ELECTRON BUNCHING

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Abstract. We consider a theory of electron dissipation in terms of martingales and derive the time evolving probability distribution of the process.

Definition 0.1. *We let* $\rho,\nu\subset {}^*(\mathcal{N})\setminus(\mathcal{N})$, $\gamma=2^\rho$, $\eta=2^\nu$, *so that* $\{\gamma,\eta\}\subset {}^*(2\mathcal{N})\setminus(2\mathcal{N})$, *and* $\rho=log_2(\gamma)$, $\nu=log_2(\eta)$. *We adopt the notation of* [2], *Definition 0.1, with* $(\overline{\Omega_\eta},\mathcal{C}_\eta,\mu_\eta)$ *and* $(\overline{T}_\nu,\mathcal{D}_\nu,\lambda_\nu)$ *denoting* $*$*-finite measure spaces, and* $(\overline{\Omega_\eta},L(\mathcal{C}_\eta),L(\mu_\eta))$, $(\overline{T}_\nu,L(\mathcal{D}_\nu),L(\lambda_\nu))$ *the associated Loeb spaces. We adapt Definitions 0.2 and 0.4, of* [2], *by defining* $\chi:\overline{\Omega_\eta}\times\overline{T}_\nu\rightarrow{}^*\mathcal{R}$;

$\chi(x,t)=\frac{1}{\gamma}({}^*\sum_{j=1}^{[\nu t]}\omega_j(x))$

where, in this case, $\omega_j:C_\nu\rightarrow\{0,-1\}$ *is the projection onto the j'th coordinate of sequences in* C_ν, *consisting of* 0*'s and* -1*'s,* ([1])

[1]More generally, if $d\in{}^*\mathcal{N}$ is prime, we let $\eta_d=d^\nu$ and let;

${}^*\mathcal{Z}_d=\{x\in{}^*\mathcal{Z}:0\leq x\leq d-1\}$

${}^*\mathcal{Z}_d^-=\{x\in{}^*\mathcal{Z}:1-d\leq x\leq 0\}$

$C_{d,\nu}=\{\overline{z}\in{}^*\mathcal{Z}^\nu:\overline{z}(j)\in{}^*\mathcal{Z}_d,1\leq j\leq\nu\}$

We let $(\overline{\Omega_{\eta_d}},\mathcal{C}_{\eta_d},\mu_{\eta_d})$ be defined as in Definition 0.1 of [2], with η_d replacing η, and corresponding Loeb space $(L(\overline{\Omega_{\eta_d}}),L(\mathcal{C}_{\eta_d}),L(\mu_{\eta_d}))$. For $0\leq i\leq d-1$ and $1\leq j\leq\nu-1$, we let $\omega_j:C_{d,\nu}\rightarrow{}^*\mathcal{Z}_d^-$ be defined by $\omega_j(\overline{z})=-\overline{z}(j)$, and let $\omega_j:\overline{\Omega_{\eta_d}}\rightarrow{}^*\mathcal{Z}_d^-$ also denote the composition $\omega_j\circ\psi_{\eta_d}$, where $\psi_{\eta_d}:\Omega_{\eta_d}\rightarrow C_{d,\nu}$ associates $x\in\overline{\Omega_{\eta_d}}$ with the d-adic representation of $[\eta_d x]$. For $1\leq j\leq\nu-1$, we let $\mu_j:\overline{\Omega_{\eta_d}}\rightarrow{}^*\mathcal{Z}_1^-$ be defined by $\mu_j(x)=0$ if $0\leq|\omega_j(x)|\leq c-1$, and, by $\mu_j(x)=-1$, if $c\leq|\omega_j(x)|\leq d-1$, where $\alpha=\frac{c}{d}$, and let $\beta=1-\alpha=\frac{d-c}{d}$. We set $\chi_\alpha(x,t)=\frac{1}{\gamma}({}^*\sum_{j=1}^{[\nu t]}\mu_j(x))$. It is easily seen that $\{\mu_j:1\leq j\leq\nu-1\}$ forms a $*$-independent, identically distributed, sequence with $E(\mu_j)=\alpha-1$ and $Var(\mu_j)=\alpha(1-\alpha)$.

We let $(\overline{P}_{2^\nu\gamma}, \mathcal{E}_{2^\nu\gamma}, \theta_{\alpha,\beta,2^\nu\gamma})$ *denote a new* $*$*-finite measure space, where, as above;*

$\overline{P}_{2^\nu\gamma} = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$, $\mathcal{E}_{2^\nu\sqrt{\nu}}$ *consists of internal unions of the intervals* $[\frac{i}{2^\nu\gamma}, \frac{i+1}{2^\nu\gamma})$, *for* $0 \leq i \leq 2^\nu\gamma - 1$, *and, for* $0 \leq \alpha \leq \beta \leq 1$, *with* $\alpha + \beta = 1$, $\{\alpha, \beta\} \subset {}^*\mathcal{R}$*;*

$\theta_{\alpha,\beta,2^\nu\gamma}([\frac{i}{2^\nu\gamma}, \frac{i+1}{2^\nu\gamma})) = \frac{\alpha}{2^{\nu-1}\gamma}$*:* $i \in {}^*(2\mathcal{N})$, $0 \leq i \leq 2^\nu\gamma - 1$

$\theta_{\alpha,\beta,2^\nu\gamma}([\frac{i}{2^\nu\gamma}, \frac{i+1}{2^\nu\gamma})) = \frac{\beta}{2^{\nu-1}\gamma}$*:* $i \in {}^*(2\mathcal{N}+1)$, $0 \leq i \leq 2^\nu\gamma - 1$

We let $(\overline{P}_{2^\nu\gamma}, L(\mathcal{E}_{2^\nu\gamma}), L(\theta_{\alpha,\beta,2^\nu\gamma}))$ *denote the associated Loeb space.*

Lemma 0.2. *The standard part mapping* $st : (\overline{P}_{2^\nu\gamma}, L(\mathcal{E}_{2^\nu\gamma}), L(\theta_{\alpha,\beta,2^\nu\gamma})) \rightarrow ([0,1], \mathfrak{B}, \mu)$ *is measurable and measure preserving, where* μ *denotes Lebesgue measure and* $\mathfrak{B}$ *is the completion of the Borel field on* $[0,1]$.

Proof. Letting $\delta = \eta\gamma$, if $\{m, n\} \subset \mathcal{N}$,$\{a, b\} \subset [0,1)$ and $[\frac{[\delta(a-\frac{1}{n})]}{\delta}, \frac{[\delta(b-\frac{1}{m})]}{\delta}) \subset {}^*[0,1]$, then;

$\theta_{\alpha,\beta,2^\nu\gamma}([\frac{[\delta(a-\frac{1}{n})]}{\delta}, \frac{[\delta(b-\frac{1}{m})]}{\delta}))$

$\simeq \frac{2\alpha}{\delta}(\frac{[\delta(b-\frac{1}{m})]-[\delta(a-\frac{1}{n})]}{2}) + \frac{2\beta}{\delta}(\frac{[\delta(b-\frac{1}{m})]-[\delta(a-\frac{1}{n})]}{2})$

$= \frac{[\delta(b-\frac{1}{m})]-[\delta(a-\frac{1}{n})]}{\delta}$

$\simeq b - a + \frac{1}{n} - \frac{1}{m}$

Hence $L(\theta_{\alpha,\beta,2^\nu\gamma})([\frac{[\eta\gamma(a-\frac{1}{n})]}{\eta\gamma}, \frac{[\eta\gamma(b-\frac{1}{m})]}{\eta\gamma})) = b - a + \frac{1}{n} - \frac{1}{m}$. The result then follows from the proof of Theorem 6.7 in [3], see also [1].

