

# A SIMPLE PROOF OF A MARTINGALE REPRESENTATION THEOREM USING NONSTANDARD ANALYSIS

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ABSTRACT. We give a proof of a Martingale Representation Theorem using the methods of nonstandard analysis.

This paper gives a nonstandard proof of the Martingale Representation Theorem. Although, a standard proof of this result is known, we consider the approach taken to be useful. The use of nonstandard analysis is important in Physics, where we can represent diffusions, either classically as in the heat equation, or in electrodynamics, using infinitesimals. The use of nonstandard analysis, probability and the method of martingales finds an application in solving the heat equation, computing the distribution of molecular velocities for ideal gases, and defining temperature, based on the work of Boltzmann. It might potentially be used in electrodynamic diffusions, satisfying the continuity equation, possibly with a view to defining pressure, temperature or establishing thermodynamic equilibrium in this context. In Lemma 0.8, we give the representation of nonstandard martingales  $\overline{X}_t$ , in terms of sums of Bernoulli random variables  $\omega_j$  and prior information defined in terms of  $\omega_k$ , for  $0 \leq k \leq j - 1$ . Lemma 0.9 gives the connection between nonstandard and standard martingales  $X_t$ , by showing that such martingales can be lifted to a nonstandard process,  ${}^\circ(\overline{X}_t) = X_{\circ t}$ , where  $\circ$  denotes the standard part mapping. In Definition 0.11, we introduce a further series of nonstandard processes, defined from  $\overline{X}_t$ , including the quadratic variation  $[\overline{S}]_t = \overline{Q}_t$ . Theorem 0.13 gives the final result, that we can represent  $X$  as  $X_t(x) = \int_0^t F(s, x) d\beta_s$ , where  $\beta_s$  is Brownian motion. The idea is to take  $F = f^{\frac{1}{2}}$ , where  ${}^\circ\overline{Q}_t = \int_0^t f(x, s) ds$ .

We introduce the following spaces;

**Definition 0.1.** *Let  $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$ , and set  $\eta = 2^\nu$ . Define;*

$$\overline{\Omega}_\eta = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$$

$$\overline{\mathcal{T}}_\nu = \{x \in {}^*\mathcal{R} : 0 \leq x \leq 1\}$$

We let  $\mathcal{C}_\eta$  consist of internal unions of the intervals  $[\frac{i}{\eta}, \frac{i+1}{\eta})$ , for  $0 \leq i \leq \eta - 1$ , and let  $\mathcal{D}_\nu$  consist of internal unions  $[\frac{i}{\nu}, \frac{i+1}{\nu})$ , for  $0 \leq i \leq \nu - 1$ , together with  $\{1\}$

We define counting measures  $\mu_\eta$  and  $\lambda_\nu$  on  $\mathcal{C}_\eta$  and  $\mathcal{D}_\nu$  respectively, by setting  $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$ ,  $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu}]) = \frac{1}{\nu}$  and  $\lambda_\nu(\{1\}) = 0$

We let  $(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$  and  $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$  be the resulting  $*$ -finite measure spaces, in the sense of [4], and let  $(\overline{\Omega}_\eta, L(\mathcal{C}_\eta), L(\mu_\eta))$ ,  $(\overline{\mathcal{T}}_\nu, L(\mathcal{D}_\nu), L(\lambda_\nu))$  be the associated Loeb spaces.

We let  $V(\mathcal{C}_\eta) = \{f : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}, f(x) = f([\frac{\eta x}{\eta}])\}$  and  $W(\mathcal{C}_\eta) \subset V(\mathcal{C}_\eta)$  be the set of measurable functions  $f : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ , with respect to  $\mathcal{C}_\eta$ , in the sense of [4]. Then  $W(\mathcal{C}_\eta)$  is a  $*$ -finite vector space over  ${}^*\mathcal{C}$ , of dimension  $\eta$ , <sup>(1)</sup>. Similarly, we let  $V(\mathcal{D}_\nu) = \{f : \overline{\mathcal{T}}_\nu \rightarrow {}^*\mathcal{C}, f(t) = f([\frac{\nu t}{\nu}])\}$  and  $W(\mathcal{D}_\nu) \subset V(\mathcal{D}_\nu)$  be the set of measurable functions  $f : \overline{\mathcal{T}}_\nu \rightarrow {}^*\mathcal{C}$ , with respect to  $\mathcal{D}_\nu$ , in the sense of [4]. Then  $W(\mathcal{D}_\nu)$  is a  $*$ -finite vector space over  ${}^*\mathcal{C}$ , of dimension  $\nu + 1$ .

**Definition 0.2.** Given  $n \in \mathcal{N}_{>0}$ , we let  $\Omega_n = \{m \in \mathcal{N} : 0 \leq m < 2^n\}$ , and let  $C_n$  be the set of sequences of length  $n$ , consisting of 1's and  $-1$ 's. We let  $\theta_n : \Omega_n \rightarrow \mathcal{N}^n$  be the map which associates  $m \in \Omega_n$  with its binary representation, and let  $\phi_n : \Omega_n \rightarrow C_n$  be the composition  $\phi_n = (\gamma \circ \theta_n)$ , where, for  $\bar{m} \in \mathcal{N}^n$ ,  $\gamma(\bar{m}) = 2 \cdot \bar{m} - \bar{1}$ . For  $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$ , we let  $\phi_\nu : \Omega_\nu \rightarrow C_\nu$  be the map, obtained by transfer of  $\phi_n$ , which associates  $i \in {}^*\mathcal{N}$ ,  $0 \leq i < 2^\nu$ , with an internal sequence of length  $\nu$ , consisting of 1's and  $-1$ 's. Similarly, for  $\eta = 2^\nu$ , we let  $\psi_\eta : \overline{\Omega}_\eta \rightarrow C_\nu$  be defined by  $\psi_\eta(x) = \phi_\nu([\eta x])$ . For  $1 \leq j \leq \nu$ , we let

