AN INVERSION THEOREM FOR LAPLACE TRANSFORMS

TRISTRAM DE PIRO

ABSTRACT. Using standard methods, we prove an inversion formula for the Laplace transformation.

Definition 0.1. *We let res* $: C(\mathcal{R}) \rightarrow C(\mathcal{R}_{>0})$ *denote the restriction mapping given by* $res(f) = f|_{[0,\infty)}$ *and* $\mathcal{S}(\mathcal{R})$ *the Schwartz class. We let* $\mathcal{S}(\mathcal{R}_{>0})$ *denote the images of* $S(\mathcal{R})$ *under res, and* $\mathcal{S}_0(\mathcal{R}_{>0}) = \{f \in \mathcal{S}(\mathcal{R}_{>0}) : f_{ext} \in S(\mathcal{R})\}$*, where* $f_{ext} : \mathcal{R} \rightarrow \mathcal{C}$ *is defined by* $f_{ext}(x) = 0$*, if* $x < 0$*, and* $f_{ext}(x) = f(x)$ *otherwise. For* $g \in S(\mathcal{R})$*, we define* $(g)^{-}(x) = g(-x)$*, for* $x \in \mathcal{R}$*. We define the real Laplace transformation* $L : \mathcal{S}(\mathcal{R}_{>0}) \rightarrow C(\mathcal{R})$ *by;*

$$L(f)(x) = \int_0^\infty f(t)exp(-itx)dt \ (x \geq 0)$$

and the inverse real Laplace transformation by;

$$L^{-}(g)(t) = \frac{1}{2\pi}\int_0^\infty g(x)exp(ixt)dx \ (t \geq 0)$$

Observe that if $f \in \mathcal{S}_0(\mathcal{R}_{>0})$*, then* $\{L(f), L^{-}(f)\} \subset S(\mathcal{R}_{>0})$*, so we can define* $\{L^2(f), L^{-2}(f), (L \circ L^{-})(f), (L^{-} \circ L)(f)\} \subset C(\mathcal{R}_{>0})$*. We define* $\mathcal{U}(\mathcal{R}_{>0}) = \{f \in S(\mathcal{R}_{>0}) : f(t)exp(st) \in S(\mathcal{R}_{>0}),\ for\ all\ s \in \mathcal{R}\}$*,* $\mathcal{V}(\mathcal{R}_{>0}) = \{f \in S(\mathcal{R}_{>0}) : L(f)exp(sx) \in S(\mathcal{R}_{>0}),\ for\ all\ s \in \mathcal{R}\}$*,* $\mathcal{U}(\mathcal{C}_{>0}) = \{f : \mathcal{C}_{>0} \rightarrow \mathcal{R}, f|_{[0,\infty)} \in \mathcal{U}(\mathcal{R}_{>0})\}$ *and* $\mathcal{V}(\mathcal{C}_{>0}) = \{f : \mathcal{C}_{>0} \rightarrow \mathcal{R}, f|_{[0,\infty)} \in \mathcal{V}(\mathcal{R}_{>0})\}$*. For* $f \in U(\mathcal{C}_{>0})$*, we abbreviate* $f|_{[0,\infty)}$ *by* f_0*, and define the complex Laplace transformation* $L_{\mathcal{C}} : \mathcal{U}(\mathcal{C}_{>0}) \rightarrow C(\mathcal{C}_{>0})$ *by;*

$$L_{\mathcal{C}}(f)(z) = \int_0^\infty f_0(t)exp(-itz)dt \ (z = (x + iy), x \geq 0)$$

and the inverse complex Laplace transformation by;

$$L_{\mathcal{C}}^{-}(g)(u) = \frac{1}{2\pi}\int_0^\infty g_0(x)exp(ixu)dx \ (u = t + is, t \geq 0)$$

Observe that, if $f \in \mathcal{V}(\mathcal{C}_{>0})$ *then* $L_{\mathcal{C}}(f) \in \mathcal{U}(\mathcal{C}_{>0})$*, and we can define* $\{L_{\mathcal{C}}^2(f), L_{\mathcal{C}}^{-2}(f), (L_{\mathcal{C}} \circ L_{\mathcal{C}}^-)(f), (L_{\mathcal{C}}^- \circ L_{\mathcal{C}})(f)\} \subset C(\mathcal{C}_{>0})$.