□

Definition 0.3. *We let* $(\overline{T}_{\nu+\rho}, \mathcal{D}_{\nu+\rho}, \lambda_{\nu+\rho})$ *denote another* $*$*-finite measure space, where;*

$\overline{T}_{\nu+\rho} = \{t \in {}^*\mathcal{R} : 0 \leq t < 1 + \frac{log_2(\gamma)}{\nu}\}$

$\mathcal{D}_{\nu+\rho}$ *consists of internal unions of the intervals* $[\frac{k}{\nu}, \frac{k+1}{\nu})$, $0 \leq k \leq \nu + \rho - 1$

$\lambda_{\nu+\rho}([\frac{k}{\nu},\frac{k+1}{\nu}))=\frac{1}{\nu}$, $0\leq k\leq\nu+\rho-1$

Again, we let $(\overline{T}_{\nu+\rho},L(\mathcal{D}_{\nu+\rho}),L(\lambda_{\nu+\rho}))$ *denote the associated Loeb space.*

We let $x:{}^{*}\mathcal{N}_{1\leq j\leq\eta}\times\overline{T}_{\nu}\rightarrow\overline{P}_{2^{\nu}\gamma}$ *be defined by;*

$x(s,\frac{k}{\nu})=\chi(\frac{s-1}{\eta},\frac{k}{\nu})-[\chi(\frac{s-1}{\eta},\frac{k}{\nu})]$

For given $(x,t)\in(\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu})$*, we let;*

$J_{(x,t)}=\{s\in{}^{*}\mathcal{N}_{1\leq j\leq\eta}:x(s,\frac{[t\nu]}{\nu})=\frac{[x\gamma]}{\gamma}\}$

and define $N:\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu}\rightarrow{}^{*}\mathcal{N}_{1\leq j\leq\eta}$ *by;*

$N(x,t)=({}^{*}Card)(J_{(x,t)})$

$Q:\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu}\rightarrow{}^{*}\mathcal{N}_{1\leq j\leq\eta}$ *by;*

$Q(x,t)=\frac{N(x,t)}{\eta}$

It is easily checked that N *and* Q *are measurable with respect to* $\mathcal{E}_{2^{\nu}\gamma}\times\mathcal{D}_{\nu}$*, ([2]).*

We define a filtration $\{\mathcal{E}^{k}_{2^{\nu}\gamma}:0\leq k\leq\nu+\rho-1\}$ *on* $\overline{P}_{2^{\nu}\gamma}$ *by letting* $\mathcal{E}^{k}_{2^{\nu}\gamma}$ *be the* $*$*-finite algebra, generated by the intervals;*

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[2] In general, we let $x_{\alpha}:{}^{*}\mathcal{N}_{1\leq j\leq\eta}\times\overline{T}_{\nu}\rightarrow\overline{P}_{2^{\nu}\gamma}$ be defined by;

$x_{\alpha}(s,\frac{k}{\nu})=\chi_{\alpha}(\frac{s-1}{\eta},\frac{k}{\nu})-[\chi_{\alpha}(\frac{s-1}{\eta},\frac{k}{\nu})]$

For given $(x,t)\in(\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu})$, we let;

$J_{\alpha,x,t}=\{s\in{}^{*}\mathcal{N}_{1\leq j\leq\eta}:x_{\alpha}(s,\frac{[t\nu]}{\nu})=\frac{[x\gamma]}{\gamma}\}$

and define $N_{\alpha}:\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu}\rightarrow{}^{*}\mathcal{N}_{1\leq j\leq\eta}$ by;

$N_{\alpha}(x,t)=({}^{*}Card)(J_{(x,t)})$

$Q_{\alpha}:\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu}\rightarrow{}^{*}\mathcal{N}_{1\leq j\leq\eta}$ by;

$Q_{\alpha}(x,t)=\frac{N_{\alpha}(x,t)}{\eta_{d}}$

It is easily checked that N_{α} and Q_{α} are measurable with respect to $\mathcal{E}_{2^{\nu}\gamma}\times\mathcal{D}_{\nu}$

$\{[\frac{i}{2^{\nu-(k+1)}\gamma}, \frac{i+1}{2^{\nu-(k+1)}\gamma}) : 0 \leq i \leq 2^{\nu-(k+1)}\gamma - 1\}$.

Lemma 0.4. *Let N be given as in Definition 0.3. Then, for $0 \leq i \leq \gamma - 2$, $0 \leq k \leq \nu - 2$;*

$$N(\tfrac{i}{\gamma}, \tfrac{k+1}{v}) = \tfrac{1}{2}(N(\tfrac{i+1}{\gamma}, \tfrac{k}{\nu}) + N(\tfrac{i}{\gamma}, \tfrac{k}{\nu}))$$

$$N(\tfrac{\gamma-1}{\gamma}, \tfrac{k+1}{v}) = \tfrac{1}{2}(N(0, \tfrac{k}{\nu}) + N(\tfrac{\gamma-1}{\gamma}, \tfrac{k}{\nu}))$$

and similarly for Q.

Proof. The proof is an easy consequence of Definition 0.3, ([3]). □

Lemma 0.5. *Let N be given as in Definition 0.3. Then, there exists $\overline{N} : \overline{P}_{2^\nu\gamma} \times \overline{T}_{\nu+\rho} \rightarrow {}^*\mathcal{N}$, such that $\frac{\gamma}{2^\nu}\overline{N}$ is a nonstandard (reverse time) martingale, in the sense of Definition 0.7 of [2], with respect to the filtration, $\{\mathcal{E}^k_{2^\nu\gamma} : 0 \leq k \leq \nu + \rho - 1\}$, and;*

$$\overline{N}(\tfrac{i}{\gamma}, \tfrac{k}{\nu}) = N(\tfrac{i}{\gamma}, \tfrac{k}{\nu}),\ 0 \leq i \leq \gamma - 1, 0 \leq k \leq \nu - 1$$

Proof. We define $\overline{N}$ inductively as follows. Let;

$$\overline{N}(x, \tfrac{\nu-1}{\nu}) = N(x, \tfrac{\nu-1}{\nu})$$

Suppose $\overline{N}$ has been defined on $\overline{P}_{2^\nu\gamma} \times (\overline{T}_{\nu+\rho} \cap [\frac{k+1}{\nu}, 1))$, for some $1 \leq k \leq \nu - 2$, then let;

$$\overline{N}(\tfrac{i}{\gamma}, \tfrac{k}{\nu}) = N(\tfrac{i}{\gamma}, \tfrac{k}{\nu}), \text{ for } 0 \leq i \leq \gamma - 1$$

$$\overline{N}(\tfrac{i(2^{\nu-(k+1)})+1}{2^{\nu-(k+1)}\gamma}, \tfrac{k}{\nu}) = N(\tfrac{i+1}{\gamma}, \tfrac{k}{\nu}), \text{ for } 0 \leq i \leq \gamma - 1$$

$$\overline{N}(\tfrac{2s}{2^{\nu-(k+1)}\gamma}, \tfrac{k}{\nu}) = \overline{N}(\tfrac{s}{2^{\nu-(k+2)}\gamma}, \tfrac{k+1}{\nu}), \overline{N}(\tfrac{2s+1}{2^{\nu-(k+1)}\gamma}, \tfrac{k}{\nu}) = \overline{N}(\tfrac{s}{2^{\nu-(k+2)}\gamma}, \tfrac{k+1}{\nu})$$

for $0 \leq s \leq \frac{\gamma}{2} - 1$, with $\{2s, 2s+1\} \cap \{i(2^{\nu-(k+1)}), i(2^{\nu-(k+1)}) + 1\} = \emptyset$, for $0 \leq i \leq \sqrt{\nu} - 1$

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[3] We have that, for $0 \leq i \leq \gamma - 2$, $0 \leq k \leq \nu - 2$;

$$N_\alpha(\tfrac{i}{\gamma}, \tfrac{k+1}{v}) = \alpha N_\alpha(\tfrac{i+1}{\gamma}, \tfrac{k}{\nu}) + (1-\alpha)N_\alpha(\tfrac{i}{\gamma}, \tfrac{k}{\nu})$$

$$N_\alpha(\tfrac{\gamma-1}{\gamma}, \tfrac{k+1}{v}) = \alpha N(0, \tfrac{k}{\nu}) + (1-\alpha)N_\alpha(\tfrac{\gamma-1}{\gamma}, \tfrac{k}{\nu})$$

and similarly for Q_α

Then define $\overline{N}$ inductively on $\overline{P}_{2^\nu\gamma}\times(\overline{T}_{\nu+\rho}\cap[1,\frac{\nu+\rho}{\nu}))$ by;

$\overline{N}(\frac{i(2^{s+1})}{\gamma},\frac{\nu+s}{\nu})=\frac{1}{2}(\overline{N}(\frac{(2i)(2^s)}{\gamma},\frac{\nu+s-1}{\nu})+\overline{N}(\frac{(2i+1)(2^s)}{\gamma},\frac{\nu+s-1}{\nu}))$,
for $0\leq s\leq\rho-1, 0\leq i\leq\frac{\gamma}{2^{s+1}}-1$.

and letting $\overline{N}(x,t)=\overline{N}(\frac{[2^{\nu-([\nu t]+1)}\gamma x]}{2^{\nu-([\nu t]+1)}\gamma},\frac{[\nu t]}{\nu})$, for $(x,t)\in\overline{P}_{2^\nu\gamma}\times\overline{T}_{\nu+\rho}$.