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<sup>1</sup> By a  $*$ -vector space, one means an internal set  $V$ , for which the operations  $+$  :  $V \times V \rightarrow V$  of addition and scalar multiplication  $\cdot$  :  ${}^*\mathcal{C} \times V \rightarrow V$  are internal. Such spaces have the property that  $*$ -finite linear combinations  ${}^*\sum_{i \in I} \lambda_i \cdot v_i$ ,  $(*)$ , for a  $*$ -finite index set  $I$ , belong to  $V$ , by transfer of the corresponding standard result for vector spaces. We say that  $V$  is a  $*$ -finite vector space, if there exists a  $*$ -finite index set  $I$  and elements  $\{v_i : i \in I\}$  such that every  $v \in V$  can be written as a combination  $(*)$ , and the elements  $\{v_i : i \in I\}$  are independent, in the sense that if  $(*) = 0$ , then each  $\lambda_i = 0$ . It is clear, by transfer of the corresponding result for finite dimensional vector space over  $\mathcal{C}$ , that  $V$  has a well defined dimension given by  $Card(I)$ , see [5], even though  $V$  may be infinite dimensional, considered as a standard vector space.

$\omega_j : C_\nu \rightarrow \{1, -1\}$  be the internal projection map onto the  $j$ 'th coordinate, and let  $\omega_j : \overline{\Omega_\eta} \rightarrow \{1, -1\}$  also denote the composition  $(\omega_j \circ \psi_\eta)$ , so that  $\omega_j \in W(\overline{\Omega_\eta})$ . By convention, we set  $\omega_0 = 1$ . For an internal sequence  $\bar{t} \in C_\nu$ , we let  $\omega_{\bar{t}} : \overline{\Omega_\eta} \rightarrow \{1, -1\}$  be the internal function defined by;

$$\omega_{\bar{t}} = \prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}$$

Again, it is clear that  $\omega_{\bar{t}} \in W(\overline{\Omega_\eta})$ .

**Lemma 0.3.** *The functions  $\{\omega_j : 1 \leq j \leq \nu\}$  are  $*$ -independent in the sense of [2], (Definition 19), in particular they are orthogonal with respect to the measure  $\mu_\eta$ . Moreover, the functions  $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$  form an orthogonal basis of  $V(\overline{\Omega_\eta})$ , and, if  $\bar{t} \neq \overline{-1}$ ,  $E_\eta(\omega_{\bar{t}}) = 0$ , and  $\text{Var}_\eta(\omega_{\bar{t}}) = 1$ , where,  $E_\eta$  and  $\text{Var}_\eta$  are the expectation and variance corresponding to the measure  $\mu_\eta$ .*

*Proof.* According to the definition, we need to verify that for an internal index set  $J = \{j_1, \dots, j_s\} \subseteq \{1, \dots, \nu\}$ , and an internal tuple  $(\alpha_1, \dots, \alpha_s)$ , where  $s = |J|$ ;

$$\begin{aligned} & \mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \dots, \omega_{j_k}(x) < \alpha_k, \dots, \omega_{j_s}(x) < \alpha_s) \\ &= \prod_{k=1}^s \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) \quad (*) \end{aligned}$$

Without loss of generality, we can assume that each  $\alpha_{j_k} > -1$ , as if some  $\alpha_{j_k} \leq -1$ , both sides of  $(*)$  are equal to zero. Let  $J' = \{j' \in J : -1 < \alpha_{j'} \leq 1\}$  and  $J'' = \{j'' \in J : 1 < \alpha_{j''}\}$ , so  $J = J' \cup J''$ . Then;

$$\begin{aligned} & \mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \dots, \omega_{j_s}(x) < \alpha_s) \\ &= \frac{1}{\eta} \text{Card}(z \in C_\nu : z(j') = -1 \text{ for } j' \in J', z(j'') \in \{-1, 1\} \text{ for } j'' \in J'') \\ &= \frac{1}{2^\nu} \text{Card}(z \in C_\nu : z(j') = -1 \text{ for } j' \in J') = \frac{2^{\nu-s'}}{2^\nu} = 2^{-s'} \end{aligned}$$

where  $s' = \text{Card}(J')$ . Moreover;

$$\prod_{k=1}^s \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) = \prod_{j' \in J'} \mu_\eta(x : \omega_{j'}(x) = -1) = 2^{-s'}$$

as  $\mu_\eta(x : \omega_j(x) = -1) = \frac{1}{2}$ , for  $1 \leq j \leq \nu$ . Hence,  $(*)$  is shown. That  $*$ -independence implies orthogonality follows easily by transfer, from the corresponding fact, for finite measure spaces, that  $E(X_{j_1} X_{j_2}) =$

$E(X_{j_1})E(X_{j_2})$ , for the standard expectation  $E$  and independent random variables  $\{X_{j_1}, X_{j_2}\}$ , (\*\*). Hence, by (\*\*);

$$E_\eta(\omega_{j_1}\omega_{j_2}) = E_\eta(\omega_{j_1})E_\eta(\omega_{j_2}) = 0, (j_1 \neq j_2) (***)$$

as clearly  $E_\eta(\omega_j) = 0$ , for  $1 \leq j \leq \nu$ . If  $\bar{t} \neq \overline{-1}$ , let  $J' = \{j' : 1 \leq j' \leq \nu, \bar{t}(j') = 1\}$ , then;

$$E_\eta(\omega_{\bar{t}}) = E_\eta\left(\prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}\right) = E_\eta\left(\prod_{j' \in J'} \omega_{j'}\right) = \prod_{j' \in J'} E_\eta(\omega_{j'}) = 0 \text{ (\#)}$$

where, in (\#), we have used the facts that  $J' \neq \emptyset$  and internal, and a simple generalisation of (\*\*), by transfer from the corresponding fact for finite measure spaces. Hence,  $1 = \omega_{\overline{-1}}$  is orthogonal to  $\omega_{\bar{t}}$ , for  $\bar{t} \neq \overline{-1}$ . If  $\bar{t}_1 \neq \bar{t}_2$  are both distinct from  $\overline{-1}$ , then, if  $J_1 = \{j : 1 \leq j \leq \nu, \bar{t}_1(j) = 1\}$  and  $J_2 = \{j : 1 \leq j \leq \nu, \bar{t}_2(j) = 1\}$ , so  $J_1 \neq J_2$  and  $J_1, J_2 \neq \emptyset$ , we have;

