \frac{[\eta\tau]}{\eta})$

$= {}^*G(a + \frac{[\eta\tau]}{\eta}, \overline{X}(a + \frac{[\eta\tau]}{\eta}))$

$\simeq G({}^{\circ}(a + \frac{[\eta\tau]}{\eta}), {}^{\circ}(\overline{X}(a + \frac{[\eta\tau]}{\eta})))$

$= G({}^{\circ}(a + \frac{[\eta\tau]}{\eta}), (\overline{x}({}^{\circ}(a + \frac{[\eta\tau]}{\eta}))))$

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[1]In the sense that $\overline{x}|_{(a,b)} \in C^{\infty}(a,b)$, and for all $n \in \mathcal{N}$, there exist $\overline{g_n} \in C[a,b]$, with $\overline{g_n}|_{(a,b)} = (\overline{x}|_{(a,b)})^{(n)}$

so that $F(\tau)\simeq G(^{\circ}\tau,\overline{x}(^{\circ}\tau))$, for $\tau\in{}^{*}[a,b]$. $(**)$

Moreover, F is measurable with respect to the $*$-algebra $\mathcal{C}_\eta$ and, as F is bounded, F is S-integrable. It follows that, using the definition $(*)$, Remarks 3.10, Theorems 3.20 of [3], Theorem 3.14 of [1] and $(**)$, that if $t\in[a,b]$;

$\overline{x}(t)={}^{\circ}\overline{X}(t)={}^{\circ}\overline{X}(a+\frac{[(t-a)\eta]}{\eta})$

$={}^{\circ}(\frac{1}{\eta}{}^{*}\sum_{i=0}^{[(t-a)\eta]-1}{}^{*}G(a+\frac{i}{\eta},\overline{X}(a+\frac{i}{\eta})))+\overline{X}(a)$

$\overline{x}(t)-\overline{x}_0$

$={}^{\circ}(\int_a^t F(\tau)d\lambda_\eta(\tau))$

$=\int_a^t {}^{\circ}F(\tau)dL(\lambda_\eta)(\tau)$

$=\int_a^t G(^{\circ}\tau,\overline{x}(^{\circ}\tau))d\mu$

$=\int_a^t G(s,\overline{x}(s))d\mu(s)$

where $d\mu$ denotes Lebesgue measure, λ_η is the counting measure on ${}^{*}[a,b]$, with respect to the $*$-algebra $\mathcal{C}_\eta$, and $L(\lambda_\eta)$ is the corresponding Loeb measure. By the Fundamental theorem of Calculus, and the condition on G, we obtain the final claim.

For the general case, we claim that that there exists a finite real constant $C>0$, such that $||\overline{X}(\tau)||_{({}^{*}\mathcal{C})^n}\leq C$, for all $\tau\in{}^{*}[a,b]$. We have that, for $0\leq i\leq[\eta(b-a)]$;

$||\overline{X}(a+\frac{i}{\eta})||_{({}^{*}\mathcal{C})^n}\leq(1+\frac{K}{\eta})^i||\overline{x}_0||_{({}^{*}\mathcal{C})^n}+L\frac{i}{\eta}+\epsilon_i$, $(***)$

where, for $0\leq i\leq[\eta(b-a)]-1$;

$\epsilon_{i+1}=(1+\frac{K}{\eta})^i\epsilon_i+\frac{KLi}{\eta^2}$, $\epsilon_0=0$, $(****)$

By internal induction; we have that $(***)$ holds for $i=0$, suppose true for $0\leq i_0<[\eta(b-a)]$. Then, using transfer of the boundedness assumption on G to $({}^{*}G)$;

$||\overline{X}(a+\frac{i_0+1}{\eta})||_{({}^{*}\mathcal{C})^n}$

$= ||\overline{X}(a+\frac{i_0}{\eta}) + \frac{1}{\eta}({}^*G)(a+\frac{i_0}{\eta}, \overline{X}(a+\frac{i_0}{\eta}))||_{({}^*\mathcal{C})^n}$

$\leq (1+\frac{K}{\eta})^{i_0}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\frac{i_0}{\eta}) + \epsilon_{i_0} + \frac{1}{\eta}[K((1+\frac{K}{\eta})^{i_0}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\frac{i_0}{\eta}) + \epsilon_{i_0}) + L]$

$= (1+\frac{K}{\eta})^{i_0+1}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\frac{i_0+1}{\eta}) + \frac{KLi_0}{\eta^2} + \epsilon_{i_0} + \frac{K\epsilon_{i_0}}{\eta}$

$= (1+\frac{K}{\eta})^{i_0+1}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\frac{i_0+1}{\eta}) + \epsilon_{i_0+1}$

giving $(***)$, with the condition $(****)$. Hence, for all $\tau \in {}^*[a,b]$;

$||\overline{X}(\tau)||_{({}^*\mathcal{C})^n} \leq (1+\frac{K}{\eta})^{[\eta+1]([b-a+1])}||\overline{x}_0||_{({}^*\mathcal{C})^n} + L(\frac{[\eta(b-a)]}{\eta}) + \epsilon$

where $\epsilon = \epsilon_{[\eta(b-a)]} \leq (1+\frac{K}{\eta})^{[\eta+1]([b-a+1])}\frac{KL[\eta(b-a)]}{\eta^2}[\eta(b-a)]$

We have that $(1+\frac{K}{\eta})^{\eta} \simeq e^K$, as $lim_{t\to\infty}(1+\frac{x}{t})^t = e^x$, for $x \in \mathcal{R}$. Hence;

$||\overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq e^{[K+1][b-a+1]}||\overline{x}_0||_{({}^*\mathcal{C})^n} + L(b-a) + KL(b-a+1)e^{[K+1][b-a+1]}$

$= e^{[K+1][b-a+1]}(||\overline{x}_0||_{({}^*\mathcal{C})^n} + KL(b-a+1)) + L(b-a) = C$

Now, clearly, by compactness, there exists a finite real constant $F > 0$, such that;

$||G(t,\overline{x})||_{({}^*\mathcal{R})^n} \leq F$, for $(t,\overline{x}) \in {}^*[a,b] \times ({}^*B)(\overline{0},C)$

Then, we can use the proof in the previous part, to give the result, as $\overline{X}(\tau) \in ({}^*B)(\overline{0},C)$, for $\tau \in {}^*[a,b]$.

□

Lemma 0.6. *Let $[a,b] \subset \mathcal{R}$ be a bounded interval, $n \in \mathcal{N}$, and, for all $\alpha > 0$, $\{c_0,\ldots,c_{n-1},d_0\} \subset C^\infty(a-\alpha,b+\alpha)$. Let $\overline{x}_0 \in \mathcal{R}^n$ be given, then there exists $x \in C^\infty([a,b])$ such that;*

$\frac{d^nx}{dt^n} + c_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots c_1(t)\frac{dx}{dt} + c_0(t)x + d_0 = 0$, $t \in (a,b)$

and $x(a) = \overline{x}_0^1, \ldots, x^{n-1}(a) = \overline{x}_0^n$ (†)

Moreover, for $s \in [a,b]$, $0 \leq j \leq n-1$, $\frac{d^jx}{dt^j}(s) = x_{j+1}(s) = {}^\circ X_{j+1}(s)$, where, for $1 \leq k \leq n$, $X_k(s) = (pr_k \circ \overline{X})(s)$, $x_k(s) = (pr_k \circ \overline{x})(s)$ and $\overline{X} : {}^*[a,b] \to {}^*\mathcal{C}^n$ is defined inductively, as in Lemma 0.5, with G given *by;*

$$G(s,x_1,\ldots,x_n)=(x_2,\ldots,x_n,-c_{n-1}(s)x_n\ldots-c_1(s)x_2-c_0x_1(s)-d_0)$$

Proof. Let $M=max_{0\leq i\leq n-1}(||c_i||_{C[a,b]},||d_0||_{C[a,b]})$. Then;

$||G(t,x_1,\ldots,x_n)||_{\mathcal{C}^n}$

$\leq\sum_{2\leq i\leq n}|x_i|+M((\sum_{1\leq i\leq n}|x_i|)+1)$

$\leq(M+1)\sum_{1\leq i\leq n}|x_i|+M$
$\leq(M+1)\sqrt{n}||\overline{x}||_{\mathcal{C}^n}+M$

Hence, G satisfies the conditions of Lemma 0.5. Therefore, if $\overline{x}={}^{\circ}\overline{X}$, then $\overline{x}$ solves the differential equation;

$\overline{x}'(t)=G(t,x_1(t),\ldots,x_n(t))$

$=(x_2(t),\ldots,x_n(t),-c_{n-1}(t)x_n\ldots-c_1(t)x_2(t)-c_0(t)),\ t\in(a,b)\ (**)$

Setting $x(t)=x_1(t)$, we obtain that $x'(t)=(x_1)'(t)=x_2(t),\ldots,(x_j)'(t)=x_{j+1}(t)$, for $1\leq j\leq n-1$, hence, $\frac{d^jx}{dt^j}(s)=x_{j+1}(t)$, for $1\leq j\leq n-1$ and;

$(x_n)'(t)=-c_{n-1}(s)x_n(t)\ldots-c_1(t)x_2(t)-c_0(t)x_1(t)-d_0(t)$

$\frac{d^nx}{dt^n}=(\frac{d^{n-1}x}{dt^{n-1}})'(s)=-c_{n-1}(s)\frac{d^{n-1}x}{dt^{n-1}}(s)\ldots-c_1(t)\frac{dx}{dt}-c_0(t)x(t)-d_0(t)$
$(*)$

as required. □

Remarks 0.7. *Given an ODE of the form (†), with solution $\overline{x}$, and $\epsilon\in\mathcal{R}_{>0}$;*

$$S_\epsilon=\{\eta:(\forall\overline{y}\in{}^*[a,b])(||^*(\overline{x})(\overline{y})-\overline{X}_\eta(\overline{y})||_{({}^*\mathcal{R})^n}<\epsilon)\}$$

is an internal set, which includes all infinite positive integers $\eta\in{}^*\mathcal{N}\setminus\mathcal{N}$. By overflow, see [3], there exists a positive integer $B(\epsilon)\in\mathcal{N}^{>0}$, with $S_\epsilon\supset\mathcal{N}^{\geq B(\epsilon)}$. By transfer, it follows that $max_{t\in[a,b]}||\overline{x}(t)-\overline{Y}_n(t)||_{({}^*\mathcal{R})^n}<\epsilon$, for $n\geq B(\epsilon)$, where $\overline{Y}_n$ is given by the $2+[n(b-a)]$-step discrete algorithm, given on $[a,b]$ by G and $(*)$ of Lemma 0.5. This guarantees the convergence of computer programs based on the

Euler method, for solving ODE's of the form (†).