By construction, for $t\in\overline{T}_{\nu+\rho}$, we have that $\overline{N}_{\frac{[\nu t]}{\nu}}$ is measurable with respect to the algebras $\mathcal{E}^{[\nu t]}_{2^\nu\gamma}$, giving condition (i) of Definition 0.7. Using Lemma 0.4 and the construction, we have that $E_{2^\nu\gamma}(\overline{N}_{\frac{[\nu s]}{\nu}}|\mathcal{E}^{\frac{[\nu t]}{\nu}}_{2^\nu\gamma})=\overline{N}_{\frac{[\nu t]}{\nu}}$, for $0\leq s\leq t<1+\frac{log_2(\gamma)}{\nu}$, giving condition (ii) of Definition 0.7 in [2]. A simple calculation gives that;

$E_{2^\nu\gamma}(\overline{N}_{\frac{\nu-1}{\nu}})$

$=\frac{1}{2^\nu\gamma}{}^*\sum_{0\leq i\leq\gamma-1}2^\nu N(\frac{i}{\gamma},\frac{\nu-1}{\nu})$

$=\frac{1}{\gamma}{}^*\sum_{0\leq i\leq\gamma-1}N(\frac{i}{\gamma},\frac{\nu-1}{\nu})=\frac{2^\nu}{\gamma}$

By condition (ii) of Definition 0.7 in [2], we have that $E_{2^\nu\gamma}(\overline{N}_{\frac{[\nu s]}{\nu}})=\frac{\eta}{\gamma}=\frac{2^\nu}{\gamma}$, and $E_{2^\nu\gamma}(\frac{\gamma}{2^\nu}\overline{N}_{\frac{[\nu s]}{\nu}})=1$, for $0\leq s<1+\frac{log_2(\gamma)}{\nu}$, so condition (iii) of Definition 0.7 holds as required, ([4]).

□

Lemma 0.6. *Let $\overline{H}:\overline{P}_{2^\nu\gamma}\times\overline{T}_{\nu+\rho}\rightarrow{}^*\mathcal{R}$ be as in Definition 0.11 of [2], with respect to $\frac{\gamma}{2^\nu}\overline{N}$, then, for $k\geq\rho$;*

$\overline{H}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})=\sqrt{\nu+\rho-1}\frac{\gamma}{2^{\nu+1}}(\overline{N}(\frac{2^\nu i+1}{2^\nu\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{2^\nu i}{2^\nu\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

Proof. We have that;

[4]Defining $\overline{N}_\alpha$ in terms of N_α, with the same definition, we obtain that $\overline{N}_\alpha$ is a nonstandard (reverse time) martingale, in the sense of Definition 0.7 of [2], with respect to the measure $\theta_{\alpha,\beta,2^\nu\gamma}$. We have that;

$E_{\alpha,\beta,2^\nu\gamma}(\overline{N}_\alpha)=\frac{\alpha}{2^{\nu-1}\gamma}{}^*\sum_{0\leq i\leq\gamma-1}2^\nu N(\frac{i}{\gamma},\frac{\nu-1}{\nu})=\frac{2\alpha d^\nu}{\gamma}$

$E_{\alpha,\beta,2^\nu\gamma}(\frac{\gamma}{2\alpha d^\nu}\overline{N}_\alpha)=1$

$\overline{H}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})=\sqrt{\nu+\rho-1}\frac{\gamma}{2^{\nu}}(\overline{N}(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu}))$

and;

$Q(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})-Q(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})$

$=\frac{1}{2}(Q(\frac{i+1}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})+Q(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu}))-Q(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})$

$=\frac{1}{2}(Q(\frac{i+1}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})-Q(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

$\overline{N}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})-\overline{N}(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})$

$=\frac{1}{2}(\overline{N}(\frac{i+1}{\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{i}{\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

$=\frac{1}{2}(\overline{N}(\frac{2^{\nu}i+1}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{2^{\nu}i}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

$\overline{H}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})$

$=\sqrt{\nu+\rho-1}\frac{\gamma}{2^{\nu+1}}(\overline{N}(\frac{2^{\nu}i+1}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{2^{\nu}i}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$, ([5])

□

Remarks 0.7. *It is straightforward to show that we can find $\{\rho,\nu\}\subset{}^{*}\mathcal{N}\setminus\mathcal{N}$ with $\sqrt{\nu+\rho-1}\frac{\gamma}{2^{\nu+1}}=c_{\rho,\nu}$, for $\gamma=2^{\rho}$, and $1\leq c_{\rho,\nu}\leq 2$. Then, for this choice of $\{\rho,\nu\}$, we obtain;*

$\overline{H}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})=c_{\rho,\nu}(\overline{N}(\frac{2^{\nu}i+1}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{2^{\nu}i}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

Tomorrow:
(i). Reweight measure to get martingale condition with $\alpha=(\frac{1}{2}-\epsilon)$, $\beta=(\frac{1}{2}+\epsilon)$

(ii). Check this specialises to Lebesgue measure.

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Let $\overline{H}:\overline{P}_{2^{\nu}\gamma}\times\overline{T}_{\nu+\rho}\rightarrow{}^{*}\mathcal{R}$ be as in Definition 0.11 of [2], with respect to $\frac{\gamma}{2\alpha d^{\nu}}\overline{N}_{\alpha}$, then, a similar calculation, shows that, for $k\geq\rho$;

$\overline{H}(\frac{i}{\gamma},\frac{\nu+\rho-(k+1)}{\nu})=\sqrt{\nu+\rho-1}\frac{\beta\gamma}{2d^{\nu}\alpha}(\overline{N}(\frac{2^{\nu}i+1}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu})-\overline{N}(\frac{2^{\nu}i}{2^{\nu}\gamma},\frac{\nu+\rho-(k+2)}{\nu}))$

(iii). Construct random walk with α, β probabilities, use $\epsilon \in {}^*\mathcal{Q}$, $\alpha = \frac{c}{d}$, $\beta = \frac{d-c}{d}$, d-adic representation of ${}^*[0,1]$, same time step $\frac{1}{\nu}$, replace 2 by d, $\eta = d^\nu$.

(iv). Choose ϵ to get rid of constant term $\sqrt{\nu+\rho-1}\frac{\sqrt{\nu}}{2^\nu}$. (now need d^ν electrons but $\rho = log_2(\sqrt{\nu})$ stays the same. Derive equation for d.

(v). Use footnote and calculation of $\overline{N}$, to get $(\overline{N}(\frac{2^\nu i+1}{2^\nu\sqrt{\nu}}, \frac{\nu+\rho-(k+2)}{\nu}) - \overline{N}(\frac{2^\nu i}{2^\nu\sqrt{\nu}}, \frac{\nu+\rho-(k+2)}{\nu})) \simeq \overline{N}(\frac{2^\nu i}{2^\nu\sqrt{\nu}}, \frac{\nu+\rho-(k+2)}{\nu})$, $(*)$

(vi). Alter last step in construction of $\overline{N}$ to get $(*)$ holding everywhere.

(vii). Martingale representation theorem gives $dX_t = cX_t dW_t$, solve for X_t and use Fokker-Planck to derive probability distribution.

$(viii)$ Nonstandard formulation of Fokker-Planck.