For $\{f, h\} \subset \mathcal{S}(\mathcal{R}_{>0})$*, we define the convolution* $(f*h) \in \mathcal{S}(\mathcal{R}_{>0})$*, by;*

$(f * h)(t) = \int_{\mathcal{R}} f_{ext}(t - w) h_{ext}(w) dw$, $(t \geq 0)$, ([1])

Lemma 0.2. *For* $f \in \mathcal{S}_0(\mathcal{R}_{>0})$*;*

$f = [L^-(L(f)) - 2\pi L(L^-(f))]$

Proof. By the Fourier Inversion Theorem, see [1], we have, for $t \in \mathcal{R}_{\geq 0}$, that;

$f(t) = \frac{1}{2\pi} \int_{\mathcal{R}} (f_{ext})\hat{}(x) exp(ixt) dx$

$= \frac{1}{2\pi} [\int_{-\infty}^0 (f_{ext})\hat{}(x) exp(ixt) dx + \int_0^\infty (f_{ext})\hat{}(x) exp(ixt) dx]$

$= \frac{1}{2\pi} [- \int_0^\infty (f_{ext})\hat{}(-u) exp(-iut) du] + L^-(((f_{ext})\hat{})(t)$ $(u = -x)$

$= -L^-((((f_{ext})\hat{})^-)(-t) + L^-(L(f))(t)$ $(*)$

We have, for $x \in \mathcal{R}_{>0}$, that;

$((f_{ext})\hat{})^-(x) = (f_{ext})\hat{}(-x)$

$= \int_0^\infty f(t) exp(-i(-x)t) dt$

$= \int_0^\infty f(t) exp(ixt) dt = 2\pi L^-(f)$ $(**)$

Hence, using $(*)$ and $(**)$;

$f(t) = -L^-(2\pi L^-(f))(-t) + L^-(L(f))(t)$

$= -2\pi L^-(L^-(f))(-t) + L^-(L(f))(t)$

$= -2\pi L(L^-(f))(t) + L^-(L(f))(t)$

$= [L^-(L(f)) - 2\pi L(L^-(f))](t)$

[1] As, we have, for $\{f_{ext}, h_{ext}\} \subset S(\mathcal{R})$, that the standard convolution $(f_{ext} * h_{ext})|_{\mathcal{R}_{<0}} = 0$ and $(f_{ext} * h_{ext}) \in S(\mathcal{R})$. Hence, $(f_{ext} * h_{ext}) = (f * h)_{ext}$.

Lemma 0.3. *For $\{f,g\}\subset\mathcal{U}(\mathcal{C}_{>0})$, we have that;*

$L_{\mathcal{C}}(f)(z)=L((f_0)_y)(x)$, $(z=x+iy)$

$L^-_{\mathcal{C}}(g)(u)=L^-((g_0)_s)(t)$, $(u=t+is)$

where $(f_0)_y(t)=f_0(t)exp(ty)$ *and* $(g_0)_s(x)=g_0(x)exp(xs)$.

Proof. By Definition 0.1, we have that;

$L_{\mathcal{C}}(f)(z)=\int_0^\infty f_0(t)exp(-it(x+iy))dt$, $(z=x+iy)$

$=\int_0^\infty (f_0(t)exp(ty))exp(-itx)dt$

$=L((f_0)_y)(x)$

A similar calculation holds for $L^-_{\mathcal{C}}(g)$. □

Lemma 0.4. *For $\{f,g\}\subset U(\mathcal{C}_{>0})$, we have;*

$[L_{\mathcal{C}}(f)]_0=L(f_0)$

$[L^-_{\mathcal{C}}(g)]_0=L^-(g_0)$

Proof. Using Definition 0.1, we have;

$[L_{\mathcal{C}}(f)]_0(x)=L_{\mathcal{C}}(f)(x)=\int_0^\infty f_0(t)exp(-i(x)t)dt=L(f_0)(x)$

and similarly for $[L^-_{\mathcal{C}}(g)]_0$.