$$\begin{aligned} & E_\eta(\omega_{\bar{t}_1}\omega_{\bar{t}_2}) \\ &= E_\eta\left(\prod_{j \in J_1} \omega_j \cdot \prod_{j \in J_2} \omega_j\right) \text{ (\#\#)} \\ &= E_\eta\left(\prod_{j \in (J_1 \setminus J_2)} \omega_j \cdot \prod_{j \in (J_2 \setminus J_1)} \omega_j\right) \text{ (\#\#\#)} \\ &= E_\eta\left(\prod_{j \in (J_1 \setminus J_2)} \omega_j\right) E_\eta\left(\prod_{j \in (J_2 \setminus J_1)} \omega_j\right) = 0 \text{ (\#\#\#\#)} \end{aligned}$$

In (\#\#), we have used the definition of  $J_1$  and  $J_2$ , and in (\#\#\#), we have used the fact that  $(J_1 \cup J_2) = (J_1 \cap J_2) \sqcup (J_1 \setminus J_2) \sqcup (J_2 \setminus J_1)$ , and  $\omega_j^2 = 1$ , for  $1 \leq j \leq \nu$ . Finally, in (\#\#\#\#), we have used the facts that  $(J_1 \setminus J_2)$  and  $(J_2 \setminus J_1)$  are disjoint, and at least one of these sets is nonempty, the result of (\#) and a similar generalisation of (\*\*). This shows that the functions  $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$  are orthogonal, (\*\*). That they form a basis for  $V(\overline{\Omega}_\eta)$  follows immediately, by transfer, from (\*\*) and the corresponding fact for finite dimensional vector spaces. The final calculation is left to the reader.  $\square$

We require the following;

**Definition 0.4.** For  $0 \leq l \leq \nu$ , we define  $\sim'_l$ , on  $C_\nu$ , to be the internal equivalence relation given by;

$$\bar{t}_1 \sim_l' \bar{t}_2 \text{ iff } \bar{t}_1(j) = \bar{t}_2(j) \ (\forall j \leq l)$$

We extend this to an internal equivalence relation on  $\bar{\Omega}_\eta$ , which we denote by  $\sim_l$ ;

$$x_1 \sim_l x_2 \text{ iff } \psi_\eta(x_1) \sim_l \psi_\eta(x_2) \ (*)$$

We let  $\mathcal{C}_\eta^l$  be the  $*$ -finite algebra generated by the partition of  $\bar{\Omega}_\eta$  into the  $2^l$  equivalence classes with respect to  $\sim_l, (*)$ . As is easily verified, we have  $\mathcal{C}_\eta^{l_1} \subseteq \mathcal{C}_\eta^{l_2}$ , if  $l_1 \leq l_2$ ,  $\mathcal{C}_\eta^0 = \{\emptyset, \bar{\Omega}_\eta\}$  and  $\mathcal{C}_\eta = \mathcal{C}_\eta^\nu$ . For  $0 \leq l \leq \nu$ , we let  $W(\mathcal{C}_\eta^l) \subseteq W(\mathcal{C}_\eta)$  be the set of measurable functions  $f : \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ , with respect to  $\mathcal{C}_\eta^l$ . We will refer to the collection  $\{\mathcal{C}_\eta^l : 0 \leq l \leq \nu\}$  of  $*$ -finite algebras, as the nonstandard filtration associated to  $\bar{\Omega}_\eta$ . We produce a standard filtration  $\{\mathfrak{D}_t : t \in [0, 1]\}$ , (\*\*), by following the method of [2], see Definition 7.14 of [5], (replacing the equivalence relation  $\sim$  there, by  $\sim_l$ , as given in (\*), and being careful to use the index  $\nu$  instead of  $\eta$ . Note that Lemma 7.15 of [5] still applies in this case.) We also require a slight modification of the construction of Brownian motion in [2]. Namely, we take;

$$\chi(t, x) = \frac{1}{\sqrt{\nu}} ({}^*\sum_{i=1}^{\lfloor \nu t \rfloor} \omega_i), \quad ({}^2)$$

$$\text{and } W(t, x) = {}^\circ\chi(t, x), \quad (t, x) \in [0, 1] \times \bar{\Omega}_\eta \quad (**).$$

One of the advantages of the non-standard approach to stochastic calculus, is that it allows one to show easily that every stochastic integral is a martingale. We follow the notation from Chapter 7 of [5], again using the filtration (\*\*) of Definition 0.4 to replace the one from Definition 7.14, and its subsequent applications;

**Theorem 0.5.** *If  $g \in \mathcal{G}_0$ , and  $f$  is a 2-lifting of  $g$ , then  $I(t, x)$ , as in Definition 7.20 of [5], is equivalent, as a stochastic process, to a martingale, with respect to the filtration  $\mathfrak{D}_t$ , ({}^3).*

<sup>2</sup> We adopt the convention that the sum is zero, when  $t = 0$

<sup>3</sup> By which I mean a function  $I : [0, 1] \times \bar{\Omega}_\eta \rightarrow \mathcal{R}$ , such that;

(i).  $I$  is  $\mathfrak{B} \times \mathfrak{D}$  measurable (complete product).

(ii).  $I_t$  is measurable with respect to  $\mathfrak{D}_t$ , for  $t \in [0, 1]$ .

(iii).  $E(|I_t|) < \infty$ , for  $t \in [0, 1]$ .

*Proof.* Let  $I'$  be the modification of  $I$ , as given in the proof of Theorem 7.25 of [5]. Then  $I'$  and agree  $I$  on  $[0, 1] \times C$ , where  $P(C) = 1$ , and  $P = L(\mu_\eta)$ , so they are equivalent as stochastic processes. We show that  $I'$  is a martingale.

(i) follows from the fact that  $I$  is  $\mathfrak{B} \times \mathfrak{D}$  measurable, and  $I = I'$  a.e  $\mu \times L(\mu_\eta)$ , (\*). Here, completeness of the product is required.