Lemma 0.8. *For* $l \leq \gamma - 1$;

$$N(\tfrac{j}{\gamma}, \tfrac{l}{\nu}) = C^l_{\gamma-1-j}2^{\nu-l}, \text{ if } \gamma-(l+1) \leq j \leq \gamma-1$$

$$N(\tfrac{j}{\gamma}, \tfrac{l}{\nu}) = 0, \text{ if } 0 \leq j < \gamma-(l+1)$$

Proof. The proof is clear from the definition of N, and a simple tabulation. □

Lemma 0.9. *If* $k \geq \rho$, $\nu \leq \gamma$ *and* $\gamma - (\nu+\rho-(k+1)) \leq j \leq \gamma - 1$, *and* $\overline{H}$ *is chosen relative to* $\overline{N}$, *then;*

$$\overline{H}(\tfrac{j}{\gamma}, \tfrac{\nu+\rho-(k+1)}{\nu}) = \frac{\sqrt{\nu+\rho-1}2^{(k+1)-\rho}(2(\gamma-1-j))-(\nu+\rho-(k+1))}{(\nu+\rho-(k+2))-(\gamma-2-j)}C^{\nu+\rho-(k+2)}_{\gamma-1-j}$$

Proof. By Lemmas 0.6, ($k \geq \rho$), 0.8, ($\nu \leq \gamma$), and footnote 16 for $k \geq \rho$, we have that;

$$\overline{H}(\tfrac{j}{\gamma}, \tfrac{\nu+\rho-(k+1)}{\nu}) = \frac{\sqrt{\nu+\rho-1}}{2}(\overline{N}(\tfrac{2^\nu j+1}{2^\nu\gamma}, \tfrac{\nu+\rho-(k+2)}{\nu}) - \overline{N}(\tfrac{2^\nu j}{2^\nu\gamma}, \tfrac{\nu+\rho-(k+2)}{\nu}))$$

$$= \frac{\sqrt{\nu+\rho-1}}{2}(N(\tfrac{j+1}{\gamma}, \tfrac{\nu+\rho-(k+2)}{\nu}) - N(\tfrac{j}{\gamma}, \tfrac{\nu+\rho-(k+2)}{\nu}))$$

$$= \frac{\sqrt{\nu+\rho-1}}{2}(C^{\nu+\rho-(k+2)}_{\gamma-2-j} - C^{\nu+\rho-(k+2)}_{\gamma-1-j})2^{(k+2)-\rho})$$

$= \frac{\sqrt{\nu+\rho-1}}{2}2^{(k+2)-\rho}(\frac{2(\gamma-1-j)}{(\nu+\rho-(k+2))-(\gamma-2-j)} - \frac{(\nu+\rho-(k+2))+1}{(\nu+\rho-(k+2))-(\gamma-2-j)})C^{\nu+\rho-(k+2)}_{\gamma-1-j}$

$= \frac{\sqrt{\nu+\rho-1}2^{(k+1)-\rho}(2(\gamma-1-j))-(\nu+\rho-(k+1))}{(\nu+\rho-(k+2))-(\gamma-2-j)}C^{\nu+\rho-(k+2)}_{\gamma-1-j}$

□

Definition 0.10. *$X : \overline{\Omega_\eta} \times \overline{T_\nu} \rightarrow \overline{\mathcal{R}_{\eta'}}$ be measurable with respect to $(\mu_\eta \times \lambda_\nu)$. If $\{t_1, t_2, t_3\} \subset \overline{T_\nu}$, with $\frac{[\nu t_1]}{\nu} < \frac{[\nu t_2]}{\nu} < \frac{[\nu t_3]}{\nu}$, $(*)$, we define the joint cumulative distribution functions $P_{t_1,t_2,t_3} : (\overline{\mathcal{R}_{\eta'+1}})^3 \rightarrow {}^*[0,1]$, $Q_{t_k,t_l} : (\overline{\mathcal{R}_{\eta'+1}})^2 \rightarrow {}^*[0,1]$, $1 \leq k < l \leq 3$, $R_{t_k} : (\overline{\mathcal{R}_{\eta'+1}}) \rightarrow {}^*[0,1]$, $1 \leq k \leq 3$, by;*

$P_{t_1,t_2,t_3}(x_1, x_2, x_3) = \mu_\eta(\cap_{1\leq k\leq 3}(X_{\frac{[\nu t_i]}{\nu}} \leq x_i))$

$Q_{t_k,t_l}(x_k, x_l) = \mu_\eta((X_{\frac{[\nu t_k]}{\nu}} \leq x_k) \cap (X_{\frac{[\nu t_l]}{\nu}} \leq x_l))$

$R_{t_k}(x_k) = \mu_\eta((X_{\frac{[\nu t_k]}{\nu}} \leq x_k))$

and the joint density functions $p_{t_1,t_2,t_3} : (\overline{\mathcal{R}_{\eta'+1}})^3 \rightarrow {}^*\mathcal{R}$, $q_{t_k,t_l} : (\overline{\mathcal{R}_{\eta'+1}})^2 \rightarrow {}^*\mathcal{R}$, $r_{t_k} : (\overline{\mathcal{R}_{\eta'+1}}) \rightarrow {}^*\mathcal{R}$ by;

$p_{t_1,t_2,t_3}(x'_1, x'_2, x'_3) = \frac{\partial^3(P_{t_1,t_2,t_3})}{\partial x_1 \partial x_2 \partial x_3}(x'_1, x'_2, x'_3)$

$q_{t_k,t_l}(x'_k, x'_l) = \frac{\partial^2(Q_{t_k,t_l})}{\partial x_k \partial x_l}(x'_k, x'_l)$

$r_{t_k}(x'_k) = \frac{dR_{t_k}}{dx_k}(x'_k)$

where the derivatives are defined as in [4], *([6])*

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[6] Observe that the partial derivatives commute and the choice of $\eta' + 1$ ensures that;

$\int_{(\overline{\mathcal{R}_{\eta'+1}})} r_{t_k} d\mu_\eta = \int_{(\overline{\mathcal{R}_{\eta'+1}})^2} q_{t_k,t_l} d\mu_\eta^2 = \int_{(\overline{\mathcal{R}_{\eta'+1}})^3} p_{t_1,t_2,t_3} d\mu_\eta^3 = 1,$

for $1 \leq k \leq 3$, $1 \leq k < l \leq 3$. Moreover, we have that, for $x_k \in \overline{\mathcal{R}_{\eta'+1}} \setminus [\eta', \eta'+1)$, letting $\kappa = \eta' + 1$;

$\int_{(\overline{\mathcal{R}_{\eta'+1}})} q_{t_k,t_l}(x_k, x_l) d\mu_\eta(x_l)$

$= \eta^* \sum_{i=-\kappa^2}^{\kappa^2-2} \frac{\partial^2(Q_{t_k,t_l})}{\partial x''_k \partial x''_l}(x_k, \frac{i}{\eta})$

$= \eta^* \sum_{i=-\kappa^2}^{\kappa^2-2} \frac{1}{\eta^2}(Q_{t_k,t_l}(x_k + \frac{1}{\eta}, \frac{i+1}{\eta}) - Q_{t_k,t_l}(x_k + \frac{1}{\eta}, \frac{i}{\eta}) - Q_{t_k,t_l}(x_k, \frac{i+1}{\eta})$

For $\{x_1',x_2'\}\subset\mathcal{R}_{\eta'}$, *with* $q_{t_1,t_2}(x_1',x_2')\neq 0$, $r_{t_2}(x_2')\neq 0$, $(**)$, *and* $\{t_1,t_2,t_3\}\subset\overline{T_\nu}$, *satisfying* $(*)$, *we define the conditional density functions,* $p_{t_3|t_2,x_2',t_1,x_1'}:\overline{\mathcal{R}_{\eta'+1}}\rightarrow {}^*\mathcal{R}$, $q_{t_3|t_2,x_2'}:\overline{\mathcal{R}_{\eta'+1}}\rightarrow {}^*\mathcal{R}$ *by;*

$$p_{t_3|t_2,x_2',t_1,x_1'}(x_3)=\frac{p_{t_1,t_2,t_3}(x_1',x_2',x_3)}{q_{t_1,t_2}(x_1',x_2')}$$

$$q_{t_3|t_2,x_2'}(x_3)=\frac{q_{t_2,t_3}(x_2',x_3)}{r_{t_2}(x_2')}$$

We say that X *is a nonstandard Markov process, if, for all* $\{t_1,t_2,t_3\}\subset\overline{T_\nu}$, *satisfying* $(*)$, *and* $\{x_1',x_2'\}\subset\mathcal{R}_{\eta'}$, *satisfying* $(**)$, *we have that;*

$$p_{t_3|t_2,x_2',t_1,x_1'}=q_{t_3|t_2,x_2'}$$

or, equivalently, with the same conditions $(*)$ *and* $(**)$*;*

$$p_{t_1,t_2,t_3}(x_1',x_2',x_3)r_{t_2}(x_2')=q_{t_2,t_3}(x_2',x_3)q_{t_2,t_3}(x_1',x_2')$$

Lemma 0.11. *Let* $Y:\overline{\Omega_\eta}\rightarrow\overline{\mathcal{R}_{\eta'}}$ *be measurable, with associated* $r_Y:\overline{\mathcal{R}_{\eta'+1}}\rightarrow {}^*\mathcal{R}$, *and let* $V\in C^2(\mathcal{R})$ *be right analytic, with* $|\frac{D^mV}{Dx^m}|\leq Km!$, *for* $m\in\mathcal{Z}_{\geq 0}$ *and some* $K\in\mathcal{R}$, *and let* $V_{\eta'+1}$ *be the measurable counterpart of* ${}^*V|_{\overline{\mathcal{R}_{\eta'+1}}}$, *then;*

$$E_\eta(Y)\simeq\int_{\overline{\mathcal{R}_{\eta'+1}}}x_{\eta'+1}r_Y d\lambda_{\eta'+1}$$

$$E_\eta(V_{\eta'+1}(Y))\simeq\int_{\overline{\mathcal{R}_{\eta'+1}}}V_{\eta'+1}r_Y d\lambda_{\eta'+1}$$

The same result holds if $V_{\eta'+1}$ *is measurable, with* $|\frac{dV_{\eta'+1}}{dx}|\leq K$.