We adopt the following notation;

Definition 0.8. *For* $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$*, we let;*

$\overline{\mathcal{H}_\eta} = \overline{\mathcal{V}_\eta} = {}^*\bigcup_{0\leq i\leq 2\eta-1}[-\pi + \pi\frac{i}{\eta}, -\pi + \pi\frac{i+1}{\eta})$

so that $\overline{\mathcal{H}_\eta} = \overline{\mathcal{V}_\eta} = {}^*[-\pi, \pi)$*. We let* $\{\mathfrak{C}_\eta, \mathfrak{D}_\eta\}$ *denote the associated* $*$*-finite algebras generated by the intervals* $[-\pi + \pi\frac{i}{\eta}, -\pi + \pi\frac{i+1}{\eta})$*, for* $0 \leq i \leq 2\eta - 1$*, and* $\{\lambda_\eta, \mu_\eta\}$ *the associated counting measures, defined by* $\lambda_\eta([-\pi + \pi\frac{i}{\eta}, -\pi + \pi\frac{i+1}{\eta})) = \mu_\eta([-\pi + \pi\frac{i}{\eta}, -\pi + \pi\frac{i+1}{\eta})) = \frac{\pi}{\eta}$*. We let* $(\overline{\mathcal{H}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta))$ *and* $(\overline{\mathcal{V}_\eta}, L(\mathfrak{D}_\eta), L(\mu_\eta))$ *denote the associated Loeb spaces, see Definition 0.5 of* []*. We let* $([-\pi, \pi], \mathfrak{B}, \mu)$ *denote the interval* $[-\pi, \pi]$*, with the completion* $\mathfrak{B}$ *of the Borel field, and* μ *the restriction of Lebesgue measure. We let* $(\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, \mathfrak{C}_\eta \times \mathfrak{D}_\eta, \lambda_\eta \times \mu_\eta)$ *be the associated product space and* $(\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, L(\mathfrak{C}_\eta \times \mathfrak{D}_\eta), L(\lambda_\eta \times \mu_\eta))$ *be the corresponding Loeb space.* $(\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, L(\mathfrak{C}_\eta) \times L(\mathfrak{D}_\eta), L(\lambda_\eta) \times L(\mu_\eta))$ *is the complete product of the Loeb spaces* $(\overline{\mathcal{H}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta))$ *and* $(\overline{\mathcal{V}_\eta}, L(\mathfrak{D}_\eta), L(\mu_\eta))$*. Similarly,* $([-\pi, \pi]^2, \mathfrak{B} \times \mathfrak{B}, \mu \times \mu)$ *is the complete product of* $([-\pi, \pi], \mathfrak{B}, \mu)$ *and* $([-\pi, \pi], \mathfrak{B}, \mu)$*.*

We let $({}^*\mathcal{R}, {}^*\mathfrak{D})$ *denote the hyperreals, with the transfer of the Borel field* $\mathfrak{D}$ *on* $\mathcal{R}$*. A function* $f : (\overline{\mathcal{H}_\eta}, \mathfrak{C}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{D})$ *is measurable, if* $f^{-1} : {}^*\mathfrak{D} \rightarrow \mathfrak{C}_\eta$*. The same definition holds for* $\mathcal{V}_\eta$ *Similarly,* $f : (\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, \mathfrak{C}_\eta \times \mathfrak{D}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{D})$ *is measurable, if* $f^{-1} : {}^*\mathfrak{D} \rightarrow \mathfrak{C}_\eta \times \mathfrak{D}_\eta$*. Observe that this is equivalent to the definition given in* [**?**]*. We will abbreviate this notation to* $f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{R}$*,* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{R}$ *or* $f : \overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{R}$ *is measurable,* $(*)$*. The same applies to* $({}^*\mathcal{C}, {}^*\mathfrak{D})$*, the hyper complex numbers, with the transfer of the Borel field* $\mathfrak{D}$*, generated by the complex topology. Observe that* $f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C}$*,* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ $f : \overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *is measurable, in this sense, iff* $Re(f)$ *and* $Im(f)$ *are measurable in the sense of* $(*)$*.*

We let $\overline{\mathcal{T}_\eta} = \overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}$ *and;*

$V(\overline{\mathcal{H}_\eta}) = \{f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C},\ f \text{ measurable } d(\lambda_\eta)\}$

and, similarly, we define $V(\overline{\mathcal{V}_\eta})$*. Let;*

$V(\overline{\mathcal{T}_\eta}) = \{f : \overline{\mathcal{T}_\eta} \rightarrow {}^*\mathcal{C},\ f\ measurable\ d(\lambda_\eta \times \mu_\eta)\}$

Lemma 0.9. *The identity;*

$i : (\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, L(\mathfrak{C}_\eta \times \mathfrak{D}_\eta), L(\lambda_\eta \times \mu_\eta))$

$\rightarrow (\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, L(\mathfrak{C}_\eta) \times L(\mathfrak{D}_\eta), L(\lambda_\eta) \times L(\mu_\eta))$

and the standard part mapping;

$st : (\overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta}, L(\mathfrak{C}_\eta) \times L(\mathfrak{D}_\eta), L(\lambda_\eta) \times L(\mu_\eta)) \rightarrow [-\pi, \pi] \times [-\pi, \pi]$

are measurable and measure preserving.

Proof. The proof is similar to Lemma 0.2 in [4], using Caratheodory's Extension Theorem and Theorem 22 of [1].

□

Definition 0.10. *Discrete Partial Derivatives*

Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable. As in [2], we define the discrete derivative f' to be the unique measurable function satisfying;

$f'(-\pi + \pi\frac{i}{\eta}) = \frac{\eta}{\pi}(f(-\pi + \pi\frac{i+1}{\eta}) - f(-\pi + \pi\frac{i}{\eta}));$

for $i \in {}^*\mathcal{N}_{0 \leq i \leq 2\eta-2}$.

$f'(\pi - \frac{\pi}{\eta}) = \frac{\eta}{\pi}(f(-\pi) - f(\pi - \frac{\pi}{\eta}))$

and, similarly, for $\overline{\mathcal{H}_\eta}$.

If $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ is measurable, then we define the shift (right);

$f^{sh}(-\pi + \pi\frac{j}{\eta}) = f(-\pi + \pi\frac{j+1}{\eta})$ *for* $0 \leq j \leq 2\eta - 2$

$f^{sh}(\pi - \frac{\pi}{\eta}) = f(-\pi)$

$f^{rsh}(-\pi + \pi\frac{j}{\eta}) = f(-\pi + \pi\frac{j-1}{\eta})$ *for* $1 \leq j \leq 2\eta - 1$

$f^{rsh}(-\pi) = f(\pi - \frac{\pi}{\eta})$

and, similarly, for $\overline{\mathcal{H}_\eta}$.

If $f:\overline{\mathcal{H}_\eta}\times\overline{\mathcal{V}_\eta}\rightarrow {}^*\mathcal{C}$ *is measurable. Then we define* $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\}$ *to be the unique measurable functions satisfying;*

$\frac{\partial f}{\partial x}(-\pi+\pi\frac{i}{\eta},y)=\frac{\eta}{\pi}(f(-\pi+\pi\frac{i+1}{\eta},y)-f(-\pi+\pi\frac{i}{\eta},y))$;

for $i\in {}^*\mathcal{N}_{0\leq i\leq 2\eta-2}, y\in\overline{\mathcal{V}_\eta}$

$\frac{\partial f}{\partial x}(-\pi+\pi\frac{2\eta-1}{\eta},y)=\frac{\eta}{\pi}(f(-\pi,y)-f(\pi-\frac{\pi}{\eta},y))$

$\frac{\partial f}{\partial y}(x,-\pi+\frac{j}{\eta})=\frac{\eta}{\pi}(f(x,-\pi+\pi\frac{j+1}{\eta})-f(x,-\pi+\pi\frac{j}{\eta}))$;

for $j\in {}^*\mathcal{N}_{0\leq i\leq 2\eta-2}, x\in\overline{\mathcal{H}_\eta}$

$\frac{\partial f}{\partial y}(x,-\pi+\pi\frac{2\eta-1}{\eta})=\frac{\eta}{\pi}(f(x,-\pi)-f(x,\pi-\frac{\pi}{\eta}))$

We define $\{f^{sh_x},f^{sh_y},f^{rsh_x},f^{rsh_y}\}$ *by;*

$f^{sh_x}(x_0,y_0)=(f_y)^{sh}(x_0)$

$f^{sh_y}(x_0,y_0)=(f_x)^{sh}(y_0)$

$f^{rsh_x}(x_0,y_0)=(f_y)^{rsh}(x_0)$

$f^{rsh_y}(x_0,y_0)=(f_x)^{rsh}(y_0)$

where, if $\{x_0,y_0\}\subset\overline{\mathcal{V}_\eta}$;

$f_y(x_0)=f_x(y_0)=f(\frac{\eta x_0}{\eta},\frac{\eta y_0}{\eta})$

Remarks 0.11. *If* f *is measurable, then so are;*

$\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial^2 f}{\partial x^2},\frac{\partial^2 f}{\partial y^2},f_x,f_y,f^{sh_x},f^{sh_y},f^{rsh_x},f^{rsh_y},f^{sh_x^2},f^{sh_y^2},f^{rsh_x^2},f^{rsh_y^2}\}$

This follows immediately, by transfer, from the corresponding result for the discrete derivatives and shifts of discrete functions $f:\mathcal{H}_n\times\mathcal{V}_n\rightarrow\mathcal{C}$, *where* $n\in\mathcal{N}$, *see Definition 0.15 and Definition 0.18 of* [5], *([2])*

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[2] If $f:\overline{\mathcal{V}_\eta}\rightarrow {}^*\mathcal{C}$ is measurable, then, as in [2], we can define the discrete derivative f' to be the unique measurable function satisfying;

Lemma 0.12. *Let $g, h : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable. Then;*

(i). $\int_{\overline{\mathcal{V}_\eta}} g'(y) d\mu_\eta(y) = 0$

(ii). $(gh)' = g'h^{sh} + gh'$

(iii). $\int_{\overline{\mathcal{V}_\eta}} (g'h)(y) d\mu_\eta(y) = -\int_{\overline{\mathcal{V}_\eta}} g^{sh} h' d\mu_\eta(y)$

(iv). $\int_{\overline{\mathcal{V}_\eta}} g(y) d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{sh}(y) d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y) d\mu_\eta(y)$

(v). $(g')^{sh} = (g^{sh})'$

(vi). $\int_{\overline{\mathcal{V}_\eta}} (g''h)(y) d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} (g^{sh^2} h'')(y) d\mu_\eta(y)$

$f'(-\pi + \pi \frac{i}{\eta}) = \frac{\eta}{\pi}(f(-\pi + \pi \frac{i+1}{\eta}) - f(-\pi + \pi \frac{i}{\eta}));$

for $i \in {}^*\mathcal{N}_{0 \leq i \leq 2\eta - 2}$.