(ii). By the construction in the proof of Theorem 7.25 of [5],  $I'_t$  is measurable with respect to  $\mathfrak{D}'_t \subset \mathfrak{D}_t$ .

(iii). We have, for  $t \in [0, 1]$ ;

$$\begin{aligned} \int_{\overline{\Omega}_\eta} I'^2(t, x) dL(\mu_\eta) &= \int_{\overline{\Omega}_\eta} I^2(t, x) dL(\mu_\eta) \\ &= \int_{\overline{\Omega}_\eta} \circ F^2(t, x) d\mu_\eta \\ &\leq \circ \int_{\overline{\Omega}_\eta} F^2(t, x) d\mu_\eta \\ &= \circ \int_{\overline{\Omega}_\eta} \int_0^t f^2(t, x) d\lambda_\nu d\mu_\eta = \|g\|_{L^2([0, t] \times \overline{\Omega}_\eta)}^2 \quad (\dagger) \end{aligned}$$

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(iv).  $E(I_t | \mathfrak{D}_s) = I_s$ , if  $s < t$  belong to  $[0, 1]$ .

(v). For  $C \subset \overline{\Omega}_\eta$ , with  $L(\mu_\eta)(C) = 1$ , and  $x \in C$ , the paths  $\gamma_x : [0, 1] \rightarrow \mathcal{R}$ , where  $\gamma_x(t) = I(t, x)$ , are continuous.

Most of this definition can be found in [7], see also [8] for a thorough discussion of discrete time martingales. We call a martingale tame if it satisfies the additional conditions that;

(vi).  $I_1 \in L^2(\overline{\Omega}_\eta, L(\mu_\eta))$  and, for  $0 \leq s < t \leq 1$ ;

$$\int_{\overline{\Omega}_\eta} (I_t^2 - I_s^2) dL(\mu_\eta) \leq C(t - s)$$

where  $C \in \mathcal{R}_{\geq 0}$

(vii) (UI) For a.a.s,  $0 \leq s < 1$  and sufficiently small  $h > 0$ ,  $\frac{[I]_{s+h} - [I]_s}{h}$  is strongly uniformly integrable in the sense that there exists  $f : \mathcal{R} \rightarrow \mathcal{R}$ , with  $f \geq 0$  and  $\lim_{x \rightarrow \infty} f(x) = 0$  such that, for  $K > 0$ ,  $K \in \mathcal{R}$ ;

$$\int_{\frac{[I]_{s+h} - [I]_s}{h} > K} \frac{[I]_{s+h} - [I]_s}{h} dL(\mu_\eta) < f(K).$$

where  $[I]$  denotes the quadratic variation of the process  $I$ .

using (\*), Definition 7.20, (see notation in Theorem 7.24), Theorem 3.16 and the proof of Theorem 7.22 in [5]. Hence  $I'_t \in L^2(\overline{\Omega}_\eta, \mathcal{C}_\eta, P)$ , so  $I'_t \in L^1(\overline{\Omega}_\eta, \mathcal{C}_\eta, P)$ , by Holder's inequality, see [6].

(iv). Suppose  $s < t$ . We first show that  $E(I'_t | \mathfrak{D}'_s) = I'_s$ , ( $\dagger\dagger$ ). Suppose  $i \in {}^*\mathcal{N}$ , with  $\frac{i}{\nu} \simeq s$ , then we claim that  $E(I'_t | \sigma(\mathcal{C}_\eta^i)^{comp}) = I'_s$ , (\*\*). As  $I_t = I'_t$  a.e  $P$ , we have  $E(I'_t | \sigma(\mathcal{C}_\eta^i)^{comp}) = E(I_t | \sigma(\mathcal{C}_\eta^i)^{comp})$ . We can also see that  $F_t \in SL^2(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$ . This follows from the calculation ( $\dagger$ ), Theorem 3.34(i) of [5], and the fact that;

$$\int_{\overline{\Omega}_\eta} I^2(t, x) dL(\mu_\eta) = \|g\|_{L^2([0, t] \times \overline{\Omega}_\eta)}^2$$

by Ito's isometry, as  $g \in \mathcal{G}_0$ . Hence, by Theorem 3.34(iv) of [5],  $F_t \in SL^1(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$ , (\*\*\*)). Applying Theorem 7.3(ii) of [5] and (\*\*\*));

$$E(I_t | \sigma(\mathcal{C}_\eta^i)^{comp}) = E({}^\circ F_t | \sigma(\mathcal{C}_\eta^i)^{comp}) = {}^\circ E(F_t | \mathcal{C}_\eta^i)$$

We have;

$$E(F_t | \mathcal{C}_\eta^i) = \sum_{j=0}^{i-1} f(\frac{j}{\nu}, x) \frac{\omega_{j+1}}{\sqrt{\nu}}$$

by \*-independence of the sequence  $\{\omega_j\}_{0 \leq j \leq [\nu t] + 1}$ . Letting  $s' = \frac{i-1}{\nu}$ , so  $s' \simeq s$ ,  $E(F_t | \mathcal{C}_\eta^i) = F_{s'}$ . We have, using Theorem 7.24 of [5], that  $I_s = I_{s'}$  a.e  $P$ , so  $I'_s = I_s = I_{s'}$  a.e  $P$ . As  $I'_s$  is  $\sigma(\mathcal{C}_\eta^i)^{comp}$ -measurable, we have  $E(I'_t | (\mathcal{C}_\eta^i)^{comp}) = I'_s$ , showing (\*\*). As  $\mathfrak{D}'_s \subset \sigma(\mathcal{C}_\eta^i)^{comp}$ , and  $I'_s$  is  $\mathfrak{D}'_s$ -measurable, we have  $E(I'_t | \mathfrak{D}'_s) = I'_s$ , showing ( $\dagger\dagger$ ).