Proof. We have that;

If $V_{\eta'+1}\geq 0$;

$$\int_{\overline{\mathcal{R}_{\eta'+1}}}V_{\eta'+1}r_Y d\lambda_{\eta'+1}$$

$+Q_{t_k,t_l}(x_k,\frac{i}{\eta}))$

$=\frac{1}{\eta}(-Q_{t_k,t_l}(x_k+\frac{1}{\eta},-\kappa^2)+Q_{t_k,t_l}(x_k+\frac{1}{\eta},\kappa^2-1)+Q_{t_k,t_l}(x_k,-\kappa^2)-Q_{t_k,t_l}(x_k,\kappa^2-1))$

$=\frac{1}{\eta}(R_{t_k}(x_k+\frac{1}{\eta})-R_{t_k}(x_k))=\frac{dR_{t_k}}{dx_k''}(x_k)=r_{t_k}(x_k)$

$= \frac{1}{\eta'+1}{}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-1} V_{\eta'+1}(\frac{j}{\eta'+1}) r_Y(\frac{j}{\eta'+1})$

$= {}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2} V_{\eta'+1}(\frac{j}{\eta'+1})(R_Y(\frac{j+1}{\eta'+1}) - R_Y(\frac{j}{\eta'+1}))$

$= {}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2} V_{\eta'+1}(\frac{j}{\eta'+1})\mu_\eta(\frac{j}{\eta'+1} < Y \leq \frac{j+1}{\eta'+1}) = \gamma$

$\leq \frac{1}{\eta}{}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2} {}^{*}\sum_{i \in J_j} V_{\eta'+1}(Y)(\frac{i}{\eta})$

where $J_j = \{i \in {}^{*}\mathcal{Z} \cap [0, \eta - 1] : Y(\frac{i}{\eta}) \in (\frac{j}{\eta'+1}, \frac{j+1}{\eta'+1}]\})$

$= \frac{1}{\eta}{}^{*}\sum_{0 \leq i \leq \eta-1} V_{\eta'+1}(Y)(\frac{i}{\eta}) = E_\eta(V_{\eta'+1}(Y))$

$\leq {}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2} V_{\eta'+1}(\frac{j+1}{\eta'+1})\mu_\eta(\frac{j}{\eta'+1} < Y \leq \frac{j+1}{\eta'+1}) = \delta$

Then $|\delta - \gamma|$

$= |{}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2}(V_{\eta'+1}(\frac{j+1}{\eta'+1}) - V_{\eta'+1}(\frac{j}{\eta'+1}))\mu_\eta(\frac{j}{\eta'+1} < Y \leq \frac{j+1}{\eta'+1})|$

$\leq \frac{1}{\eta'+1}{}^{*}\sum_{-(\eta'+1)^2 \leq j \leq (\eta'+1)^2-2} |\frac{dV_{\eta'+1}}{dx}|(\frac{j}{\eta'+1})\mu_\eta(\frac{j}{\eta'+1} < Y \leq \frac{j+1}{\eta'+1})$

$\leq \frac{K+\epsilon}{\eta'+1} \simeq 0$

where $\epsilon \in \mathcal{R}_{>0}$, as $\mu_\eta(\overline{\Omega_\eta}) = 1$ and $\frac{dV_{\eta'+1}}{dx} \simeq (\frac{DV}{Dx})_{\eta'+1}$, ([7]). Hence, $E_\eta(V_{\eta'+1}(Y)) \simeq \int_{\overline{\mathcal{R}_{\eta'+1}}} V_{\eta'+1} r_Y d\lambda_{\eta'+1}$. For arbitrary $V_{\eta'+1}$, we have;

$$E_\eta(V^+_{\eta'+1}(Y)) \simeq \int_{\overline{\mathcal{R}_{\eta'+1}}} V^+_{\eta'+1} r_Y d\lambda_{\eta'+1}$$

$$E_\eta(V^-_{\eta'+1}(Y)) \simeq \int_{\overline{\mathcal{R}_{\eta'+1}}} V^-_{\eta'+1} r_Y d\lambda_{\eta'+1}$$

$$E_\eta(V_{\eta'+1}(Y)) = E_\eta(V^+_{\eta'+1}(Y)) - E_\eta(V^-_{\eta'+1}(Y))$$

$$\simeq \int_{\overline{\mathcal{R}_{\eta'+1}}} V^+_{\eta'+1} r_Y d\lambda_{\eta'+1} - \int_{\overline{\mathcal{R}_{\eta'+1}}} V^-_{\eta'+1} r_Y d\lambda_{\eta'+1}$$

$$= \int_{\overline{\mathcal{R}_{\eta'+1}}} V_{\eta'+1} r_Y d\lambda_{\eta'+1}$$

as required.

□

Lemma 0.12. *There exists $W \in C^2(\mathcal{R})$, which is right analytic, ([8]), $K \geq 0$, such that $|\frac{D^m W}{Dx^m}| \leq K$, ([9]), for $m \in \mathcal{Z}_{\geq 0}$, and, $lim_{|x|\to\infty} W(x) =$*

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[7] We have, using the assumption of right analyticity, that, for $x_0 \in \mathcal{R}$;

$$|V(x_0) - V(x_0 + \tfrac{1}{n})|$$

$$\leq \sum_{m=1}^{\infty} n^{-m} \frac{|\frac{DV^m}{Dx^m}(x_0)|}{m!}$$

$$\leq K \sum_{m=1}^{\infty} n^{-m} = \frac{K}{n-1}$$

and

$$|\frac{DV}{Dx}(x_0) - \frac{V(x_0+\frac{1}{n}) - V(x_0)}{\frac{1}{n}}|$$

$$\leq \sum_{m=2}^{\infty} n^{-(m-1)} \frac{|\frac{DV^m}{Dx^m}(x_0)|}{m!}$$

$$\leq K \sum_{m=2}^{\infty} n^{-(m-1)} = \frac{K}{n-1}$$

Then $\mathcal{R} \models (\forall x \in \mathcal{R})(\forall n \in \mathcal{N})(max(|\frac{DV}{Dx} - \frac{dV_n}{dx}|, |V - V^{sh,\frac{1}{n}}) \leq \frac{K}{n-1})$, hence, ${}^*\mathcal{R}$ satisfies the transferred statement. In particular, for any infinite $\eta \in {}^*\mathcal{N}$, we obtain that $(\frac{DV}{Dx})_\eta \simeq \frac{dV_\eta}{dx}$, as $\frac{K}{\eta-1} \simeq 0$, and ${}^*(\frac{DV}{Dx})$ is S-continuous on ${}^*\mathcal{R}$.

[8] By which we mean that, for any $x_0 \in \mathcal{R}$, there exists $h > 0$, such that $W|_{[x_0, x_0+h)}$ is defined by a power series.

[9] We define $\frac{D^m W}{Dx^m}$ inductively, by setting;

$$\frac{D^{m+1}W}{Dx^{m+1}}(x_0) = lim_{h\to 0,+} \frac{D^m(x_0+h) - D^m(x_0)}{h}.$$

$lim_{|x|\rightarrow\infty}W'(x)=0$. In particular, for infinite η, $(\frac{DW}{Dx})_\eta\simeq(\frac{dW_\eta}{dx})$, and, moreover, $(\frac{D^2W}{Dx^2})_\eta\simeq(\frac{d^2W_\eta}{dx^2})$ on $\overline{\mathcal{R}}_\eta$.