$f'(-\pi + \pi \frac{2\eta - 1}{\eta}) = 0$

and, similarly, for $\overline{\mathcal{H}_\eta}$. We can define the shift by;

$f^{sh}(-\pi + \pi \frac{j}{\eta}) = f(-\pi + \pi \frac{j+1}{\eta})$ for $0 \leq j \leq 2\eta - 2$

$f^{sh}(\pi - \frac{\pi}{\eta}) = 0.$

Similar extensions can be made to $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f^{sh_x}, f^{sh_y}, f^{rsh_x}, f^{rsh_y}\}$.

The same results hold, if $g,h:\overline{\mathcal{H}_\eta}\rightarrow {}^*\mathcal{C}$ are measurable, ([3])

Proof. In the first part, for (i), we have, using Definitions 0.10 and 0.8, that;

$\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y)$

$=\frac{\pi}{\eta}[{}^*\sum_{0\leq j\leq 2\eta-2}\frac{\eta}{\pi}[g(-\pi+\pi(\frac{j+1}{\eta}))-g(-\pi+\pi(\frac{j}{\eta}))]$

$+\frac{\eta}{\pi}[g(-\pi)-g(\pi-\frac{\pi}{\eta}]]=0$

The proofs of $(ii),(iii)$ are as in Lemma 0.12 of [2]. (iv) is clear. (v) follows easily from Definitions 0.10 and (vi) follows, repeating the

[3]Similar results hold for $\{sh_x, sh_y, \frac{\partial}{\partial x},\frac{\partial}{\partial y}\}$. Namely, if $g,h:\overline{\mathcal{T}_\eta}\rightarrow {}^*\mathcal{C}$ are measurable. Then;

(i). $\int_{\overline{\mathcal{T}_\eta}}\frac{\partial g}{\partial x}d(\lambda_\eta\times\mu_\eta)=0$

(ii). $\frac{\partial gh}{\partial x}=\frac{\partial g}{\partial x}h^{sh_x}+g\frac{\partial h}{\partial x}$

(iii). $\int_{\overline{\mathcal{T}_\eta}}\frac{\partial g}{\partial x}hd(\lambda_\eta\times\mu_\eta)=-\int_{\overline{\mathcal{T}_\eta}}g^{sh_x}\frac{\partial h}{\partial x}d(\lambda_\eta\times\mu_\eta)$

(iv). $\int_{\overline{\mathcal{T}_\eta}}gd(\lambda_\eta\times\mu_\eta)=\int_{\overline{\mathcal{T}_\eta}}g^{sh_x}d(\lambda_\eta\times\mu_\eta)=\int_{\overline{\mathcal{T}_\eta}}g^{rsh_x}d(\lambda_\eta\times\mu_\eta)$

(v). $(\frac{\partial g}{\partial x})^{sh_x}=\frac{\partial(g^{sh_x})}{\partial x}$

(vi). $\int_{\overline{\mathcal{T}_\eta}}(\frac{\partial^2 g}{\partial x^2}h)d(\lambda_\eta\times\mu_\eta)=\int_{\overline{\mathcal{T}_\eta}}(g^{sh_x^2}\frac{\partial^2 h}{\partial x^2})d(\lambda_\eta\times\mu_\eta)$ $(*)$

and, with sh_y replacing sh_x, and $\frac{\partial}{\partial y}$, replacing $\frac{\partial}{\partial x}$, in $(*)$.

For (i), using (i) from the argument in the main proof, we have;

$\int_{\overline{\mathcal{T}_\eta}}\frac{\partial g}{\partial x}d(\lambda_\eta\times\mu_\eta)$

$=\int_{\overline{\mathcal{V}_\eta}}(\int_{\overline{\mathcal{H}_\eta}}(\frac{\partial g}{\partial x})_y d\lambda_\eta)d\mu_\eta(y)$

$=\int_{\overline{\mathcal{V}_\eta}}(\int_{\overline{\mathcal{H}_\eta}}(\frac{\partial g_y}{\partial x})d\lambda_\eta)d\mu_\eta(y)$

$=\int_{\overline{\mathcal{V}_\eta}}0d\mu_\eta(y)=0$

The proofs of $(ii),(iii),(iv)$ are similar to the main proof, relying on the result of (i). (v) follows easily from Definitions 0.10 and (vi) follows, repeating the result of (iii), and applying (v).

result of (iii), and applying (v).

□

Definition 0.13. *We define the nonstandard Laplacian operator* $\Delta : V(\overline{\mathcal{T}_\eta}) \rightarrow V(\overline{\mathcal{T}_\eta})$ *by;*

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

and $<,>: V(\overline{\mathcal{T}_\eta}) \times V(\overline{\mathcal{T}_\eta}) \rightarrow {}^*\mathcal{C}$ *by;*

$< f, g >= \int_{\overline{\mathcal{T}_\eta}} f\overline{g} d(\lambda_\eta \times \mu_\eta)$

$||.||_1 : V(\overline{\mathcal{T}_\eta}) \rightarrow V(\overline{\mathcal{T}_\eta})$ *by;*

$||f||_1 = \int_{\overline{\mathcal{T}_\eta}} |f| d(\lambda_\eta \times \mu_\eta)$

Lemma 0.14. *We have that* $\Delta^* = \Delta + S$

where $S : V(\overline{\mathcal{T}_\eta}) \rightarrow V(\overline{\mathcal{T}_\eta})$ *is given by;*

$S(g) = (g_{yy}^{rsh_y^2} - g_{yy}^{rsh_x^2}) = (g_{xx}^{rsh_x^2} - g_{xx}^{rsh_y^2})$

Proof. We have that, for $\{f, g\} \subset V(\overline{\mathcal{T}_\eta})$, using $(iv), (vi)$ of Lemma 0.12;

$< \Delta f, g >= \int_{\overline{\mathcal{T}_\eta}} (\Delta f)\overline{g} d(\lambda_\eta \times \mu_\eta)$

$= \int_{\overline{\mathcal{T}_\eta}} (f_{xx} + f_{yy})\overline{g} d(\lambda_\eta \times \mu_\eta)$

$= \int_{\overline{\mathcal{T}_\eta}} (f^{sh_x^2}\overline{g}_{xx} + f^{sh_y^2}\overline{g}_{yy}) d(\lambda_\eta \times \mu_\eta)$

$= \int_{\overline{\mathcal{T}_\eta}} f^{sh_x^2}(\Delta\overline{g}) d(\lambda_\eta \times \mu_\eta) + \int_{\overline{\mathcal{T}_\eta}} (f^{sh_y^2} - f^{sh_x^2})\overline{g}_{yy} d(\lambda_\eta \times \mu_\eta)$

$=< f, \Delta g > + < f^{sh_y^2} - f^{sh_x^2}, g_{yy} >$

$=< f, \Delta g > + < f, (g_{yy})^{rsh_y^2} - (g_{yy})^{rsh_x^2} >=< f, (\bigtriangledown + S)(g) >$

By symmetry, we obtain the result.

□

Definition 0.15. *If* $\{f,g\}\subset V(\overline{\mathcal{T}_\eta})$*, with* $||f||_2=||g||_2=1$*, we say that* f *and* g *are almost orthogonal, if* $<f,g>\simeq 0$*.*

Lemma 0.16. *If* $\{\lambda,\mu\}$ *are distinct eigenvalues of* Δ *on* $V(\overline{\mathcal{T}_\eta})$*, with* $(\lambda-\mu)\notin\mu(0)$*, and corresponding eigenvectors* $\{f_\lambda,f_\mu\}\subset V_{n,g,r}$*, ([4]) with* $||f||_2=||g||_2=1$*,* $n\geq 2$ *then* f_λ *and* f_μ *are almost orthogonal. Moreover, if* $\{\lambda,\mu\}$ *are distinct generalised eigenvalues of* Δ *on* $V(\overline{\mathcal{T}_\eta})$*, with* $(\lambda-\mu)\notin\mu(0)$*, and corresponding generalised eigenvectors* $\{f_\lambda,f_\mu\}\subset V_{n,g,r}$*,* $n\geq 2$*, then* f_λ *and* f_μ *are almost orthogonal. (...?In particular, there exists a pairwise almost orthogonal basis of ?? consisting of generalised eigenvectors of* $\bigtriangledown$ *on* $V(\overline{\mathcal{T}_\eta})$*?...)*

Proof. For the first part, we have that;

$<\lambda f_\lambda,f_\mu>$

$=<\Delta f_\lambda,f_\mu>$

$=<f_\lambda,\Delta^* f_\mu>$

$=<f_\lambda,(\Delta+S)f_\mu>$

$=<f_\lambda,\mu f_\mu>+<f_\lambda,S(f_\mu)>$

Hence;

---

[4]For $g\in C(\mathcal{R})$, with $g(x)\geq Cx$, for sufficiently large $x\in\mathcal{R}$, $C>0$, and $n\in{}^*\mathcal{N}_{\geq 0}$, we let;

$W_{0,g}(\overline{\mathcal{H}_\eta})=\{f\in V(\overline{\mathcal{H}_\eta}):|f(-\pi+\pi\frac{i+1}{\eta})-f(-\pi+\pi\frac{i}{\eta})|<\frac{1}{g(\eta)}\}$

(with the convention that $2\eta=0$)

$W_{n,g}(\overline{\mathcal{H}_\eta})=\{f\in V(\overline{\mathcal{H}_\eta}):\{f,f',\ldots,f^{(n)}\}\subset W_{0,g}(\overline{\mathcal{H}_\eta})\}$

$W_{n,g,r}(\overline{\mathcal{H}_\eta})=\{f\in W_{n,g}(\overline{\mathcal{H}_\eta}):max(|f|,|f'|,\ldots,|f^{(n)}|)\leq r\}$

where $r\in\mathcal{R}$, and, similarly, we define $\{W_{0,g}(\overline{\mathcal{V}_\eta}),W_{n,g}(\overline{\mathcal{V}_\eta}),W_{n,g,r}(\overline{\mathcal{V}_\eta})\}$. For $f\in V(\overline{\mathcal{T}_\eta})$, we let $||f||_2=[\int_{\overline{\mathcal{T}_\eta}}|f|^2d(\lambda_\eta\times\mu_\eta)]^{\frac{1}{2}}$.