If  $A \in \mathfrak{D}_s$ , then, by Lemma 7.15(i) of [5],  $A \in \mathfrak{D}'_{s_1}$ , for  $s < s_1 < t$ . As  $E(I'_t | \mathfrak{D}'_{s_1}) = I'_{s_1}$ , to show (iv), it is sufficient to prove that;

$$\int_A I'_s dL(\mu_\eta) = \lim_{s_1 \rightarrow s} \int_A I'_{s_1} dL(\mu_\eta) \quad (\dagger\dagger\dagger)$$

To show ( $\dagger\dagger\dagger$ ), observe that  $\|I'_{s_1} - I'_s\|_2^2 \leq \|g_{[0, s_1]} - g_{[0, s]}\|_2^2$  by ( $\dagger$ ), where  $g_{[0, s_1]}$  is obtained by truncating the function  $g$  to the interval  $[0, s_1]$ , (<sup>4</sup>). Using Holder's inequality and the DCT, we have  $\lim_{s_1 \rightarrow s} \|I'_{s_1} - I'_s\|_1 \leq \lim_{s_1 \rightarrow s} \|g_{[0, s_1]} - g_{[0, s]}\|_1 = 0$ . Therefore, ( $\dagger\dagger\dagger$ ) is shown. This proves (iv).

<sup>4</sup>Technically, you need to show that  $I_{s_1}$  is the non standard stochastic integral of  $g_{[0, s_1]}$ , and then apply Theorem 7.22 of [5], however, this is clear by truncating the corresponding lift of  $g$ .

(v). This is Theorem 25 of [2].

□

We proceed to show the converse, that every martingale can be represented as a stochastic integral, using the nonstandard approach.

**Lemma 0.6.** *For  $0 \leq l \leq \nu$ , a basis of the  $*$ -finite vector space  $W(\mathcal{C}_\eta^l)$  is given by  $D_l = \bigcup_{0 \leq m \leq l} B_m$ , where, for  $1 \leq m \leq \nu$ ,  $B_m = \{\omega_{\bar{t}} : \bar{t}(m) = 1, \bar{t}(m') = -1, m < m' \leq \nu\}$ , and  $B_0 = \{\omega_{-1}\}$ .*

*Proof.* The case when  $l = 0$  is clear as  $\omega_{-1} = 1$ , and using the description of  $\mathcal{C}_\eta^0$  in Definition 0.4. Using the observation (\*) there, we have, for  $1 \leq l \leq \nu$ , that  $W(\mathcal{C}_\eta^l)$  is a  $*$ -finite vector space of dimension  $2^l$ . Using Lemma 0.3, and the fact that  $\text{Card}(D_l) = 2^l$ , it is sufficient to show each  $\omega_{\bar{t}} \in D_l$  is measurable with respect to  $\mathcal{C}_\eta^l$ . We have that, for  $1 \leq j \leq l$ ,  $\omega_j$  is measurable with respect to  $\mathcal{C}_\eta^j \subseteq \mathcal{C}_\eta^l$ . Hence, the result follows easily, by transfer of the result for finite measure spaces, that the product  $X_{j_1} X_{j_2}$ , of two measurable random variables  $X_{j_1}$  and  $X_{j_2}$  is measurable.

□

**Definition 0.7.** *We define a nonstandard martingale to be a  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable function  $Y : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow *C$ , such that;*

(i). *For  $t \in \overline{\mathcal{T}}_\nu$ ,  $Y_{\lfloor \frac{\nu t}{\nu} \rfloor}$  is measurable with respect to  $\mathcal{C}_\eta^{\lfloor \frac{\nu t}{\nu} \rfloor}$ .*

(ii).  *$E_\eta(Y_{\lfloor \frac{\nu t}{\nu} \rfloor} | \mathcal{C}_\eta^{\lfloor \frac{\nu s}{\nu} \rfloor}) = Y_{\lfloor \frac{\nu s}{\nu} \rfloor}$ , for  $(0 \leq s \leq t \leq 1)$ .*

(iii).  *$E_\eta(|Y_{\lfloor \frac{\nu t}{\nu} \rfloor}|)$  is finite.*

*We say that  $Y$  is  $S$ -continuous, if there exists  $C \subset \overline{\Omega}_\eta$  with  $L(\mu_\eta)(C) = 1$ , such that for  $x \in C$ ,  $Y(t, x) \simeq Y(s, x)$ , when  $s \simeq t$ , and each  $Y(t, x)$  is near standard. We say that  $Y$  has infinitesimal increments if, for all  $x \in \overline{\Omega}_\eta$ , and  $t \in \overline{\mathcal{T}}_\nu$ ,  $t \neq 1$ ,  $Y(\frac{\lfloor \nu t \rfloor + 1}{\nu}, x) \simeq Y(\frac{\lfloor \nu t \rfloor}{\nu}, x)$ .*

**Lemma 0.8.** *Let  $Y : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow *R$  be a  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable function, satisfying (i) and (ii) of Definition 0.7, then;*

$$Y_t(x) = \sum_{j=0}^{\lfloor \frac{\nu t}{\nu} \rfloor} c_j(t, x) \omega_j(x) \quad (*)$$

*where  $c_0 : [0, 1] \times \overline{\Omega}_\eta \rightarrow *C$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta^0$ -measurable,  $c_j : [\frac{j}{\nu}, 1] \times \overline{\Omega}_\eta \rightarrow *C$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$ -measurable, for  $1 \leq j \leq \nu$ , and  $c_0(s, x) =$*

$c_0(t, x)$ , for  $0 \leq s \leq t \leq 1$ ,  $c_j(s, x) = c_j(t, x)$ , for  $\frac{j}{\nu} \leq s \leq t \leq 1$ . Conversely, if  $\{c_j : 0 \leq j \leq \nu\}$  is a collection of functions satisfying the above conditions, then the definition (\*) produces a  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable function, satisfying (i) and (ii) of Definition 0.7.