Proof. Let $W_1^{\overline{a}}(x)=\sum_{i=0}^5 a_ix^i$, with $\overline{a}\subset\mathcal{R}^6$, then $W'(x)=\sum_{i=1}^5 ia_ix^{i-1}$, $W''(x)=\sum_{i=2}^5 i(i-1)a_ix^{i-2}$. The condition that $W_1^{\overline{a}}(1)=(W_1^{\overline{a}})''(1)=W_1^{\overline{a}}(-1)=(W_1^{\overline{a}})'(-1)=(W_1^{\overline{a}})''(-1)=\frac{1}{e}$, $(W_1^{\overline{a}})'(1)=\frac{-1}{e}$, is given by $\overline{M}(\overline{a})=\overline{b}_0$, $(*)$ where;

$$\overline{M}=\begin{pmatrix}1&1&1&1&1&1\\0&1&2&3&4&5\\0&0&2&6&12&20\\1&-1&1&-1&1&-1\\0&1&-2&3&-4&5\\0&0&2&-6&12&-20\end{pmatrix}$$

and

$$\overline{b}_0=\frac{1}{e}\begin{pmatrix}1\\-1\\1\\1\\1\\1\end{pmatrix}$$

The equation $(*)$ is solvable, with solution $\overline{a}_0$, as $\overline{M}$ is invertible. Let $W(x)=W_1^{\overline{a}_0}(x)$, if $|x|\leq 1$, $W(x)=e^{-x}$, if $x\geq 1$, $W(x)=e^x$ if $x\leq -1$, then $W\in C^2(\mathcal{R})$, and is right analytic as $\{e^{-x}|_{(1-\epsilon,\infty)},e^x|_{(-\infty,-1)},W_1^{\overline{a}_0}|_{(-1-\epsilon,1)}\}$ are analytic, for $\epsilon>0$. We clearly have that;

$|\frac{D^mW}{Dx^m}|=|W^{(m)}|=|(-1)^me^{-x}|\leq 1$, $(x\geq 1)$

$|\frac{D^mW}{Dx^m}|=|W^{(m)}|=|e^x|\leq 1$, $(x<1)$

$|\frac{D^mW}{Dx^m}|=|(W_1^{\overline{a}_0})^{(m)}|\leq max_{x\in[-1,1],0\leq m\leq 5}(|(W_1^{\overline{a}_0})^{(m)}|)=K_1$,

$(x\in[-1,1))$

as $\frac{D^6(W_1^{\overline{a}_0})}{Dx^6}=0$ and $(W_1^{\overline{a}_0})^{(m)}\in C([-1,1]$ for $0\leq m\leq 5$. Taking $K=max(1,K_1)$ gives $|\frac{D^mW}{Dx^m}|\leq K$, for $m\in\mathcal{Z}_{\geq 0}$. We clearly have that $lim_{|x|\rightarrow\infty}W(x)=lim_{|x|\rightarrow\infty}W'(x)=0$. $(\frac{DW}{Dx})_\eta\simeq(\frac{dW_\eta}{dx})$ follows from footnote 7, in Lemma 0.11, or by a simple calculation, ([10]) Moreover, $(\frac{D^2W}{Dx^2})_\eta \simeq (\frac{d^2W_\eta}{dx^2})$ is again a simple calculation, see footnote 10.

□

Lemma 0.13. *Let $t_0 \in [0,1]$, then there exists $V_{t_0} \in C^2([0,t_0])$, which is right analytic on $[0,t_0)$, and $K \geq 0$, such that $|\frac{D^m V_{t_0}}{Dx^m}| \leq Km!$, for $m \in \mathcal{Z}_{\geq 0}$, and $V_{t_0}(0) = V_{t_0}(t_0) = 0$ and $V_{t_0}(t) > 0$, for $t \in (0,t_0)$. In particular $(\frac{DV_{t_0}}{Dt})_\nu \simeq \frac{d(V_{t_0})_\nu}{dt}$, on $(\overline{T}_\nu \cap {}^*[0,t_0])$.*

Proof. Let $V_{t_0}(x) = sin(\frac{\pi x}{t_0})$, then clearly V_{t_0} is right analytic, $V_{t_0}(0) = V_{t_0}(t_0) = 0$, $V_{t_0}(t) > 0$, for $t \in (0,t_0)$, and;

$$|\frac{D^m V_{t_0}}{Dx^m}| \leq (\frac{\pi}{t_0})^m |V_{t_0}| \leq (\frac{\pi}{t_0})^m \leq (\frac{\pi}{t_0})^a m!$$

where $a \in \mathcal{Z}_{\geq 0}$, is chosen so that $a+1 \geq \frac{\pi}{t_0}$. The last claim follows from footnote 7, in Lemma 0.11, or by a simple calculation, ([11]).

---

[10] Namely, for $x_0 \in \overline{\mathcal{R}}_{\eta,\geq 1}$;

$\frac{dW_\eta}{dx}|_{x_0} = \eta({}^*exp(-(\frac{[\eta x_0]}{\eta}+\frac{1}{\eta})) - {}^*exp(-(\frac{[\eta x_0]}{\eta})))$

$= \eta({}^*exp(-\frac{1}{\eta})-1){}^*exp(-(\frac{[\eta x_0]}{\eta}))$

$\simeq -{}^*exp(-(\frac{[\eta x_0]}{\eta}))$, (as $\eta({}^*exp(-\frac{1}{\eta})-1) \simeq -1$, and $|{}^*exp(-(\frac{[\eta x_0]}{\eta}))| \leq 1$)

$= (\frac{DW}{Dx})_\eta(x_0)$.

A similar calculation holds for $x_0 \in \overline{\mathcal{R}}_{\eta,\leq 1}$. For $x_0 \in (\overline{\mathcal{R}}_\eta \cap {}^*[-1,1))$, we have that;

$\frac{d(W_1^{\overline{a}_0})_\eta}{dx}|_{x_0}$

$= \eta({}^*W_1^{\overline{a}_0}((\frac{[\eta x_0]}{\eta}+\frac{1}{\eta})) - {}^*W_1^{\overline{a}_0}((\frac{[\eta x_0]}{\eta})))$

$= \eta(\sum_{i=0}^5 a_{i,0}({}^*x^i((\frac{[\eta x_0]}{\eta}+\frac{1}{\eta})) - {}^*x^i((\frac{[\eta x_0]}{\eta}))))$

$= \eta(\sum_{i=0}^5 a_{i,0}(\sum_{j=0}^i (C_j^i){}^*x^{i-j}((\frac{[\eta x_0]}{\eta})){}^*x^j(\frac{1}{\eta}) - {}^*x^i((\frac{[\eta x_0]}{\eta}))))$

$\simeq (\sum_{i=0}^5 a_{i,0})i{}^*x^{i-1}((\frac{[\eta x_0]}{\eta}))$

$= (\frac{DW_1^{\overline{a}_0}}{Dx})_\eta((\frac{[\eta x_0]}{\eta}))$

$\simeq (\frac{DW_1^{\overline{a}_0}}{Dx})_\eta(x_0)$

[11] That $(V_{t_0})_\nu(t) = {}^*Im(exp_\nu(i\frac{\pi}{t_0}t))$;

□

([12])

$\frac{d(V_{t_0})_\nu}{dt}|_{t_1} = {}^*Im(\nu({}^*exp(\frac{i\pi}{t_0}(\frac{[\nu t_1]}{\nu}+\frac{1}{\nu})) - {}^*exp(\frac{i\pi}{t_0}(\frac{[\nu t_1]}{\nu}))))$

$= {}^*Im(\nu({}^*exp(\frac{i\pi}{\nu t_0})-1){}^*exp(\frac{i\pi}{t_0}(\frac{[\nu t_1]}{\nu})))$

$\simeq {}^*Im(\frac{i\pi}{t_0}{}^*exp(\frac{i\pi}{t_0}(\frac{[\nu t_1]}{\nu})))$

$= \frac{\pi}{t_0}{}^*cos(\frac{\pi}{t_0}\frac{[\nu t_1]}{\nu}) = (\frac{DV_{t_0}}{Dt})_\nu(t_1)$

[12]Not sure about this....