$V_{n,g,r}(\overline{\mathcal{T}_\eta})=span_{{}^*\mathcal{C}}\{f\in V(\overline{\mathcal{T}_\eta}),||f||=1,$

$f^x\in W_{n,g,r}(\overline{\mathcal{V}_\eta}),f^y\in W_{n,g,r}(\overline{\mathcal{H}_\eta}),(x,y)\in\overline{\mathcal{T}_\eta}\}$

$(\lambda-\overline{\mu})<f_\lambda,f_\mu>=<f_\lambda,S(f_\mu)>$

We have, as $|f_\lambda|\leq r$ that;

$|<f_\lambda,S(f_\mu)>|\leq r||((f_\mu)_{yy}^{rsh_y^2}-(f_\mu)_{yy}^{rsh_x^2})||$

For $(x,y)\in\overline{\mathcal{T}_\eta}$, $((f_\mu)_{yy})_x$ and $((f_\mu)_{yy})_y$ are S-continuous, (explain this), hence;

$||((f_\mu)_{yy}^{rsh_y^2}-(f_\mu)_{yy}^{rsh_x^2})||_1\leq max((f_\mu)_{yy}^{rsh_y^2}-(f_\mu)_{yy}^{rsh_x^2})=\epsilon\simeq 0$

Hence;

$|<f_\lambda,f_\mu>|\leq\frac{|<f_\lambda,S(f_\mu)>|}{|(\lambda-\overline{\mu})|}\leq\frac{r\epsilon}{(\lambda-\overline{\mu})}\simeq 0$

Sketch..For the second part, observe that $\Delta:V_{n,g,r}\rightarrow V_{n-2,g,r}$ and $S:V_{n,g,r}\rightarrow V_{n-2,g,r}$, for $n\geq 2$. If $\kappa\in{}^*\mathcal{N}_{\geq 2}$, we can choose a basis $\{\overline{v}_\kappa\}$ for $V_{\kappa,g,r}$, and extend it to bases $\{\overline{v}_\lambda:2\leq\lambda\leq\kappa\}$ for $\{V_{\lambda,g,r}:2\leq\lambda\leq\kappa\}$, with the property that $\overline{v_{\lambda_1}}\subset\overline{v_{\lambda_2}}$, if $2\leq\lambda_2\leq\lambda_1$. Let $B_\kappa:V_{2,g,r}\rightarrow V_{2,g,r}$ be defined by;

$B_\kappa(f)=\Delta(f)$ if $f\in\bigcup_{4\leq\lambda\leq\kappa}V_{\lambda,g,r}$

$B_\kappa(f)=f_{2,3}$, if $f\in V_{3,g,r}$, where $\Delta(f)=f_{2,3}+f_1$, $f_{2,3}\in span(\overline{v_2}\setminus\overline{v_4})$, $f_1\in span(\overline{v_1}\setminus\overline{v_2})$

$B_\kappa(f)=f_2$, if $f\in V_{2,g,r}$, where $\Delta(f)=f_2+f_{0,1}$, $f_2\in span(\overline{v_2})$, $f_{0,1}\in span(\overline{v_0}\setminus\overline{v_2})$

We have that $B_\kappa^*=B_\kappa+S_1$, where $S_1(g)\simeq 0$, for $g\in V_{2,g,r}$, $||g||_2=1$. Hence $B_\kappa=B_{\kappa,s}+S_2$, where $B_{\kappa,s}$ is symmetric and $S_2(g)\simeq 0$, for $g\in V_{2,g,r}$. Suppose that λ is a generalised eigenvector of Δ, with corresponding eigenvector $f_\lambda\in W_2$, such that, for n minimal with $(\Delta-\lambda I)^n(f)=0$, $(\Delta-\lambda I)^m(f)\in W_2$, for $0\leq m<n$. Then, clearly;

$0=(B_\kappa-\lambda I)^n(f_\lambda)$

$(B_{\kappa,s}+S_2-\lambda I)^n(f_\lambda)$

$=\sum_{j=0}^n(B_{\kappa,s}-\lambda I)^{n-j}S_2^j(f_\lambda)$

$\simeq (B_{\kappa,s}-\lambda I)^n f_\lambda$

As $B_{\kappa,s}$ is diagonalizable, it follows there exists $\lambda'\in\mu(\lambda)$, with λ' an eigenvalue of $B_{\kappa,s}$, such that $(B_{\kappa,s}-\lambda' I)f_\lambda\simeq 0$. It follows that;

$(\Delta-\lambda I)f_\lambda$

$=(B_\kappa-\lambda I)f_\lambda$

$=(B_{\kappa,s}+S_2-\lambda I)f_\lambda$

$\simeq (B_{\kappa,s}-\lambda I)f_\lambda$

$=(B_{\kappa,s}-\lambda' I)f_\lambda+(\lambda'-\lambda)f_\lambda\simeq 0$

Hence, using the same argument to show that $((\Delta-\mu I)f_\mu\simeq 0$;

$0\simeq<(\bigtriangledown-\lambda I)f_\lambda,f_\mu>$

$=<f_\lambda,(\bigtriangledown-\lambda I)^*f_\mu>$

$=<f_\lambda,(\bigtriangledown+S-\overline{\lambda})f_\mu>$

$\simeq<f_\lambda,\bigtriangledown f_\mu>-\overline{\lambda}<f_\lambda,f_\mu>$

$=<f_\lambda,(\bigtriangledown-\mu I)f_\mu>+(\mu-\overline{\lambda})<f_\lambda,f_\mu>$

$\simeq(\mu-\overline{\lambda})<f_\lambda,f_\mu>$

$<f_\lambda,f_\mu>\simeq 0.$

□

Strategy:

(i). Show that for finite λ, a generalised eigenvalue of Δ in $C=\{f\in V(\overline{T}_\eta): f(-\pi,y)=f(\pi-\frac{\pi}{\eta},y),\frac{\partial f}{\partial x}(-\pi,y)=\frac{\partial f}{\partial x}(\pi-\frac{\pi}{\eta},y),f(x,-\pi)=f(x,\pi),\frac{\partial f}{\partial y}(x,-\pi)=\frac{\partial f}{\partial y}(x,\pi-\frac{\pi}{\eta})\}$, ${}^\circ\lambda$ is an eigenvalue of Δ on $([-\pi,\pi]\times[-\pi,\pi]$, same question with $V(\overline{T}_\eta)$ replacing C?.

(ii). Show that for finite λ, an eigenvalue of Δ in $C^\infty([-\pi,\pi]\times[-\pi,\pi]$, there exists $g\in C$, $\lambda'\in\mu(\lambda)$ an eigenvector of Δ in C, with eigenvalue λ'. (Show, using S, there exists ! $\lambda' \in \mu(\lambda)$. If, using $V(\overline{T}_\eta)$, there might exist multiple...or use Lemma 0.17 and second question in (i), extending solution to all of $\overline{T}_\eta$.)

(iii). For corresponding eigenvector $f_{\lambda'}$, $||f_{\lambda'}|| = 1$, $\{f_{\lambda'}, (f_{\lambda'})_{xx}, (f_{\lambda'})_{yy}\}$ are S-continuous, in particularly $S(f) \simeq 0$, and ${}^\circ f_{\lambda'}$ is an eigenvector of Δ on $([-\pi,\pi] \times [-\pi,\pi]$, ${}^\circ((f_\lambda)_{xx})$, ${}^\circ((f_\lambda)_{yy}) = \frac{\partial^2({}^\circ f_\lambda)}{\partial x^2}$, ${}^\circ((f_\lambda)_{yy}) = \frac{\partial^2({}^\circ f_\lambda)}{\partial y^2}$. Extension of Lemma 0.26, differentiate nonstandard equation, satisfied by f_λ, use same specialisation argument in (i) to prove the last two claims.

(iv). JCF implies basis of generalised eigenvectors e_i in $V(\overline{T}_\eta)$.

(v). Eigenvectors for finite λ are almost orthogonal, as shown above.

(vi). Generalised eigenvectors for finite λ are almost orthogonal, as shown above.

(vii). Decay rates and spacing; show that $| < f, e_i > | \leq \frac{1}{\lambda_i^2}$, for $f \in C$, $|\lambda_i - \lambda_j| \geq D \in \mathcal{R}$, need a transfer argument, result for sufficiently large finite n, (finitely many (finite) eigenvalues.

$(viii)$ $f \simeq {}^*\sum < f, e_i > e_i$, using GS+(vi).

(ix) ${}^\circ f = \sum_{i \in \mathcal{Z}} < f, {}^\circ e_i > {}^\circ e_i$, uniform convergence in Hodge Theorem for the torus.

$(i)'(ii)'(iii)'$. Although S is not Δ-invariant, can replace Δ by nonstandard approximation $e^{t\Delta}$, and find integral representation to show that S is $e^{t\Delta}$-invariant. Similar questions to $(i), (ii), (iii)$...

a. Evaluate eigenvalues of Δ in $V(\overline{H}_\eta)$, using linear algebra argument.

b. Show that, for finite m if $(\Delta - \lambda I)^m f = 0$, $f \neq 0$, in $V(\overline{H}_\eta)$, then ${}^\circ\lambda$ is an eigenvalue of Δ on $[-\pi,\pi]$. Hence, for finite $m \in \mathcal{N}$, n sufficiently large, $(\Delta_n - \lambda' I)^m f = 0$, implies $|\lambda - \lambda'| < \epsilon$, $\lambda \in Spec(\Delta|_{[-\pi,\pi]})$ Use transfer, to show that $\lambda' \simeq \lambda$ for generalised eigenvalues λ' of λ.

Lemma 0.17. *Given measurable boundary conditions* $\{h_1, h_2\} \subset V(\overline{\mathcal{V}_\eta})$, *there exists a unique measurable* $f \in V(\overline{\mathcal{T}_\eta})$, *satisfying the nonstandard Laplace eigenvalue problem, for* $\lambda \in {}^*\mathcal{C}$;

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f$

on $(\overline{\mathcal{H}_\eta} \setminus [\pi\frac{\eta-2}{\eta}, \pi)) \times \overline{\mathcal{V}_\eta}$

with $f(-\pi, y) = h_1(y)$, $\frac{\partial f}{\partial x}(-\pi, y) = h_2(y)$ *for* $y \in \overline{\mathcal{V}_\eta}$, $(*)$.

Proof. Observe that, by Definition 0.10, if $f : \overline{\mathcal{T}_\eta} \rightarrow {}^*\mathcal{C}$ is measurable, then;

$\frac{\partial^2 f}{\partial x^2}(-\pi + \pi\frac{i}{\eta}, y) = \frac{\eta^2}{\pi^2}(f(-\pi + \pi\frac{i+2}{\eta}, y) - 2f(-\pi + \pi\frac{i+1}{\eta}, y) + f(-\pi + \pi\frac{i}{\eta}, y))$

$(0 \leq i \leq 2\eta - 3), y \in \overline{\mathcal{V}_\eta}$

$\frac{\partial f}{\partial x}(-\pi, y) = \frac{\eta}{\pi}(f(-\pi + \pi\frac{1}{\eta}, y) - f(-\pi, y))$, $y \in \overline{\mathcal{V}_\eta}$

Therefore, if f satisfies $(*)$, we must have;

$f(-\pi, y) = h_1(y)$, $(y \in \overline{\mathcal{V}_\eta})$

$f(-\pi + \pi\frac{1}{\eta}, y) = \frac{\pi h_2(y)}{\eta} + f(-\pi, y) = \frac{\pi h_2(y)}{\eta} + h_1(y)$, $(y \in \overline{\mathcal{V}_\eta})$

$f(-\pi + \pi\frac{i+2}{\eta}, y)$

$= -f(-\pi+\pi\frac{i}{\eta}, y) + 2f(-\pi+\pi\frac{i+1}{\eta}, y) + \frac{\pi^2}{\eta^2}(\lambda f(-\pi+\pi\frac{i}{\eta}, y) - \frac{\partial^2 f}{\partial y^2}(-\pi + \pi\frac{i}{\eta}, y))$

$(0 \leq i \leq 2\eta - 3, y \in \overline{\mathcal{V}_\eta})$

See also the proof of Lemma 0.5 in [4].