□

Lemma 0.5. *For $\{f,g\}\subset\mathcal{V}(\mathcal{C}_{>0})$, we have;*

$L^-_{\mathcal{C}}(L_{\mathcal{C}}(f))(u)=L^-(L(f_0)(x)exp(xs))(t)$, $(u=t+is)$

$L_{\mathcal{C}}(L^-_{\mathcal{C}}(g))(z)=L^-(L(g_0)(t)exp(xy))(t)$, $(z=x+iy)$

Proof. Using Lemmas 0.3 and 0.4, we have;

$L^-_{\mathcal{C}}(L_{\mathcal{C}}(f))(u)$ $(u=t+is)$

$= L^{-}(([L_{\mathcal{C}}(f)]_0)_s)(t)$

$= L^{-}((L(f_0))_s)(t) = L^{-}((L(f_0)(x)exp(xs))(t)$

$L_{\mathcal{C}}(L_{\mathcal{C}}^{-}(g))(z)$ $(z = x + iy)$

$= L(([L_{\mathcal{C}}^{-}(g)]_0)_y)(x)$

$= L((L^{-}(g_0))_y)(x) = L(L^{-}(g_0)(t)exp(ty))(x)$

□

Lemma 0.6. *For* $\{f, h\} \subset \mathcal{S}(\mathcal{R}_{>0})$*, we have;*

$L(f * h) = L(f)L(h)$

Proof. We have, for $x \in \mathcal{R}_{>0}$, using footnote 1, the standard convolution theorem for Fourier transforms, and Definition 0.1, that;

$L(f * h)(x) = \int_0^\infty (f * h)(t)exp(-itx)dt$

$= \int_{\mathcal{R}} (f * h)_{ext}(t)exp(-itx)dt$

$= \int_{\mathcal{R}} (f_{ext} * h_{ext})(t)exp(-itx)dt$

$= \hat{(}f_{ext} * g_{ext})(x) = \hat{f}_{ext}(x)\hat{g}_{ext}(x) = [L(f)L(h)](x)$

□

Lemma 0.7. *For* $s < 0$*, there exists a unique solution* $h_s \in S(\mathcal{R}_{>0})$ *to the equation* $L(h_s) = exp(xs)$*, given by;*

$h_s(t) = \sum_{n=0}^\infty (-2\pi)^n (LL^{-})^n L^{-}(exp(xs))(t)$

Proof. By Lemma 0.2;

We have $(1 + 2\pi(LL^{-}))(h_s) = (L^{-}L)(h_s) = L^{-}(exp(xs))$

$h_s(t) = (1 + 2\pi(LL^{-}))^{-1}(L^{-}(exp(xs)))(t)$

$= \sum_{n=0}^\infty (-2\pi)^n (LL^{-})^n L^{-}(exp(xs))(t)$

□

Lemma 0.8. *For $f \in \mathcal{S}_0(\mathcal{C}_{>0})$, we have that;*

$$(L_{\mathcal{C}}^{-} L_{\mathcal{C}} - 2\pi L_{\mathcal{C}} L_{\mathcal{C}}^{-})(f)(u) = (f * h_s)(t),\ (u = t + is)$$

where h_s is given by Lemma 0.7.

Proof. By Lemma 0.2, Lemma 0.6, Lemma 0.7, and Lemma 0.3, for $s \in \mathcal{R}_{>0}$, and h_s as in Lemma 0.7, we have;

$$(f * h_s) = (L^{-}L - 2\pi(LL^{-})(f * h_s)$$

$$= L^{-}(L(f)L(h_s)) - 2\pi L(L^{-}(f)L(h_s))$$

$$= L^{-}(L(f)exp(sx)) - 2\pi L(L^{-}(f)exp(sx))$$

$$= (L_{\mathcal{C}}^{-} L_{\mathcal{C}})(f)(t + is) - 2\pi(L_{\mathcal{C}}^{+} L_{\mathcal{C}})(f)(t + is)$$

□

□

References

[1] A Simple Proof of the Fourier Inversion Theorem Using Nonstandard Analysis, Tristram de Piro, submitted to the Logic Journal Quarterley, (2014)

Mathematics Department, The University of Exeter, Exeter
E-mail address: tdpd201@exeter.ac.uk