*Proof.* Using (ii), we have that  $E_\eta(Y_t) = E_\eta(Y_t | \mathcal{C}_\eta^0) = Y_0$ . Replacing  $Y_t$  by  $Y_t - Y_0$ , we can, without loss of generality, assume that  $E_\eta(Y_t) = 0$ , for  $t \in {}^*[0, 1]$ . By (i) and Lemma 0.6;

$$Y_t = \sum_{j=1}^{[\nu t]} c_j(t, x) \omega_j(x)$$

where;

$$c_j(t, x) = \sum_{a=0}^{j-1} \sum_{i_0 < \dots < i_a; 0}^{j-1} p_j^{(i_0, \dots, i_a)}(t) \omega_{i_0} \dots \omega_{i_a}(x)$$

Clearly,  $c_j$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$ -measurable. Again, using (ii), and the fact that  $c_k(t, x) \omega_k$  is orthogonal to the basis  $D_{[\nu s]}$  of  $W(\mathcal{C}_\eta^{[\nu s]})$ , for  $[\nu s] < k \leq [\nu t]$ , ( $\dagger$ ), we have;

$$\sum_{j=1}^{[\nu s]} c_j(t, x) \omega_j(x) = \sum_{j=1}^{[\nu s]} c_j(s, x) \omega_j(x)$$

Equating coefficients, and using the fact that  $D_j$  is a basis for  $W(\mathcal{C}_\eta^j)$ , for  $1 \leq j \leq [\nu s]$ , we obtain  $c_j(s, x) = c_j(t, x)$ , for all  $\frac{j}{\nu} \leq s \leq t \leq 1$ .

The converse is easy to check. (i) is obtained, observing that for  $t \in \mathcal{T}_\nu$ , all the functions  $c_{j,t}$  and  $\omega_j$  are measurable with respect to  $\mathcal{C}_\eta^{[\nu t]}$ , for  $0 \leq j \leq [\nu t]$ . To obtain (ii), just take the conditional expectation of (\*) and make the observation  $\dagger$  again.

□

**Lemma 0.9.** *Let  $X$  be a martingale, see footnote 3 for the definition, with the extra condition that  $X_1 \in L^2(\overline{\Omega}_\eta)$ , then there exists a nonstandard martingale  $\overline{X}$ , see Definition 0.7, with  ${}^\circ(\overline{X}_t) = X_{\circ t}$ , for  $t \in \overline{\mathcal{T}}_\nu$ , a.e  $L(\mu_\eta)$ , and such that the sequence  $\{\overline{X}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\overline{\Omega}_\eta, \mu_\eta)$ . Moreover,  $\overline{X}$  is  $S$ -continuous, and we can take  $\overline{X}$  to have infinitesimal increments.*

*Proof.* By (i) of footnote 3, we have  $X$  is  $\mathfrak{B} \times \mathfrak{D}$ -measurable. We claim that  $X \in L^1([0, 1] \times \overline{\Omega}_\eta)$ , (\*). Without loss of generality, we can assume

that  $X \geq 0$ , <sup>(5)</sup> Then  $(*)$  follows from the fact that, for  $0 \leq t \leq 1$ ,  $E(X_t) = E(X_t | \mathfrak{D}_0) = X_0$ , by  $(iv)$  of footnote 3, and so;

$$\int_{[0,1] \times \overline{\Omega}_\eta} X(t, x) d(L(\lambda_\nu) \times L(\mu_\eta)) = X_0 < \infty$$

by  $(iii)$  of footnote 3 and Fubini's theorem, see [6]. By the hypothesis that  $X_1 \in L^2(\overline{\Omega}_\eta)$ , and using Theorem 7 of [2], see also Theorems 3.31 and 3.34 of [5], we can find  $\overline{V} \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$ , with  $(\circ\overline{V}) = X_1$ , a.e  $L(\mu_\eta)$ ,  $(\dagger)$ . We now define  $\overline{X} : \overline{T}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$  by taking  $\overline{X}(t, x) = (E_\eta(\overline{V} | \mathcal{C}_\eta^{[\nu t]}))(x)$ . We may assume that  $\overline{X}$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta$  measurable, by the definition of  $E_\eta(\cdot)$ , see footnote 25 of Chapter 7, [5], and transfer of the corresponding result for finite measure spaces. Then, by Theorem 7.3 of [5];

$$(\circ\overline{X})(t, x) = \circ(E_\eta(\overline{V} | \mathcal{C}_\eta^{[\nu t]}))(x) = E((\circ\overline{V}) | \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}) (**)$$

Moreover, if  $A \in \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}$ , we have;

$$\int_A X_{\circ t} dL(\mu_\eta) = \lim_{t' \rightarrow \circ t} \int_A X_{t'} dL(\mu_\eta) = \int_A X_1 dL(\mu_\eta) (***)$$

using  $(iv), (v)$  of footnote 3 and the result of  $(*)$  to apply the *DCT*. Hence, as  $\mathfrak{D}_{\circ t} \subset \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp} \subset \mathfrak{D}_{t'}$ , for  $0 \leq \circ t < t'$ , using  $(**)$  in Definition 0.4, we have;

$$E((\circ\overline{V}) | \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}) = E((\circ\overline{V}) | \mathfrak{D}_{\circ t}) = E(X_1 | \mathfrak{D}_{\circ t}) = X_{\circ t}$$

by  $(***), (\dagger)$  and  $(iv)$  of footnote 3. By  $(**)$ , we then have  $(\circ\overline{X}_t) = X_{\circ t}$ , a.e  $L(\mu_\eta)$ . We now verify conditions  $(i), (ii), (iii)$  of Definition 0.7.  $(i)$  is clear by Definition of  $\overline{X}$  and footnote 25 of Chapter 7, [5].  $(ii)$  follows by transfer of the tower law for the conditional expectation  $E_\eta(\cdot)$ , see again footnote 25 of Chapter 7.  $(iii)$  follows immediately