□

Definition 0.18. *We let* $\overline{\mathcal{Z}_\eta} = \{m \in {}^*\mathcal{Z} : -\eta \leq m \leq \eta - 1\}$ *Given a measurable* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$, *we define, for* $m \in \mathcal{Z}_\eta$, *the* $m'th$ *discrete Fourier coefficient to be;*

$\hat{f}_\eta(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(y)exp_\eta(-iym)d\mu_\eta(y)$

(and, similarly, for $\overline{\mathcal{V}_\eta}$*).*

Transposing Lemma 0.9 of [2], *(*[5]*);*

$f(y) = \sum_{m\in\mathcal{Z}_\eta} \hat{f}_\eta(m)exp_\eta(iym)$ $(*)$

Given a measurable $f : \overline{\mathcal{H}_\eta} \times \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard vertical Fourier transform* $\hat{f} : \overline{\mathcal{H}_\eta} \times \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *by;*

$\hat{f}(x,m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(x,y)exp_\eta(-iym)d\mu_\eta(y)$

and, given a measurable $g : \overline{\mathcal{H}_\eta} \times \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard inverse vertical Fourier transform by;*

$\check{g}(x,y) = \sum_{m\in\mathcal{Z}_\eta} g(x,m)exp_\eta(iym)$

so that, by $(*)$*,* $f = \check{\hat{f}}$

Similar to Definition 0.20 of [2]*, for* $f \in \overline{\mathcal{V}_\eta}$*, we let* $\phi_\eta, \psi_\eta : \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *be defined by;*

$\phi_\eta(m) = \frac{\eta}{\pi}(exp_\eta(-im\frac{\pi}{\eta}) - 1)$

$\psi_\eta(m) = \frac{\eta}{\pi}(exp_\eta(im\frac{\pi}{\eta}) - 1)$

$R_\eta(m) = \frac{1}{2\pi\eta}f(\pi(1-\frac{1}{\eta}))(exp_\eta(-im\pi(1-\frac{2}{\eta}))-exp_\eta(\frac{im\pi}{\eta})exp_\eta(-im\pi(1-\frac{1}{\eta}))$

$R'_\eta(m) = \frac{1}{2\pi\eta}f'(\pi(1-\frac{1}{\eta}))(exp_\eta(-im\pi(1-\frac{2}{\eta}))-exp_\eta(\frac{im\pi}{\eta})exp_\eta(-im\pi(1-\frac{1}{\eta}))$.

$T_\eta(m) = \phi_\eta(m)R_\eta(m)$

$S_\eta(m) = \phi_\eta(m)\psi_\eta(m)R_\eta(m)) + \phi_\eta(m)R'_\eta(m)$

[5] *We have there that the measure on* $\overline{\mathcal{S}_\eta} = \lambda_\eta$. The result follows using the scalar map $p : \overline{\mathcal{V}_\eta} \rightarrow \overline{\mathcal{S}_\eta}$, $p(x) = \frac{x}{\pi}$, and the fact that $p_*(\mu_\eta) = \lambda_\eta$

The following is the analogue of Lemma 0.14 in [2], using the definition of the discrete derivative in Definition 0.10 and the discrete Fourier coefficients from Definition 0.18;

Lemma 0.19. *Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable; then, for $m \in \mathcal{Z}_\eta$,*

$$\hat{f}''(m) + S_\eta(m) = \psi_\eta^2(m)\hat{f}(m)$$

In particular, if $\{f, f'\}$ are finite, which is the case when $f = {}^*h$, for $h \in C^1[-\pi,\pi]$, or $f \in W_{2,g,r}$, $S_\eta(m) \simeq 0$, for $m \in \mathcal{Z}_\eta$, and, similarly, for $\overline{\mathcal{H}_\eta}$.

Proof. We have, using Lemma 0.12(iii), that;

$$(\hat{f'})(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f'(y)exp_\eta(-iym)d\mu_\eta(y)$$

$$= -\frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f^{sh}(y)(exp_\eta)'(-iym)d\mu_\eta(y)$$

A simple calculation shows that;

$$(exp_\eta)'(-iym) = exp_\eta(-iym)\phi_\eta(m)$$

where $\phi_\eta(m) = \frac{\eta}{\pi}(exp_\eta(-im\frac{\pi}{\eta}) - 1)$.

Therefore;

$$(\hat{f'})(m) = -\phi_\eta(m)\hat{f^{sh}}(m)$$

We have $\hat{f^{sh}}(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f^{sh}(y)exp_\eta(-iym)d\mu_\eta(y)$

$$= \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(y)exp_\eta^{rsh}(-iym)d\mu_\eta(y)$$

$$= exp_\eta(im\frac{\pi}{\eta})\hat{f}(m) + R_\eta(m)$$

Hence, $(\hat{f'})(m) = -\phi_\eta(m)\hat{f^{sh}}(m) = -\phi_\eta(m)exp_\eta(im\frac{\pi}{\eta})\hat{f}(m) - \phi_\eta(m)R_\eta(m)$

$$= \psi_\eta(m)\hat{f}(m) - T_\eta(m)$$

Then $\hat{f}''(m)$

$$= \psi_\eta(m)\hat{f}'(m) - \phi_\eta(m)R'_\eta(m)$$

$= \psi_\eta(m)(\psi_\eta(m)\hat{f}(m) - \phi_\eta(m)R_\eta(m)) - \phi_\eta(m)R'_\eta(m)$

$= \psi_\eta^2(m)\hat{f}(m) - \phi_\eta(m)\psi_\eta(m)R_\eta(m)) - \phi_\eta(m)R'_\eta(m)$

$= \psi_\eta^2(m)\hat{f}(m) - S_\eta(m)$

as required. □

We also obtain an improvement of Lemmas 0.15 from [2];

Lemma 0.20. *If $f \in C^\infty([-\pi,\pi])$, then the functions $\hat{f''}_{n,st}(m)$ are uniformly bounded, independently of n, for $n \geq 1$, and, if η is infinite, $\hat{f''}_{\eta,st}(m)$ is bounded. The same result holds, if $f \in W_{2,g,r}(\overline{\mathcal{V}_\eta})$.*

Proof. For the first part, we follow through the proof of Lemma 0.22 in [5], ignoring the constant D_n there, to obtain that $|\hat{f''}_{n,st}(m)| \leq M$, for $m \in \mathcal{Z}_n$, where $M = ||f''||_{L^1([-\pi,\pi])}$. The second claim follows by transfer. For the last part, we have, using the first part of Lemma 0.22 in [5], that;

$|\hat{f''}|(t)$

$\leq (\sum_{0\leq j\leq 2n-2} |f'(-\pi+\pi(\frac{j+1}{n})) - f'(-\pi+\pi(\frac{j}{n}))|$

$+|f'(-\pi) - f'(-\pi+\pi(\frac{2n-1}{n}))|$

$\leq \frac{2\eta}{g(\eta)} \leq \frac{2}{C}$, by definition of $W_{2,g,r}$.

□

We improve Lemma 0.16 from [2];

Lemma 0.21. *Let $f \in W_{2,g,r}(\overline{\mathcal{V}_\eta})$, then ${}^\circ f \in C^1([-\pi,\pi])$, ${}^\circ((\hat{f})(m)) = ({}^\circ f)\hat{}(m)$, for $m \in \mathcal{Z}$, and, there exists a constant $W \in \mathcal{R}$, such that $|\hat{f}(m)| \leq \frac{W}{m^2}$, for $m \in \overline{\mathcal{Z}_\eta}$. In particular;*

${}^\circ(\sum_{m\in\mathcal{Z}_\eta} \hat{f}(m)exp_\eta(imx'))$

$= \sum_{m\in\mathcal{Z}} {}^\circ((\hat{f})(m))exp(im{}^\circ(x'))$

$= \sum_{m\in\mathcal{Z}} ({}^\circ f)\hat{}(m)exp(im{}^\circ(x'))$

Proof. For the first part, by definition of g, we have that $g(\eta) \geq K\eta$, for some $K \in \mathcal{R}$. Then, if $\{x,y\} \subset \overline{\mathcal{V}_\eta}$, with $x \simeq y$, we have $|x-y| = \frac{\pi j}{\eta}$, with $j < \frac{\epsilon\eta}{\pi}$, for all $\epsilon > 0$. Then $|f(x) - f(y)| \leq \frac{j}{g(\eta)} < \frac{\epsilon\eta}{g(\eta)\pi} \leq \frac{\epsilon}{K}$. Hence, $f(x) \simeq f(y)$, and f is S-continuous in the sense of Definition 2.35 of [6]. As f is finite, by Theorem 2.36 of [6], we have that ${}^{\circ}f \in C[-\pi,\pi]$, ([6]). Similarly, ${}^{\circ}(f') \in C[-\pi,\pi]$, and, using the fact that $({}^{\circ}f)' = {}^{\circ}(f')$, ([7]) we obtain ${}^{\circ}f \in C^1([-\pi,\pi])$. We have ${}^{\circ}((\hat{f})(m)) = ({}^{\circ}f)\hat{}(m)$, for $m \in \mathcal{Z}$, using Theorem 3.20 of [6], as $f(y)exp_\eta(-iym)$ is finite, and, therefore S-integrable, with ${}^{\circ}[f(y')exp_\eta(-iy'm)] = {}^{\circ}f(y)e^{-iym}$, $y = {}^{\circ}y'$. For the next claim, we have, by Lemma 0.20, that $\hat{f''}_{\eta,st}(m)$ is bounded, for $m \in \overline{\mathcal{Z}_\eta}$, and, by Lemma 0.19, that $S_\eta(m) \simeq 0$. Hence, again using Lemma 0.19, and the estimate on ψ_η^2, from Lemma 0.16 of [2];

$$|\hat{f}(m)| \leq \frac{|\hat{f''}(m)+S_\eta(m)|}{|\psi_\eta^2(m)|} = \frac{|\hat{f''}(m)}{|\psi_\eta^2(m)|} \leq \frac{W}{m^2}$$

The rest is clear, from the proof of Lemma 0.16, in [2].

