A NOTE ON CONVERGENCE OF FOURIER SERIES

TRISTRAM DE PIRO

Abstract. We make some observations on the uniform convergence of Fourier series to symmetric and asymmetric functions.

Definition 0.1. *Let $C^\infty(S^1)$ denote the smooth functions on $[-1,1]$, with endpoints identified. If $f \in C^\infty(S^1)$, we define $f^r \in C^\infty(S^1)$ by;*

$$f^r(x) = f(-x)$$

We say that $f \in C^\infty(S^1)$ is symmetric if $f(x) = f(-x)$ for $x \in S^1$, with the convention that $-1 = 1$, and f is asymmetric if $f(x) = -f(-x)$ for $x \in S^1$.

Lemma 0.2. *If $f \in C^\infty(S^1)$, then $(f^r)^r = f$. If $g \in C^\infty(S^1)$, then $g^r = g$, iff g is symmetric, and $g^r = -g$ iff g is asymmetric. Moreover, $f + f^r$ is symmetric and $f - f^r$ is asymmetric.*

Proof. We have that $(f^r)^r(x) = -(f^r)(-x) = -(-(f(-(-x)))) = f(x)$, $(*)$. We have that $g^r = g$ iff $g(x) = g^r(x) = g(-x)$. Similarly, we have that $g^r = -g$ iff $g(x) = -g^r(x) = -g(-x)$, $(**)$. $(f + f^r)^r = f^r + (f^r)^r = f + f^r$, by $(*)$. $(f - f^r)^r = f^r - (f^r)^r = f^r - f = -(f - f^r)$, $(**)$, again by $(*)$. Hence, by $(**)$, $f + f^r$ is symmetric and $f - f^r$ is asymmetric, as required. ∎

Lemma 0.3. *$f \in C^\infty(S^1)$, then f is symmetric iff f' is asymmetric, moreover $f^{(2n+1)}(0) = f^{(2n+1)}(1) = f^{(2n+1)}(-1) = 0$, for $n \in \mathcal{Z}_{\geq 0}$, and f is asymmetric iff f' is symmetric, moreover $f^{(2n)}(0) = f^{(2n)}(1) = f^{(2n)}(-1) = 0$, for $n \in \mathcal{Z}_{\geq 0}$.*

Suppose f is symmetric, then;

$$f'(x) = lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = lim_{h\to 0}\frac{f(-x-h)-f(-x)}{h}$$

$$= -lim_{h\to 0}\frac{(f(-x)-f(-x-h))}{h} = -f'(-x).$$

Conversely, suppose f' is asymmetric, then, using the FTC;

$f(x)-f(-1)=\int_{-1}^{x}f'(y)dy=\int_{-1}^{x}-f'(-y)dy=\int_{1}^{-x}-f'(x)(-dx)$

$=\int_{1}^{-x}f'(x)dx=-\int_{-x}^{1}f'(x)dx=-(f(1)-f(-x))$

hence, $f(x)=f(-x)$, as $f(1)=f(-1)$. The second part is similar. For the last part, we have, if f is symmetric, then the odd derivatives f^{2n+1}, for $n\in\mathcal{Z}_{\geq 0}$ are antisymmetric. Hence, $f^{2n+1}(0)=-f^{2n+1}(0)$ and $f^{2n+1}(1)=-f^{2n+1}(-1)=-f^{2n+1}(1)$, therefore, $f^{2n+1}(0)=f^{2n+1}(1)$ $=f^{2n+1}(-1)=0$. The last part, for f asymmetric, is the same.

Lemma 0.4. *Let $f\in C^{\infty}(S^1)$ be symmetric, and let $g\in C^{\infty}(S^1)$ be antisymmetric, then the series;*

$\sum_{m\geq 0}a_m cos(\pi xm)$

where $a_m=\int_{-1}^{1}f(x)cos(\pi xm)dx$, $m\geq 1$, $a_0=\frac{1}{2}\int_{-1}^{1}f(x)dx$

$\sum_{m\geq 1}b_m sin(\pi xm)$

where $b_m=\int_{-1}^{1}g(x)sin(\pi xm)dx$, converge uniformly to f and g respectively on S^1.