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<sup>5</sup> In order to see this, it is sufficient to show that  $X^+$  is a martingale,  $(*)$ . We have  $X = X^+ - X^-$ , and, by  $(iv)$ , for  $0 \leq t \leq 1$ ;

$$X_t = X_t^+ - X_t^- = E(X_1 | \mathfrak{D}_t) = E(X_1^+ - X_1^- | \mathfrak{D}_t) = Y_t - Y_t' (**)$$

where  $Y_t = E(X_1^+ | \mathfrak{D}_t)$  and  $Y_t' = E(X_1^- | \mathfrak{D}_t)$ . It follows easily, modifying  $Y$  to  $Y^1$ , and  $Y'$  to  $Y'^1$ , a.e  $L(\lambda_\nu) \times L(\mu_\eta)$ , if necessary, and, using the tower law and definition of conditional expectations, see [8], that  $Y, Y'$  are martingales and  $Y, Y' \geq 0$ . We then have, by  $(**)$ , that  $X_t^+ = Y_t$  and  $X_t^- = Y_t'$  a.e  $L(\mu_\eta)$ . Hence,  $(*)$  is shown.

from the fact that  $\bar{V} \in SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , and;

$$|E_\eta(\bar{X}_t)| = |E_\eta(\bar{V})| \leq E_\eta(|\bar{V}|) \leq \|\bar{V}\|_{SL^2} \simeq \|X_1\|_{L^2} < \infty \text{ (for } t \in \bar{\mathcal{T}}_\nu)$$

by transfer of Holders inequality, the definition of  $E_\eta(|\cdot|)$ , and property (ii) in Definition 0.7. Finally, using Theorem 7.3 of [5], we have that the sequence  $\{\bar{X}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\bar{\Omega}_\eta, \mu_\eta)$ . The  $S$ -continuity claim follows from the proof of Theorem 8.1 in [3]. We omit the details. For the final claim, we modify  $\bar{X}$  to obtain the final condition, while preserving the other properties. For  $n \in {}^*\mathcal{N}$ , we let;

$$V_n = \{x : \exists t(|\Delta\bar{X}(t, x)| \geq \frac{1}{n})\}, \text{ (}^6\text{)}$$

By  $S$ -continuity of  $\bar{X}$ , we have that the internal set  $A = \{n \in {}^*\mathcal{N} : \mu_\eta(V_n) \leq \frac{1}{n}\}$  contains  $\mathcal{N}$ , hence, it contains an infinite element  $\kappa$ . For  $x \in V_\kappa$ , we let  $\tau(x)$  be the first  $t$  such that  $|\Delta\bar{X}(t, x)| \geq \frac{1}{\kappa}$  and let  $\tau(x) = 1$  otherwise. We let  $\bar{W}$  be the internal process defined by;

$$\bar{W}_0 = \bar{X}_0$$

$$\Delta\bar{W}(x, t) = \Delta\bar{X}(x, t), \text{ if } t < \tau(x).$$

$$\Delta\bar{W}(x, t) = 0, \text{ if } t \geq \tau(x).$$

We claim that  $\bar{W}$  is a nonstandard martingale in the sense of Definition 0.7. For (i), by hyperfinite induction, and the fact that  $\bar{W}_0 = \bar{X}_0$ , it is sufficient to show that if  $\bar{W}_{\frac{i-1}{\nu}}$  is measurable with respect to  $\mathcal{C}_\eta^{i-1}$ , then  $\bar{W}_{\frac{i}{\nu}}$  is measurable with respect to  $\mathcal{C}_\eta^i$ , for  $1 \leq i \leq \nu$ , ( $\dagger\dagger$ ). If  $x \sim_i x'$ , we have that  $\frac{i-1}{\nu} < \tau(x)$  iff  $\frac{i-1}{\nu} < \tau(x')$ , as this is an internal definition depending only on information up to time  $\frac{i}{\nu}$ , hence must contain the equivalence class  $[x]_{\sim_i}$ . In this case, we have that  $\bar{W}(x, \frac{i}{\nu}) = \bar{W}(x, \frac{i-1}{\nu}) + \Delta\bar{X}(x, \frac{i-1}{\nu})$ , which is constant on  $[x]_{\sim_i}$ , using the inductive hypothesis and measurability of  $\bar{X}$ . The case when  $\frac{i-1}{\nu} \geq \tau(x)$  is similar. Hence, ( $\dagger\dagger$ ) and (i) are shown. For (ii), it is sufficient to show that if  $x \in \bar{\Omega}_\eta$ , then;

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<sup>6</sup>We use the notation  $\Delta\bar{X}(t, x)$  to denote the increment  $\bar{X}(t + \frac{1}{\nu}, x) - \bar{X}(t, x)$ , for  $0 \leq t \leq 1 - \frac{1}{\nu}$

$$\int_{[x]_{\sim_{i-1}}} \overline{W}_{\frac{i-1}{\nu}} d\mu_\eta = \int_{[x]_{\sim_{i-1}}} \overline{W}_{\frac{i}{\nu}} d\mu_\eta, \text{ for } 1 \leq i \leq \nu \ (\dagger\dagger\dagger)$$

Clearly, if  $\frac{i-1}{\nu} \geq \tau(x')$ , for all  $x' \in [x]_{\sim_{i-1}}$ , then  $\Delta \overline{W}(x, \frac{i-1}{\nu})|_{[x]_{\sim_{i-1}}} = 0$ , and the result  $(\dagger\dagger\dagger)$  follows trivially. Similarly, if  $\frac{i-1}{\nu} < \tau(x')$ , for all  $x' \in [x]_{\sim_{i-1}}$ , then  $\overline{W}|_{[x]_{\sim_{i-1}} \times [\frac{i-1}{\nu}, \frac{i+1}{\nu})} = \overline{X}|_{[x]_{\sim_{i-1}} \times [\frac{i-1}{\nu}, \frac{i+1}{\nu})}$ , and the result  $(\dagger\dagger\dagger)$  follows from the martingale property of  $\overline{X}$ . We can, therefore, write  $[x]_{\sim_{i-1}} = [x_1]_{\sim_i} \cup [x_2]_{\sim_i}$ , and assume that  $\frac{i-1}{\nu} < \tau(x')$ , for all  $x' \in [x_1]_{\sim_i}$ , and  $\frac{i-1}{\nu} \geq \tau(x')$ , for all  $x' \in [x_2]_{\sim_i}$ . If  $\frac{i-2}{\nu} \geq \tau(x')$ , for all  $x' \in [x_2]_{\sim_i}$ , then the same must hold for all  $x' \in [x_1]_{\sim_i}$ , contradicting the assumption. Hence, we can also assume that  $\frac{i-2}{\nu} < \tau(x')$ , for all  $x' \in [x_2]_{\sim_i}$ . It follows that  $|\Delta \overline{X}(x, \frac{i-1}{\nu})|_{[x_1]_{\sim_i}} \leq \frac{1}{\kappa}$  and  $|\Delta \overline{X}(x, \frac{i-1}{\nu})|_{[x_2]_{\sim_i}} > \frac{1}{\kappa}$ , but this contradicts the martingale property  $(\dagger\dagger\dagger)$  for  $\overline{X}$ . Hence, this case can't happen, so  $(\dagger\dagger\dagger)$  and  $(ii)$  is shown. Property  $(iii)$  follows from the fact that  $\overline{W}_1 \in SL^2(\overline{\Omega}_\eta)$ ,  $(\dagger\dagger\dagger\dagger)$ , which we show below, and the inequality;