□

Lemma 0.22. *If $f : \overline{\mathcal{T}}_\eta \rightarrow {}^*\mathcal{C}$ is measurable then;*

(i). $[(\frac{\partial^2 f}{\partial x^2})]^m = \frac{d^2(\hat{f})^m}{dx^2}$

(ii). $[(\frac{\partial^2 f}{\partial y^2})]^m = \psi_\eta^2(m)(\hat{f})^m - S_\eta(m)$

so, if f is an eigenvector for the Laplace operator, given by Lemma 0.17, with eigenvalue $\lambda \in {}^*\mathcal{C}$;

$$\frac{d^2(\hat{f})^m}{dx^2}(x') + (\psi_\eta^2(m) - \lambda)(\hat{f})^m(x') - S_\eta(m) = 0, \ (x' \in \overline{\mathcal{H}}_\eta \setminus [-\pi(1+\tfrac{2}{\eta}),\pi))$$

$$\frac{d(\hat{f})^m}{dx}(-\pi + \tfrac{\pi}{\eta}) - \hat{g}_2(m) = 0$$

$$(\hat{f})^m(-\pi) = \hat{g}_1(m)$$

$(*)$

[6]Observe that f^{add} is S-continuous and finite on ${}^*[-\pi,\pi]$, for the compact interval $[-\pi,\pi]$, where $f^{add}(\pi) = f(-\pi)$, $f^{add}|_{[-\pi,\pi)} = f|_{[-\pi,\pi)}$.

[7]To see this, observe that we can adapt the proof of Lemma 0.6, taking $G(t) = f'(t)$, for $t \in [-\pi,\pi)$, so $|G| \leq L$, G is S-integrable and ${}^{\circ}G \in C[-\pi,\pi]$. We then have that $f^{sch}(t)$ satisfies the defining schema $(*)$, where $f^{sch}(\pi) = f(\pi(1-\frac{1}{\eta}))$ and $f^{sch}|_{[-\pi,\pi)} = f|_{[-\pi,\pi)}$. Then;

$${}^{\circ}f(t) - {}^{\circ}f(-\pi) = {}^{\circ}f^{sch}(t) - {}^{\circ}f^{sch}(-\pi) = \int_{-\pi}^{t} {}^{\circ}G(s)ds$$

Therefore, ${}^{\circ}f \in C^1[-\pi,\pi]$, and $({}^{\circ}f)' = {}^{\circ}G = {}^{\circ}(f')$

for $m \in \mathcal{Z}_\eta$, $x' \in \overline{H}_\eta$.

Proof. The proof that $\hat{[(\frac{\partial f}{\partial x})]}^m = \frac{d[(\hat{f})^m]}{dx}$, for $m \in \overline{\mathcal{Z}_\eta}$, is a simple adaptation of the proof of Lemma 0.8 in [2], using the new definition in 0.12. Hence, $\hat{[(\frac{\partial^2 f}{\partial x^2})]}^m = \frac{d^2[(\hat{f})^m]}{dx^2}$. Using Lemma 0.22, we have that $\hat{(\frac{\partial^2 f}{\partial y^2})} = \psi_\eta^2 \hat{f} - S_\eta$, hence, substituting for $m \in \overline{\mathcal{Z}_\eta}$, we obtain $\hat{[(\frac{\partial^2 f}{\partial y^2})]}^m = \psi_\eta^2(m)(\hat{f})^m - S_\eta(m)$. The final result is then clear. □

Lemma 0.23. *If* $\{\lambda, a, b\} \subset {}^*\mathcal{C}$ *are finite,* $m \in \mathcal{Z}_\eta$ *is finite, then if* $g \in \overline{\mathcal{H}_\eta}$*, and* $g''(x) = (\lambda - \psi_\eta^2(m))g$*, with* $g(-\pi) = a$, $g'(-\pi) = b$*, then* $h = {}^\circ g$ *satisfies the ODE;*

$$h''(x) = ({}^\circ\lambda + m^2)h(x),\ (x \in (-\pi, \pi))$$

with initial condition $h(-\pi) = {}^\circ a$, $h'(-\pi) = {}^\circ b$.

Proof. Let $H : {}^*[-\pi, \pi) \times {}^*\mathcal{C}^2 \to {}^*\mathcal{C}^2$ be given by;

$$H(\tau, x_1', x_2') = (x_2', (\lambda - \psi_\eta^2(m))x_1')$$

Observing that, $H({}^\circ\tau, {}^\circ x_1', {}^\circ x_2') = ({}^\circ x_2', ({}^\circ\lambda + m^2)({}^\circ x_1'))$, a simple adaptation of Lemmas 0.5 and 0.6, with $c_0(s) = -({}^\circ\lambda + m^2)$, H replacing ${}^*G$, where $G(s, x_1, x_2) = (x_2, ({}^\circ\lambda + m^2)x_1)$, and (a, b) replacing the initial condition $\overline{x_0} \in \mathcal{C}^2$, (noting that, in Lemma 0.5, ${}^\circ(\overline{X}(-\pi)) = ({}^\circ a, {}^\circ b)$) gives the result.

□

Lemma 0.24. *Let* $\{\mu, e, f\} \subset {}^*\mathcal{C}$*, and let* $h \in \overline{\mathcal{H}_\eta}$ *be the unique solution to* $\frac{d^2h}{dx^2} = \mu h$*, (∗), on* $\overline{\mathcal{H}_\eta} \setminus [\pi(\frac{\eta-2}{\eta}), \pi)$*, with boundary condition* $h(-\pi) = e$ *and* $h'(-\pi) = f$*, ([8]). Then, there exist ∗-polynomials in* $x^{\frac{1}{2}}$, $\{P_{j,\eta}^i(x), Q_{j,\eta}^i(x)\} \subset {}^*\mathcal{C}[x]$*, with* $1 \leq i \leq 2$*;*

---

[8]Given $g_0 \in \overline{\mathcal{H}_\eta}$, we can also consider the equation $\frac{d^2h}{dx^2} = \mu h + g_0$, (∗∗) on $\overline{\mathcal{H}_\eta} \setminus [\pi(\frac{\eta-2}{\eta}), \pi)$, with boundary condition $h(-\pi) = e$ and $h'(-\pi) = f$. Then H is defined by $H(\tau, x', y') = (y', \mu x' + g_0(\tau))$, (check convergence, Lemma **??**). The solution to (∗∗), with initial condition $h(-\pi) = e, h'(-\pi) = f$, is then given by, using Definition 0.10 and the method in 0.6;

$$h(-\pi) = e, h'(-\pi) = f$$

$$h(-\pi + \pi(\tfrac{(j+1)}{\eta})) = h(-\pi + \pi(\tfrac{j}{\eta})) + \tfrac{\pi}{\eta}h'(-\pi + \pi(\tfrac{j}{\eta})),\ 0 \leq j \leq \eta - 2,\ (i)$$

with $h(-\pi+\pi(\frac{j}{\eta}))=P^1_{j,\eta}(\mu)e+Q^1_{j,\eta}(\mu)f$

$h'(-\pi+\pi(\frac{j}{\eta}))=P^2_{j,\eta}(\mu)e+Q^2_{j,\eta}(\mu)f$

Moreover, let $\{\lambda,c,d\}\subset\mathcal{C}$ *and let* $m\in\mathcal{Z}$*, then, if* $g\in C^\infty[-\pi,\pi]$*, and* $g''(x)=(\lambda+m^2)g$*, with* $g(-\pi)=c$*,* $g'(-\pi)=d$*, then there exists* $\lambda'\in{}^*\mathcal{C}\cap\mu(\lambda)$ *and* $h\in\overline{\mathcal{H}_\eta}$*, satisfying the ODE;*

$h''(x)=(\lambda'-\psi^2_\eta(m))h(x)$*,* $(*)$*,* $x\in(\mathcal{H}_\eta\setminus[\pi-\frac{\pi}{\eta},\pi))$

such that ${}^\circ h=g$*, with* $h(-\pi)=h(\pi-\frac{\pi}{\eta})=c$ *and* $h'(-\pi)=h'(\pi-\frac{\pi}{\eta})=d$

with initial condition $h(-\pi)={}^\circ a$*,* $h'(-\pi)={}^\circ b$*.*

Proof. Suppose $\{\mu,e,f\}\subset{}^*\mathcal{C}$, and let $H:\overline{\mathcal{H}_\eta}\times{}^*\mathcal{C}\times{}^*\mathcal{C}\rightarrow{}^*\mathcal{C}\times{}^*\mathcal{C}$ be defined by;

$h'(-\pi+\pi(\frac{(j+1)}{\eta}))=h'(-\pi+\pi(\frac{j}{\eta}))+\frac{\pi}{\eta}(\mu h(-\pi+\pi(\frac{j}{\eta}))+g_0(-\pi+\pi(\frac{j}{\eta})))$, $0\leq j\leq\eta-2$, (ii)

$\overline{w}_{j+1}=\overline{A}\overline{w}_j+\overline{c}_j$, $0\leq j\leq\eta-2$, where;

$$\overline{w}_j=\begin{pmatrix}h(-\pi+\pi(\frac{j}{\eta}))\\ h'(-\pi+\pi(\frac{j}{\eta}))\end{pmatrix}$$

$$\overline{c}_j=\begin{pmatrix}0\\ \frac{\pi g_0(-\pi+\pi(\frac{j}{\eta}))}{\eta}\end{pmatrix}$$

and;

$$\overline{A}=\begin{pmatrix}1 & \frac{\pi}{\eta}\\ \frac{\pi\mu}{\eta} & 1\end{pmatrix}$$

$\overline{w}_j=(\overline{A})^j(\overline{w}_0)+{}^*\sum_{k=1}^j(\overline{A})^{j-k}\overline{c}_{k-1}$, $1\leq j\leq\eta-2$

As in the main proof, we obtain that, for $1\leq j\leq\eta-2$;

$\overline{w}_j=\overline{B}^{-1}diag((\frac{\pi\sqrt{\mu}}{\eta}+1)^j,(\frac{\pi\sqrt{\mu}}{\eta}-1)^j)\overline{B}(\overline{w}_0)$

$+{}^*\sum_{k=1}^j\overline{B}^{-1}diag((\frac{\pi\sqrt{\mu}}{\eta}+1)^{j-k},(\frac{\pi\sqrt{\mu}}{\eta}-1)^{j-k})\overline{B}(\overline{c}_{k-1})$

$h(-\pi+\pi(\frac{j}{\eta}))=P^1_{j,\eta}(\mu)e+Q^1_{j,\eta}(\mu)f+{}^*\sum_{k=1}^jQ^1_{j-k,\eta}(\mu)\frac{\pi}{\eta}g_0(-\pi+\pi(\frac{k-1}{\eta}))$

$h'(-\pi+\pi(\frac{j}{\eta}))=P^2_{j,\eta}(\mu)e+Q^2_{j,\eta}(\mu)f+{}^*\sum_{k=1}^jQ^2_{j-k,\eta}(\mu)\frac{\pi}{\eta}g_0(-\pi+\pi(\frac{k-1}{\eta}))$

