A LEMMA ON POLYNOMIAL ROOTS

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Abstract. We prove a simple lemma on polynomial roots, which is a special case of Cauchy's bound on their magnitude.

Lemma 0.1. *For any monic polynomial of degree n, with real coefficients $\{c_i : 0 \leq |c_i| \leq a\}$, then for any solution y_i, $1 \leq i \leq n$, we have $|y_i| \leq a+1$.*

Proof. Let A be the $n \times n$ matrix, defined by $A_{ij} = \delta_{ij} y_i$, for $1 \leq i, j \leq n$. By the spectral radius formula, we have that $|y_i| \leq lim_{k\rightarrow\infty}||A^k||^{\frac{1}{k}}$, where $||.||$ is the spectral norm, given by $max_{1\leq i\leq n}\lambda_i$, and, for $1 \leq i \leq n$, λ_i is an eigenvalue of AA^*. We have, $||A^k|| \leq (\sum_{i=1}^n |y_i|^{2k})^{\frac{1}{2}}$, so $|y_i| \leq lim_{k\rightarrow\infty}||A^k||^{\frac{1}{k}} \leq lim_{k\rightarrow\infty}(\sum_{i=1}^n |y_i|^{2k})^{\frac{1}{2k}}$. We claim that for any $\epsilon > 0$, we can choose $k(\epsilon) \in \mathcal{N}_{>0}$, such that $arg(y_i^{k(\epsilon)}) \leq \epsilon$, for $1 \leq i \leq n$, $(*)$. It then follows that we can find infinitely many k, for which $Re(y_i^k) > 0$. For such k, using the fact that the roots $\{y_i : 1 \leq i \leq n\}$ are closed under complex conjugation, we have that;

$$(\sum_{i=1}^n |y_i|^{2k}) \leq (\sum_{i=1}^n y_i^k)^2$$

$$(\sum_{i=1}^n |y_i|^{2k})^{\frac{1}{2k}} \leq (\sum_{i=1}^n y_i^k)^{\frac{1}{k}} =\leq a+1$$

where we have used the bound $|\sum_{i=1}^n y_i^k| \leq (a+1)^k$, that follows from the Newton identities for power sums. This gives the result. We now show $(*)$. We let $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ denote the unit circle, $T = S^1 \times S^1$ the torus, and $T_n = (T)^n$ the n torus. T is naturally isomorphic to a complex projective irreducible algebraic group of dimension 1, using the fact that $\mathcal{C}/\Delta$ is a complex analytic group, and the $GAGA$ principle. In order to show $(*)$, we first consider the closed irreducible algebraic subgroups of T_m for $m \in \mathcal{Z}_{>0}$. We claim that if $G_1 \leq G$ is such a group, then, up to reordering;

$$G_1 = \{(\overline{g}, \overline{h}, \overline{g}', \overline{e}) : \overline{g} \in T_r, \overline{h} = \overline{g}^{-1}, \overline{g}' \in T_{r'}\}$$

where, $\overline{g}$ has length r, and $\overline{e}$ is a repeated tuple of the identity, with length r'', so $2r+r'+r''=m$, $(\dagger)$. Let $dim(G_1)=s$, and choose a projection $pr_{m,s}:T_m\rightarrow T_s$, such that $pr_{m,s}|_{G_1}$ is generically finite. As G_1 is closed, we have $pr_{m,s}(G_1)=T_s$. If $pr_{m,s}|G_1$ has an infinite fibre, then, by purely group theoretic considerations, $dim(Ker(pr_{m,s}|G_1))\geq 1$, and $dim(G_1)\geq s+1$. Therefore, $pr_{m,s}|G_1$ has finite fibres, and, by Zariski's Main Theorem, $pr_{m,s}|G_1$ is a finite morphism. By the same group theoretic considerations, for all $g,h\in T_s$, $Card(pr_{m,s}|G_1^{-1}(g))=Card(pr_{m,s}|G_1^{-1}(h))$. If $deg(pr_{m,s}|G_1)=l\geq 2$, then, using the fact that the ramification locus of $pr_{m,s}|G_1$ has codimension 1, and a Zariski geometries argument, see [1], we can find $g\in T_s$, with $1\leq Card(pr_{m,s}|G_1^{-1}(g)=l'<l$, hence $deg(pr_{m,s}|G_1)=1$. Now the result follows by straightforward consideration of maps on lattices. We now show, that, for any $x\in T_m$, and $\epsilon>0$, there exists $k_{x,\epsilon}$, with $|x^{k_{x,\epsilon}}-\overline{1}|\leq\epsilon$, $(**)$. We prove this by induction on m. If $m=1$, and $x\in T_1$, consider the orbit $G_x=\{x^k:k\in\mathcal{Z}\}$, and the Zariski closure $H_x=\overline{G_x}$. If $H_x=T_1$, then clearly the result $(**)$ follows for $m=1$, otherwise, G_x is finite, and we can take $k_{x,\epsilon}=Card(G_x)$. Suppose the result is true for m. Taking $x\in T_{m+1}$, we again consider the orbit G_x and its closure H_x. If $H_x=T_{m+1}$, the result follows. Otherwise, H_x is a proper algebraic subgroup of T_{m+1}. If H_x is connected, we can write x in the form given by $(\dagger)$, with corresponding tuples $\{\overline{g},\overline{h},\overline{e}\}$. As H_x is a proper subgroup, we can assume that $length(\overline{g})=r<m+1$, $length(\overline{g}')=r'<m+1$, then, using the induction hypothesis, we can find $k_{x,\delta}$ such that $|\overline{g}^k_{x,\delta}-\overline{e}_r|<\delta$, and, $|\overline{g'}^k_{x,\delta}-\overline{e}_{r'}|<\delta$. Then $|x^{k_{x,\delta}}-\overline{e}_{m+1}|\leq 3\delta$. Taking $\delta=\frac{\epsilon}{3}$ gives the required result. If H_x is not connected, we can write H_x as a finite union $\bigcup_{i=1}^p a_iH$, where H is the connected component. First, observe that we can write a coset a_iH in the form rH, where r is a w'th root of unity in the group T_{m+1}, $(***)$. In order to see this, observe that the cosets $\{a_i^nH:n\in\mathcal{Z}\}$ cannot all be distinct, as H_x has only finitely many connected components. Then $a_i^w\in H$, for some $w\in\mathcal{Z}_{>0}$, and, hence, $a_i^ws^w=(a_is)^w\in H$, where s is a $w'th$ root of unity. Then, using the description of connected subgroups in $(**)$, $a_is=(\overline{c},\overline{v})$, where $\overline{v}^w=\overline{e}_{r''}$, and $\overline{c}\in pr_{m+1,2r+r'}(H)$ (this last group being closed under taking roots). It follows that we can find a further w'th root of unity $q=(\overline{e}_{m+1,2r+r'},\overline{v}^{-1})$, such that $a_isq\in H$. Taking $r=sq$ gives $(***)$. The result $(**)$ now follows from the previous claim, replacing x with x', taking $k_{x,\epsilon}=wk_{x',\delta}$, where $x=rx'$, with $x'\in H$, $r^w=1$.

□

References

[1] Zariski Geometries, B. Zilber, LMS Lecture Note Series, (2010).

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