Lemma 0.14. *There exists $W \in C^\infty(\mathcal{R})$, with $lim_{|x|\rightarrow\infty}W(x) = lim_{|x|\rightarrow\infty}W'(x) = 0$, such that, for infinite η, $(\frac{dW}{dx})_\eta \simeq (\frac{dW_\eta}{dx})$, and, moreover, $(\frac{d^2W}{dx^2})_\eta \simeq (\frac{d^2W_\eta}{dx^2})$ on $\overline{\mathcal{R}}_\eta$.*

Proof. Let $W(x) = e^{-x^2}$, so that $\frac{dW}{dx} = -2xe^{-x^2}$ and $\frac{d^2W}{dx^2} = -2e^{-x^2} + 4x^2e^{-x^2}$. Clearly, $lim_{|x|\rightarrow\infty}W(x) = lim_{|x|\rightarrow\infty}W'(x) = 0$. We have that, for infinite η and $x_0 \in \overline{\mathcal{R}}_\eta$;

$\frac{dW_\eta}{dx}(x_0) = \eta({}^*exp(-(\frac{[x_0\eta]}{\eta}+\frac{1}{\eta})^2) - {}^*exp(-(\frac{[x_0\eta]}{\eta})^2)$

$= ({}^*exp(-(\frac{[x_0\eta]}{\eta})^2)(\eta({}^*exp(-2(\frac{[x_0\eta]}{\eta})-\frac{1}{\eta^2})-1))$

Using Taylor's Theorem, we have, for $n \in \mathcal{Z}_{\geq 1}$ and $x \in \mathcal{R}$, that;

$n(e^{-2\frac{x}{n}-\frac{1}{n^2}}-1) = n\sum_{m=1}^\infty \frac{(-1)^m(2\frac{x}{n}+\frac{1}{n^2})^m}{m!}$

$= -2x - \frac{1}{n} + \sum_{m=2}^\infty \frac{(-1)^m(2\frac{x}{n}+\frac{1}{n^2})^m}{m!}$

$|n(e^{-2\frac{x}{n}-\frac{1}{n^2}}-1)+2x| \leq \frac{1}{n} + \sum_{m=2}^\infty \frac{(2x+\frac{1}{n})^m}{n^m m!}$

□

Lemma 0.15. *If $W \in C^\infty([0,1])$ and analytic, $K \in \mathcal{R}$, with $|\frac{d^mW}{dx^m}| \leq K$, for $m \geq 0$, then, for infinite η, $(\frac{dW}{dx})_\eta \simeq \frac{dW_\eta}{dx}$ and $(\frac{d^2W}{dx^2})_\eta \simeq \frac{d^2W_\eta}{dx^2}$.*

Proof. The first part follows from footnote 7. For the second part, we have, if $x_0 \in [0,1]$ that;

$(\frac{d^2W_n}{dx^2})^{lsh,\frac{1}{n}}(x_0) = n^2(W(x_0+\frac{1}{n}) - 2W(x_0) + W(x_0-\frac{1}{n}))$

$(\frac{D^2W}{Dx^2})(x_0) = 2n^2(W(x_0+\frac{1}{n}) - W(x_0) - W'(x_0)\frac{1}{n}) + ...O(\frac{1}{n})$ (analytic)

$|(\frac{D^2W}{Dx^2})(x_0) - (\frac{d^2W_n}{dx^2})^{lsh,\frac{1}{n}}(x_0)|$

$= 2n^2(W(x_0+\frac{1}{n}) - W(x_0) - W'(x_0)\frac{1}{n}) - n^2(W(x_0+\frac{1}{n}) - 2W(x_0) + W(x_0-\frac{1}{n}))$

Lemma 0.16. *Let* $X : \overline{\Omega_\eta} \times \overline{T_\nu} \rightarrow \overline{\mathcal{R}_{\eta'}}$ *be a nonstandard martingale in the sense of* [2], *with* $E_\eta(X_t) = 0$, *for* $t \in \overline{T_\nu}$, *where* $|X| \leq \eta'$, $\eta = 2^\nu$, *and* $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, *with associated* $H : \overline{\Omega_\eta} \times \overline{T_\nu} \rightarrow \overline{\mathcal{R}_{\eta''}}$, *where* $\eta'' = [\frac{2\eta'}{\sqrt{\nu}} + 1]$, *such that* $H^2 = 2D(X,t)$, *where* $D : \overline{\mathcal{R}_{\eta'}} \times \overline{T_\nu} \rightarrow \overline{\mathcal{R}_{\eta'''}}$, *with* $\eta''' = (\eta'')^2$, *then the nonstandard cumulative probability distribution* $p : \overline{\mathcal{R}_{\eta'}} \times \overline{T_\nu} \rightarrow {}^*[0,1]$, *defined by* $p(x',t) = \frac{\partial P}{\partial x} P(x',t)$ *where* $P(x,t) = \mu_\eta((X_{\frac{[\nu t]}{\nu}} \leq x))$ *satisfies the differential equation;*

$= n^2(W(x_0 + \frac{1}{n}) - 2W'(x_0)\frac{1}{n} - W(x_0 - \frac{1}{n})) + O(\frac{1}{n})$

$= n^2(W(x_0 + \frac{1}{n}) - W(x_0) - 2W'(x_0)\frac{1}{n} + W(x_0) - W(x_0 - \frac{1}{n})) + O(\frac{1}{n})$

$= n(\frac{dW_n}{dx} - W'(x_0)) + n(\frac{dW_n}{dx}^{lsh,\frac{1}{n}} - W'(x_0)) + O(\frac{1}{n})$

Using footnote 7, we have that;

$|W'(x_0) - \frac{dW_n}{dx}| \leq \frac{K}{n-1}$

Hence;

$|(\frac{d^2W}{dx^2})(x_0) - (\frac{d^2W_n}{dx^2})^{lsh,\frac{1}{n}}(x_0)| \leq \frac{2Kn}{n-1} + O(\frac{1}{n})$

It follows, (using transfer), and $(\frac{d^2W_n}{dx^2})^{lsh,\frac{1}{\eta}} \simeq \frac{d^2W_n}{dx^2}$, that, as $(\frac{d^2W}{dx^2})_\eta$ is bounded, $G_\eta = \frac{d^2W_\eta}{dx^2}$ is bounded, hence S-integrable. We have that, for $t \in {}^*[0,1]$;

$\frac{dW_\eta}{dx}(t) - \frac{dW_\eta}{dx}(0) = \int_0^t \frac{d^2W_\eta}{dx^2} d\lambda_\nu = \int_0^t G_\eta d\lambda_\nu$

Using the first part, we have that;

$(\frac{dW}{dx})_\eta(t) - (\frac{dW}{dx})_\eta(0) \simeq \int_0^t G_\eta d\lambda_\nu$

Taking standard parts, we obtain that;

$(\frac{dW}{dx})(^\circ t) - (\frac{dW}{dx})(0) = \int_0^t {}^\circ G_\eta dL(\lambda_\nu)$

By the FTC and the fact that $st : ({}^*[0,t], L(\lambda_\nu)) \rightarrow ([0,t], d\mu)$ is measure preserving, for Lebesgue measure $d\mu$, we have that;

$(\frac{dW}{dx})(^\circ t) - (\frac{dW}{dx})(0) = \int_0^{circ_t} (\frac{d^2W}{dx^2}) d\mu = \int_0^t ({}^\circ(\frac{d^2W}{dx^2})_\eta dL(\lambda_\nu)$

Hence, $\int_0^t ({}^\circ G_\eta - {}^\circ(\frac{d^2W}{dx^2})_\eta) dL(\lambda_\nu) = 0$

It follows, as $t \in {}^*[0,1]$ is arbitrary, that $G_\eta \simeq (\frac{d^2W}{dx^2})_\eta$ as required.

□

$\frac{\partial p}{\partial t}-\frac{\partial(Dp)}{\partial x^2}=g$

where the partial derivatives $\frac{\partial}{\partial t},\frac{\partial}{\partial x}$ *are defined as in* [4], *and* $g:\overline{\mathcal{R}_{\eta'}}\times\overline{T_\nu}\rightarrow {}^*\mathcal{R}$, *with* $|g|_{U_{epsilon}}|\leq\epsilon$, *with* $\epsilon\simeq 0$, *and* $(\mu_{\eta'}\times\lambda_\nu)((\overline{\mathcal{R}_{\eta'}}\times\overline{T_\nu}\setminus U_\epsilon))\simeq 0$.