$H(\tau, x', y') = (y', \mu x')$

The solution to $(*)$, with initial condition $h(-\pi) = e, h'(-\pi) = f$, is then given by, using Definition 0.10 and the method in 0.6;

$h(-\pi) = e, h'(-\pi) = f$

$h(-\pi+\pi(\frac{(j+1)}{\eta})) = h(-\pi+\pi(\frac{j}{\eta}))+\frac{\pi}{\eta}h'(-\pi+\pi(\frac{j}{\eta})), 0 \leq j \leq 2\eta-2, (i)$

$h'(-\pi+\pi(\frac{(j+1)}{\eta})) = h'(-\pi+\pi(\frac{j}{\eta}))+\frac{\pi}{\eta}\mu h(-\pi+\pi(\frac{j}{\eta})), 0 \leq j \leq 2\eta-2,$ (ii)

$\overline{w}_{j+1} = \overline{A}\overline{w}_j$, $0 \leq j \leq 2\eta - 2$

$\overline{w}_j = \overline{A}^j\overline{w}_0$, $0 \leq j \leq 2\eta - 2$

where;

$\overline{w}_j = \begin{pmatrix} h(-\pi + \pi(\frac{j}{\eta})) \\ h'(-\pi + \pi(\frac{j}{\eta})) \end{pmatrix}$

and;

$\overline{A} = \begin{pmatrix} 1 & \frac{\pi}{\eta} \\ \frac{\pi\mu}{\eta} & 1 \end{pmatrix}$

The eigenvalues of $\overline{A}$ are given by $\{\frac{\pi\sqrt{\mu}}{\eta} + 1, \frac{-\pi\sqrt{\mu}}{\eta} + 1\}$, with eigenvectors $\{\overline{v}_1, \overline{v}_2\}$, where;

$\overline{v}_1 = \begin{pmatrix} 1 \\ \sqrt{\mu} \end{pmatrix}$

and;

$\overline{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{\mu} \end{pmatrix}$

so that $\overline{B}^{-1}\overline{AB} = diag(\frac{\pi\sqrt{\mu}}{\eta} + 1, \frac{-\pi\sqrt{\mu}}{\eta} + 1)$, where;

$\overline{B} = \begin{pmatrix} 1 & 1 \\ \sqrt{\mu} & -\sqrt{\mu} \end{pmatrix}$

$$\overline{B}^{-1}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2\sqrt{\mu}}\\ \frac{1}{2} & \frac{-1}{2\sqrt{\mu}}\end{pmatrix}$$

$$\overline{w}_j=\overline{B}diag(\tfrac{\pi\sqrt{\mu}}{\eta}+1,\tfrac{-\pi\sqrt{\mu}}{\eta}+1)^j\overline{B}^{-1}(\overline{w}_0)$$

$$=\overline{B}diag((\tfrac{\pi\sqrt{\mu}}{\eta}+1)^j,(\tfrac{\pi\sqrt{\mu}}{\eta}-1)^j)\overline{B}^{-1}(\overline{w}_0)$$

$$h(-\pi+\pi(\tfrac{j}{\eta}))=\frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j+(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}e+\frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j-(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2\sqrt{\mu}}f$$

$$h'(-\pi+\pi(\tfrac{j}{\eta}))=\frac{\sqrt{\mu}[(1+\frac{\pi\sqrt{\mu}}{\eta})^j-(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}e+\frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j+(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}f$$

so that;

$$P^1_{j,\eta}(x)=\tfrac{1}{2}[(1+\tfrac{\pi\sqrt{x}}{\eta})^j+(1-\tfrac{\pi\sqrt{x}}{\eta})^j]$$

$$Q^1_{j,\eta}(x)=\tfrac{1}{2\sqrt{x}}[(1+\tfrac{\pi\sqrt{x}}{\eta})^j-(1-\tfrac{\pi\sqrt{x}}{\eta})^j]$$

$$P^2_{j,\eta}(x)=\tfrac{\sqrt{x}}{2}[(1+\tfrac{\pi\sqrt{x}}{\eta})^j-(1-\tfrac{\pi\sqrt{x}}{\eta})^j]$$

$$Q^2_{j,\eta}(x)=\tfrac{1}{2}[(1+\tfrac{\pi\sqrt{x}}{\eta})^j+(1-\tfrac{\pi\sqrt{x}}{\eta})^j]$$

...work out G_η, initial terms, formula for h'. Obtain algebraic $P_\eta(e,f,\mu)=h(\pi)$. Use multiplicity argument to get result, for finite n, and transfer to get result for η.

□

Lemma 0.25. *Let $A_\lambda=\{\lambda\in{}^*\mathcal{C},\exists h\in V(\overline{\mathcal{H}_\eta})_{\neq 0},h(-\pi)=h(\pi(1-\frac{1}{\eta})),h'(-\pi)=h'(\pi(1-\frac{1}{\eta}))\}$, then $A_\lambda\neq\emptyset$, and $Card(A_\lambda)\leq 2(2\eta-1)$.*

Proof. By 0.24, we have that $\lambda\in A_\mu$ iff $det(B-I)=0$, where;

$$B=\begin{pmatrix}P^1_{2\eta-1,\eta}(\mu) & Q^1_{2\eta-1,\eta}(\mu)\\ P^2_{2\eta-1,\eta}(\mu) & Q^2_{2\eta-1,\eta}(\mu)\end{pmatrix}$$

Clearly, $degree(det(B-I))\leq degree(P^1_{2\eta-1,\eta})+degree(Q^2_{2\eta-1,\eta})\leq 2(2\eta-1)$ as required.

□

Theorem 0.26. *Let notation be as in 0.3. If $\lambda\in Z$ and $g\in V^{\Delta,\lambda}(T)$, then there exists f satisfying the nonstandard eigenvalue problem from Lemma 0.17, (with eigenvalue λ) such that ${}^\circ f(x',y')=g(x,y)$, for $(x',y')\in\overline{\mathcal{T}_\eta}$, $(x,y)\in[-\pi,\pi]\times[-\pi,\pi]$ with ${}^\circ(x',y')=(x,y)$.*

Proof. Let $g_1(y)=g(-\pi,y)$ and $g_2(y)=\frac{\partial g}{\partial x}(-\pi,y)$, and extend the initial conditions $\{g_1,g_2\}$ to smooth functions $\{h_1,h_2\}$ on $[-2\pi,2\pi]$, with;

$\{h_1,h_2\}|_{[-2\pi,-2\pi+\delta)\cup[2\pi-\delta,2\pi]}=0$

and $\{h_1,h_2\}|_{[-\delta-\pi,-\pi)\cup[\pi,\pi+\delta]}=\{g_1,g_2\}|_{[\pi-\delta,\pi)\cup[-\delta-\pi,-\pi)}$, $(*)$

for some $\delta\in\mathcal{R}_{>0}$. We let $\overline{\mathcal{V}'_\eta}={}^*\bigcup_{0\leq i\leq 2\eta-1}[-2\pi+2\pi\frac{i}{\eta},-2\pi+2\pi\frac{i+1}{\eta})$, so that $\overline{\mathcal{V}'_\eta}={}^*[-2\pi,2\pi)$. Choose η' infinite with $\eta'<\frac{1}{2}[\frac{\delta\eta}{2\pi}]-1$. Using Definition 0.8, we replace $\overline{\mathcal{H}_\eta}$ with $\overline{\mathcal{H}_{\eta'}}$, and let $\{\overline{\mathcal{C}}_{\eta'},\overline{\mathcal{D}}'_\eta,\}$ denote the corresponding $*$-finite algebras with counting measures $\{\lambda_{\eta'},\mu'_\eta\}$, defined by $\lambda_{\eta'}([-\pi+\pi\frac{i}{\eta'},-\pi+\pi\frac{i+1}{\eta'}))=\frac{\pi}{\eta'}$ and $\mu'_\eta([-2\pi+2\pi\frac{i}{\eta},-2\pi+2\pi\frac{i+1}{\eta}))=\frac{2\pi}{\eta}$, and corresponding Loeb spaces, ([9]). We let $f\in V(\overline{\mathcal{T}_{\eta',\eta}})$ be given by Lemma 0.17, for the corresponding eigenvalue problem on $\mathcal{T}_{\eta',\eta}$, with initial conditions $\{h_{1,\eta},h_{2,\eta}\}\subset V(\overline{\mathcal{V}'_\eta})$, being the measurable counterparts of $\{h_1,h_2\}$. By the choice of η', we have that $f(x,2\pi-\frac{\pi}{\eta})=0$ and $\frac{\partial f}{\partial y}(x,2\pi-\frac{\pi}{\eta})=0$, for $x\in\overline{\mathcal{H}_{\eta'}}$, hence $S_{f_x,\eta,1}(m)=0$, for $m\in\overline{\mathcal{Z}_\eta}$, $x\in\overline{\mathcal{H}_{\eta'}}$ $(*)$, see notation in 9. It follows, using $(*)$

[9] It is a simple exercise to verify Lemma 0.9, replacing $[-\pi,\pi]$ by $[-2\pi,2\pi]$ where necessary. We extend Definition 0.10 in the obvious way to these spaces, and similarly Lemma 0.12 goes through. Similarly to Definition 0.18, if $f\in\overline{\mathcal{V}'_\eta}$, and $m\in\overline{\mathcal{Z}_\eta}$, we let $\hat{f}(m)=\frac{1}{4\pi}\int_{\overline{\mathcal{V}'_\eta}}f(y)exp_\eta(\frac{-imy}{2})d\lambda'_\eta$, so that $f(y)=\sum_{m\in\mathcal{Z}_\eta}\hat{f}(m)exp_\eta(\frac{imy}{2})$. We define;

$\phi_{\eta,1}=\frac{\phi_\eta}{2}$, $\psi_{\eta,1}=\frac{\psi_\eta}{2}$

$R_{f,\eta,1}(m)=\frac{1}{4\pi\eta}f(2\pi(1-\frac{1}{\eta}))(exp_\eta(-im\pi(1-\frac{2}{\eta}))-exp_\eta(\frac{im\pi}{\eta})exp_\eta(-im\pi(1-\frac{1}{\eta}))$

$R'_{f,\eta,1}(m)=\frac{1}{4\pi\eta}f'(2\pi(1-\frac{1}{\eta}))(exp_\eta(-im\pi(1-\frac{2}{\eta}))-exp_\eta(\frac{im\pi}{\eta})exp_\eta(-im\pi(1-\frac{1}{\eta}))$.