Moreover, if $h\in C^{\infty}(S^1)$, then the series;

$\sum_{m\geq 0}c_m cos(\pi xm)+\sum_{m\geq 1}d_m sin(\pi xm)$

where $c_m=\int_{-1}^{1}h(x)cos(\pi xm)dx$, $m\geq 1$, $c_0=\frac{1}{2}\int_{-1}^{1}h(x)dx$ and $d_m=\int_{-1}^{1}h(x)sin(\pi xm)dx$, $m\geq 1$, converge uniformly to h on S^1.

Proof. By the result of [1], if $h\in C^{\infty}(S^1)$, we have that the series;

$\frac{1}{2}\sum_{m\in\mathcal{Z}}c_m e^{\pi xm}$

where $c_m=\int_{-1}^{1}h(x)e^{-\pi ixm}dx$

converges uniformly to h on S^1. If f is symmetric, we have, for $m\geq 1$, $c_m=\int_{-1}^{1}f(x)cos(\pi xm)dx=a_m$, $c_{-m}=\int_{-1}^{1}f(x)cos(-\pi xm)dx=c_m=a_m$, $c_0=\int_{-1}^{1}d(x)dx=2a_0$. Then;

$f=\frac{1}{2}\sum_{m\geq 1}c_m(e^{\pi ixm}+e^{-\pi ixm})+a_0=\sum_{m\geq 1}a_m cos(\pi xm)$

If f is asymmetric, we have, for $m \geq 1$, $c_m = -i\int_{-1}^{1} f(x)sin(\pi xm)dx = -ib_m$, $c_{-m} = i\int_{-1}^{1} f(x)sin(\pi xm)dx = -c_m = ib_m$. Then;

$f = \frac{1}{2}\sum_{m\geq 1} c_m(e^{\pi ixm} - e^{-\pi ixm}) = \sum_{m\geq 1}(b_m sin(\pi xm)$

as required for the first part. For the second part, if $h \in C^{\infty}(S^1)$, then, using the Lemma 0.2, we have that;

$h = \frac{(h+h^r)+(h-h^r)}{2} = h^{sym} + h^{asym}$

with $h^{sym} = \frac{h+h^r}{2}$ symmetric and $\frac{h-h^r}{2}$ asymmetric. By the first part;

$h^{sym} = \sum_{m\geq 0} a_m cos(\pi xm)$

where for $m \geq 1$;

$a_m = \int_{-1}^{1} h^{sym}(x)cos(\pi xm)dx$

$= \int_{-1}^{1} h(x)cos(\pi xm)dx$

as $\int_{-1}^{1} h^{asym}(x)cos(\pi xm)dx = 0$

and $a_0 = \frac{1}{2}\int_{-1}^{1} h^{sym}(x)dx$

$= \frac{1}{2}\int_{-1}^{1} h(x)dx$

as $\frac{1}{2}\int_{-1}^{1} hasym(x)dx = 0$

$h^{asym} = \sum_{m\geq 1} b_m sin(\pi xm)$

For $m \geq 1$;

$b_m = \int_{-1}^{1}(x)h^{asym}(x)sin(\pi xm)dx$

$= \int_{-1}^{1}(x)h(x)sin(\pi xm)dx$

as $\int_{-1}^{1}(x)h^{sym}(x)sin(\pi xm)dx = 0$

It follows that;

$h = h^{sym} + h^{asym}$

$= \sum_{m \geq 0} a_m cos(\pi x m) + \sum_{m \geq 1} b_m sin(\pi x m)$

and the series converge uniformly to h on S^1, by the first result.

$\square$

References

[1] A Simple Proof of the Uniform Convergence of Fourier Series Using Nonstandard Analysis, Tristram de Piro, (2012).