Proof. For $t_0\in[0,1]$, let $V_{t_0}\in C^2([0,t_0])$, as in Lemma 0.13, ([13]), with $V_{t_0}(0)=V_{t_0}(t_0)=0$ and $V_{t_0}(t)>0$, for $t\in[0,t_0]$. Let $W_{t_0}(x,t)=V_{t_0}(t)W(x)$, where W is given by Lemma 0.12, then $W_{t_0}\in C^2(\mathcal{R}\times[0,t_0])$, ([14]), $W_{t_0}(x,0)=W_{t_0}(x,t_0)=0$, for $x\in\mathcal{R}$, $W_{t_0}(x,t)>0$, for $(x,t)\in(\mathcal{R}\times[0,t_0])$. We have that $|\frac{DW_{t_0}}{Dx}|=|V_{t_0}||\frac{DW}{Dx}|\leq K$, and, for $t\in[0,t_0]$, $lim_{|x|\rightarrow\infty}W_{t_0}(x,t)=lim_{|x|\rightarrow\infty}\frac{DW_{t_0}}{Dx}(x,t)=0$. Let $\{{}^*W_{t_0},{}^*\frac{DW_{t_0}}{Dt},{}^*\frac{DW_{t_0}}{Dx},{}^*\frac{D^2W_{t_0}}{Dx^2}\}:{}^*\mathcal{R}\times{}^*[0,t_0]\rightarrow{}^*\mathcal{R}$ denote their transfers, and $\{W_{\eta'+1,t_0},(\frac{DW_{t_0}}{Dt})_{\eta'+1},(\frac{DW_{t_0}}{Dx})_{\eta'+1},(\frac{D^2W_{t_0}}{Dx^2})_{\eta'+1}\}:\overline{\mathcal{R}_{\eta'+1}}\times\overline{\mathcal{T}_{\nu,t_0}}\rightarrow\overline{\mathcal{R}_{\eta''''}}$ denote their measurable counterparts, ([15]). Let $S_{t_0}:\overline{\Omega_\eta}\times\overline{T_{\nu,t_0}}\rightarrow\overline{\mathcal{R}_{\eta''''}}$ be the measurable function defined by $S_{t_0}(x,t)=V_{t_0,\eta'+1}(X(x,t),t)$, then, we claim that, for $t\in(\mu(t_0)\cap\overline{\mathcal{T}_{\nu,t_0}})$;

$$S_{t_0,\eta'}(x,t)\simeq\int_0^t[(\frac{\partial V_{t_0}}{\partial t})_{\eta'+1}|_{(X(x,s),s)}+D(\frac{\partial^2V_{t_0}}{\partial x^2})_{\eta'+1}|_{(X(x,s),s)}]d\lambda_\nu(s)$$

$$+\int_0^t H(\frac{\partial V_{t_0}}{\partial x})_{\eta'+1}|_{(X(x,s),s)}d\chi(x,s)\ (*)$$

(Truncate functions appropriately and use [3], proof of Ito's lemma, (AB) gives dt,dW terms, (B) gives square term in dt, $D_2=t$, $F_2=0$, $D_1=0$, $F_1=\int_0^s H(t,x)d\chi(t,x)$, cross terms and error terms vanish up to $\simeq$, to get $(*)$.)

Follow proof of Rouah

From $(*)$, Definition 0.7(ii), the assumption that $E_\eta(X_t)=0$, Lemma 0.11, (obtain bound in ns derivative of D term, for suitable choice of ν

[13]By which we mean that $V_{t_0}\in C([0,t_0])$, $V|_{(0,t_0)}\in C^2(0,t_0)$, and the derivatives $\{\frac{d^iV_{t_0}}{dt^i}\}$ extend to $g_{i,t_0}\in C([0,t_0])$, for $0\leq i\leq 2$

[14]By which we mean that $W_{t_0}\in C(\mathcal{R}\times[0,t_0])$, $W_{t_0}|_{\mathcal{R}\times(0,t_0)}\in C^\infty(\mathcal{R}\times(0,t_0))$, and the partial derivatives $\{\frac{\partial^i\partial^jW_{t_0}}{\partial t^i\partial x^j}\}$ extend to $g_{i,j,t_0}\in C(\mathcal{R}\times[0,t_0])$, for $0\leq i\leq i\leq 2$.

[15]Here, $\overline{\mathcal{T}_{\nu,t_0}}$ denotes the interval $[0,\frac{[t_0\eta']+1}{\eta'})$, with the $*$-algebra $\mathcal{A}_{t_0}$ generated by internal unions of the intervals $[\frac{j}{\eta'},\frac{j+1}{\eta'})$, and measure λ_ν defined by $\lambda_\nu([\frac{j}{\eta'},\frac{j+1}{\eta'}))=\frac{1}{\eta'}$, for $0\leq j\leq[t_0\eta']$

and $\rho..D(x)=o(\frac{x}{\nu+\rho-1}),\eta'?$, we obtain that;

$E_\eta(S_{t_0,\eta'}(x,t_0)\simeq E_\eta(\int_0^{t_0}[(\frac{\partial V_{t_0}}{\partial t})_{\eta'+1}|_{(X(x,s),s)}+D(\frac{\partial^2 V_{t_0}}{\partial x^2})_{\eta'+1}|_{(X(x,s),s)}]d\lambda_\nu(s))$

$+E_\eta(\int_0^{t_0}H(\frac{\partial V_{t_0}}{\partial x})_{\eta'+1}|_{(X(x,s),s)}d\chi(x,s))\ (*)$

$=E_\eta(\int_0^{t_0}[(\frac{\partial V_{t_0}}{\partial t})_{\eta'+1}|_{(X(x,s),s)}+D(\frac{\partial^2 V_{t_0}}{\partial x^2})_{\eta'+1}|_{(X(x,s),s)}]d\lambda_\nu(s))$

$\simeq\int_{\mathcal{R}_{\eta'+1}}\int_0^{t_0}[(\frac{\partial V_{t_0}}{\partial t})_{\eta'+1}|_{(X(x,s),s)}+D(\frac{\partial^2 V_{t_0}}{\partial x^2})_{\eta'+1}|_{(X(x,s),s)}]p(x,s)d\lambda_\nu(s)d\lambda_{\eta'+1}(x)$

Observe that $|(\frac{\partial V}{\partial t})_{\eta'}-(\frac{\partial V_{\eta'}}{\partial t})|_{(x,t)}\leq\epsilon e^{-x^2}$, compute $|(\frac{\partial V}{\partial x})_{\eta'}-(\frac{\partial V_{\eta'}}{\partial x})|_{(x,t)}$ $|(\frac{\partial^2 V}{\partial x^2})_{\eta'}-(\frac{\partial^2 V_{\eta'}}{\partial x^2})|_{(x,t)}$. Alter $\simeq$ $(\frac{\partial V}{\partial t})_{\eta'}$ to $(\frac{\partial V_{\eta'}}{\partial t})$, $(\frac{\partial^2 V}{\partial x^2})_{\eta'}$ to $(\frac{\partial^2 V_{\eta'}}{\partial x^2})$, and use ns integration by parts.

to obtain;

$\int_{\overline{\mathcal{R}_{\eta'}}}\int_{\overline{\mathcal{T}_\nu}_{0\leq t\leq t_0}}V_{\eta'}(x,t)(\frac{\partial p}{\partial t}-\frac{\partial(Dp)}{\partial x^2})d\lambda_\nu d\mu_{\eta'}\simeq 0$, for all $t_0\in\overline{\mathcal{T}_\nu}$. Use properties of Lebesgue measure and continuity of V, to get $(\lambda_\nu\times\mu_{\eta'})((a\geq\frac{1}{n}))\leq\frac{1}{n^2}$, where $a(x,t)=\frac{\partial p}{\partial t}-\frac{\partial(Dp)}{\partial x^2}$, giving result.

□

([16])

[16] We have that;

$C^n_m-C^n_{m+1}=\frac{n!}{(n-m)!m!}-\frac{n!}{(n-(m+1))!(m+1)!}$

$=\frac{n...(n-(m-1))}{m!}-\frac{n...(n-m)}{(m+1)!}$

$=\frac{n...(n+1-m)(m+1)-n...(n-m)}{(m+1)!}$

$=C^n_m\frac{(m+1)-(n-m)}{(m+1)}$

$=C^n_m(2-\frac{1+n}{1+m})$

$\frac{C^n_m}{C^n_{m+1}}=\frac{n!}{(n-m)!m!}\frac{(n-(m+1))!(m+1)!}{n!}=\frac{(m+1)}{(n-m)}$

$C^n_m-C^n_{m+1}=\frac{(m+1)}{(n-m)}(2-\frac{1+n}{1+m})C^n_{m+1}$

$=(\frac{2(m+1)}{(n-m)}-\frac{(1+n)}{(n-m)})C^n_{m+1}$

Then, for $n>>m$, $C^n_m-C^n_{m+1}\simeq -C^n_{m+1}$, $C^n_{m+1}-C^n_m\simeq C^n_{m+1}$

References

[1] A Non-Standard Representation for Browninan Motion and Ito Integration, R. Anderson, Israel Journal of Mathematics, Vol. 25, (1976).

[2] A Simple Proof of the Martingale Representation Theorem using Nonstandard Analysis, T.de Piro, (2014).

[3] Advances in Nonstandard Analysis, T.de Piro, (2014).

[4] A Simple Proof of the Fourier Inversion Theorem Using Nonstandard Analysis, T. de Piro, (2013).