$T_{f,\eta,1}(m)=\phi_{\eta,1}(m)R_{f,\eta,1}(m)$

$S_{f,\eta,1}(m)=\phi_{\eta,1}(m)\psi_{\eta,1}(m)R_{f,\eta,1}(m))+\phi_\eta(m)R'_{f,\eta,1}(m)$

As in Lemma 0.19, we obtain, for $m\in\overline{\mathcal{Z}_\eta}$, that $\hat{f''}(m)+S_{f,\eta,1}(m)=\psi^2_{\eta,1}(m)\hat{f}(m)$. We let $\overline{\mathcal{T}_{\eta',\eta}}=\overline{\mathcal{H}_{\eta'}}\times\overline{\mathcal{V}'_\eta}$, with the obvious modifications $\{W_{n,g,r}(\overline{\mathcal{H}_{\eta'}}),W_{n,g,r}(\overline{\mathcal{V}'_\eta}),V_{n,g,r}(\overline{\mathcal{T}_{\eta',\eta}})\}$, and obtain corresponding versions of Lemmas 0.20,0.21, 0.22 and 0.23. We adapt the notation of Definition 0.1, for $f\in C^\infty([-2\pi,2\pi])$ and $m\in\mathcal{Z}$, we define the standard Fourier coefficient $\mathcal{F}(f)(m)=\frac{1}{4\pi}\int_{-2\pi}^{2\pi}f(y)e^{\frac{-\pi iym}{2}}dy$, to obtain the analogous result of 0.2, $f(x,y)=\sum_{m\in\mathcal{Z}}\mathcal{F}(f)(x,m)e^{imy}$.

and the corresponding result of Lemma 0.22, see footnote 9, that, for $x' \in \overline{\mathcal{H}_{\eta'}}$, $m \in \overline{\mathcal{Z}_\eta}$;

$$\frac{d^2(\hat{f})^m}{dx^2}(x') + (\psi^2_{\eta,1}(m) - \lambda)(\hat{f})^m(x') - S_{f_{x'},\eta,1}(m)$$

$$= \frac{d^2(\hat{f})^m}{dx^2}(x') + (\psi^2_{\eta,1}(m) - \lambda)(\hat{f})^m(x') = 0 \ (**)$$

We let $\overline{T} = [-\pi, \pi] \times [-2\pi, 2\pi]$, and extend g to $g_{ext} \in C(\overline{T})$, solving problem on $\overline{T}$, with $g_x \in C^\infty([-2\pi, 2\pi])$, for $x \in [-\pi, \pi]$, ($\sharp$), (check this), and $g_{ext,-\pi} = g_1$, $(\frac{\partial g_{ext}}{\partial y})_{-\pi} = g_2$, so that $g_{ext}|_T = g$. Then, for $x \in (-\pi, \pi)$ and $m \in \mathcal{Z}$;

$$(\mathcal{F}(g_{ext}))^m_{xx} - m^2(\mathcal{F}(g_{ext}))^m = \lambda(\mathcal{F}(g_{ext}))^m \ (***)$$

with initial conditions;

$$(\mathcal{F}(g_{ext}))^m(-\pi) = \mathcal{F}(h_1)(m)$$

$$(\mathcal{F}(g_{ext}))^m_x(-\pi) = (\mathcal{F}(\frac{dg_{ext}}{dx}))^m(-\pi) = (\mathcal{F}(h_2))^m \ (****)$$

Using $(**)$ and the corresponding result of Lemma 0.23, we have that, for $m \in \mathcal{Z}$, $f_{1,m} : [-\pi, \pi] \rightarrow \mathcal{C}$, defined by $f_{1,m}(x) = {}^\circ\hat{f}^m(x')$, ${}^\circ x' = x$, satisfies the differential equation $(f_{1,m})_{xx} - m^2 f_{1,m} = \lambda f_{1,m}$, with initial conditions;

$$f_{1,m}(-\pi) = {}^\circ(\hat{f}^m)(-\pi) = {}^\circ(\hat{h_{1,\eta}}(m)) = \mathcal{F}(h_1)(m)$$

$$(f_{1,m})_x(-\pi) = {}^\circ(\frac{d(\hat{f})^m}{dx}(-\pi)) = {}^\circ(\hat{\frac{df}{dx}})^m(-\pi) = {}^\circ(\hat{h_{2,\eta}}(m)) = \mathcal{F}(h_2)(m)$$

$$(*****)$$

where we have used the fact that $\{h_{1,\eta}, h_{2,\eta}\}$ are S-integrable. Comparing $(****)$ and $(*****)$, and, using Peano's Theorem, we obtain that $f_{1,m} = (\mathcal{F}(g))_m$, for $m \in \mathcal{Z}$. By $\sharp$, we have that, for $x \in [-\pi, \pi]$ and $m \in \mathcal{Z}_{\neq 0}$, $|(\mathcal{F}(g_x))(m)| \leq \frac{C_x}{m^2} \leq \frac{E}{m^2}$, where $E = ||\frac{\partial^2 g}{\partial y^2}||_{C(\overline{T})}$.

Then, using Lemma 0.19 and footnote 9, we obtain that, for $x' \in \mathcal{H}_\eta$, $m \in \mathcal{Z}_{\neq 0}$;

$$(\hat{f_{x'}})''(m) = \psi^2_{\eta,1}(m)\hat{f_{x'}}(m)$$

$^{\circ}(\hat{f_{x'}})''(m) = -m^{2\circ}\hat{f_{x'}}(m) = -m^2 f_{1,m}(^{\circ}x') = -m^2(\mathcal{F}(g))_m(^{\circ}x')$

$|^{\circ}(\hat{f_{x'}})''(m)| \leq m^2\frac{C_{^{\circ}x'}}{m^2} = C_{^{\circ}x'}$

$|(\hat{f_{x'}})''(m)| \leq C_{^{\circ}x'} + 1$

and, for $m \in \mathcal{Z}_{\neq 0}$;

$|\hat{f_{x'}}(m)| = \frac{|(\hat{f_{x'}})''(m)|}{|\psi^2_{\eta,1}(m)|} \leq \frac{C_{^{\circ}x'}+1}{(m-\epsilon)^2} \leq \frac{D}{m^2}$ (♯♯)

where $\epsilon \in \mu(0)$, $D \in \mathcal{R}_{>0}$ and $D \geq 2(E+1) \geq 2(C_{^{\circ}x'}+1)$.

For given $m \in \mathcal{Z}_{\neq 0}$, we let $\phi(m,\eta) \in {}^*\mathcal{N}_{<\frac{1}{2}[\frac{\delta\eta}{2\pi}]-1}$ be minimal with the property that;

$|_{\eta}\hat{f}(x,m)| \leq \frac{E}{m^2}$, for $x \in \overline{H}_{\eta''}$, $\phi(m,\eta) \leq \eta'' < \frac{1}{2}[\frac{\delta\eta}{2\pi}] - 1$

By the above $\phi(m,\eta)$ exists, for all infinite $\eta \in {}^*\mathcal{N}$, hence, by underflow, there exists a finite $n(m)$, (need independent of m, work through calculation (♯♯), using explicit expression for $\psi_{\eta,1}$ and $\{P^i_{j,\eta}(\mu), Q^i_{j,\eta}(\mu)\}$, for given j, when η, μ are finite. Use fact that these are polynomials in $\frac{\mu}{\eta^2}$? (check this), hence can remove the dependence in μ with $\frac{1}{\eta}$ to get a uniform bound.), such that, for all $k \geq n(m)$;

$|_k\hat{f}(x,m)| \leq \frac{E}{m^2}$, for $x \in \overline{H}_r$, $\phi(m,k) \leq r < \frac{1}{2}[\frac{\delta k}{2\pi}] - 1$.

Then;

$\mathcal{R} \models (\forall m \in \mathcal{Z}_{\neq 0})(\exists n(m))(\forall k \geq n(m))(|_k\hat{f}(x,m)| \leq \frac{E}{m^2}; x \in \overline{H}_r, \phi(m,k) \leq r < \frac{1}{2}[\frac{\delta k}{2\pi}] - 1)$

By transfer, if $m \in \mathcal{Z}_{\eta,\neq 0}$,
.... Let $g_1(y) = g(-\pi, y)$ and $g_2(y) = \frac{\partial g}{\partial x}(-\pi, y)$. Let $f \in V(\overline{\mathcal{T}_\eta})$ be given by Lemma 0.17, with eigenvalue λ and initial conditions $\{g_1, g_2\} \subset V(\overline{\mathcal{V}_\eta})$. We have that $\hat{f}$ satisfies the equations in $(*)$ in Lemma 0.22, and $\mathcal{F}(g)$ satisfies the requirement in lemma 0.3. In particular, for $m \in \mathcal{Z}$, we have that;

$\frac{d^2(\hat{f})^m}{dx^2}(x') + (\psi^2_\eta(m) - \lambda)(\hat{f})^m(x') - S_\eta(m) = 0$

$(\mathcal{F}(g))^m_{xx} - m^2(\mathcal{F}(g))^m = \lambda(\mathcal{F}(g))^m$, on $(-\pi,\pi)$

Sketch proof...
Extend i.c's $\{g_1, g_2\}$ to smooth functions on $[-2\pi, 2\pi]$, with;

$\{h_1, h_2\}|_{[-2\pi,-2\pi+\delta)\cup[2\pi-\delta,2\pi]} = 0$

and $\{h_1, h_2\}|_{[-\delta-\pi,-\pi)\cup[\pi,\pi+\delta]} = \{g_1, g_2\}|_{[\pi-\delta,\pi)\cup[-\delta-\pi,-\pi)}$, $(*)$

for some $\delta > 0$. Alter x-step, to get $f(x, 2\pi(1-\frac{1}{\eta})), f'(x, 2\pi(1-\frac{1}{\eta})) = 0$, for $x \in \overline{\mathcal{H}_\eta}$, so error term $S_\eta = 0$, (as in [4]). Hence, using Lemma 0.19, adapted to include error term, ${}^\circ(\hat{f}(m)) = \mathcal{F}(g)(m)$ for $m \in \mathcal{Z}$. Moreover, by $(*)$, we have that $f|[-\pi,\pi)$ solves original problem. Have $(\frac{d^2}{dx^2})\hat{(}f_n)(m) = \hat{(}\frac{d^2}{dx^2})(f_n)(m) \leq C$ (C finite), as $max(\hat{h_1}(m), \hat{h_2}(m)) \leq D$, for $m \in \mathcal{Z}$, and $sine/cosine$ solution to $z'' + (\lambda - m^2)z = 0$. For $m \in Z_\eta \setminus Z$, consider $\{n : M \models (\forall m \in (N)) : (\frac{d^2}{dx^2})\hat{(}f_n)(m) = \hat{(}\frac{d^2}{dx^2})(f_n)(m) \leq C$ (C finite), holds for all infinite n,so for $n \geq \kappa$, transfer to get result for $m \in Z_\eta$. Gives decay rate for $m \in Z_\eta \setminus \mathcal{Z}$, $\leq \frac{1}{m^2}$, by Lemma 0.15. Then ${}^\circ f = g$, and, therefore ${}^\circ f|_{[-\pi,\pi]} = g_{[-\pi,\pi]}$.

□

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