$$E_\eta(|\overline{W}_{\frac{\nu t}{\nu}}|) \leq E_\eta(\overline{W}_{\frac{\nu t}{\nu}}^2)^{\frac{1}{2}} \leq E_\eta(\overline{W}_1^2)^{\frac{1}{2}}.$$

which uses Cauchy-Schwartz, and the martingale property  $(ii)$ . By construction  $\overline{W}$  has infinitesimal increments. As we are only modifying  $\overline{X}$  inside  $V_\kappa \times \mathcal{T}_\nu$ , where  $L(\mu_\eta)(V_\kappa) = 0$  it is clear that  $S$ -continuity is preserved. Similarly, we must have that  ${}^\circ(\overline{W}_t) = X_{\circ t}$ , for  $t \in \overline{\mathcal{T}}_\nu$ , a.e  $L(\mu_\eta)$ . It remains to show  $(\dagger\dagger\dagger\dagger)$ . By the above remark on modification, it is sufficient to show that  $\int_{V_\kappa} \overline{W}_1^2 d\mu_\eta \simeq 0$ . We can define a relation on  $\overline{\Omega}_\eta$  by  $x \sim x'$  if  $x' \in [x]_{\tau(x)-1}$ . If  $x \sim x'$ , then, by the above discussion,  $\tau(x) = \tau(x')$ , and so  $\sim$  defines an equivalence relation. We clearly have that  $V_\kappa = \bigcup_{1 \leq j \leq r} [x_j]_{\sim}$  is an internal union of such equivalence classes. A simple calculation gives that;

$$\begin{aligned} \int_{V_\kappa} \overline{W}_1^2 d\mu_\eta &= {}^* \sum_{1 \leq j \leq r} \int_{[x_j]_{\sim}} \overline{W}_1^2 d\mu_\eta \\ &= {}^* \sum_{1 \leq j \leq r} \int_{[x_j]_{\tau(x_j)-1}} \overline{X}_{\tau(x_j)-1}^2 d\mu_\eta \\ &\leq {}^* \sum_{1 \leq j \leq r} \int_{[x_j]_{\tau(x_j)-1}} \overline{X}_1^2 d\mu_\eta \\ &= \int_{V_\kappa} \overline{X}_1^2 d\mu_\eta \simeq 0 \end{aligned}$$

where we have used the definition of  $\overline{W}$ , and the calculation of Theorem 12(ii) in [2]. This gives the result.

□

**Lemma 0.10.** *Let  $X$  be a tame martingale, and let  $\bar{X}$  be as in Lemma 0.9. Then we can find  $\kappa \in {}^*\mathcal{N} \setminus \mathcal{N}$  such that  $\kappa|\nu$ , and for all  $t \in \mathcal{T}_\nu$ ;*

$$\int_{\bar{\Omega}_\eta} (\bar{X}_{t+\frac{1}{\kappa}}^2 - \bar{X}_t^2) d\mu_\eta \leq \frac{C+1}{\kappa}$$

where  $C \in \mathcal{R}_{\geq 0}$  is as given in footnote 3. Moreover, we can find  $D \subset \bar{\Omega}_\eta$ , with  $\mu_\eta(D) \simeq 1$ ,  $E \subset \mathcal{T}_\nu$  with  $\mu_\eta(E) \simeq 0$ , such that for all  $t \in \mathcal{T}_\nu \setminus E$ ;

$$1_D \kappa([\bar{X}]_{t+\frac{1}{\kappa}} - [\bar{X}]_t) \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$$

*Proof.* Without loss of generality we can assume that  $n|\nu$ , for all  $n \in \mathcal{N}$ . If  $t \in \mathcal{T}_\nu$  and  $n \in \mathcal{N}$ , we have that  $\{\bar{X}_t, \bar{X}_{t+\frac{1}{n}}\} \subset SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , hence  $(\bar{X}_{t+\frac{1}{n}}^2 - \bar{X}_t^2) \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$ . We, therefore, have that;

$$\begin{aligned} & \circ(\int_{\bar{\Omega}_\eta} (\bar{X}_{t+\frac{1}{n}}^2 - \bar{X}_t^2) d\mu_\eta) \\ &= \int_{\bar{\Omega}_\eta} (\circ(\bar{X}_{t+\frac{1}{n}})^2 - \circ(\bar{X}_t)^2) dL(\mu_\eta) \\ &= \int_{\bar{\Omega}_\eta} (X_{\circ t+\frac{1}{n}}^2 - X_{\circ t}^2) dL(\mu_\eta) \leq \frac{C}{n} \end{aligned}$$

It follows that;

$$\int_{\bar{\Omega}_\eta} (\bar{X}_{t+\frac{1}{n}}^2 - \bar{X}_t^2) d\mu_\eta \leq \frac{C+1}{n}$$

As this holds for all  $n \in \mathcal{N}$ , and the property is internal, we can find an infinite  $\kappa|\nu$ , such that;

$$\int_{\bar{\Omega}_\eta} (\bar{X}_{t+\frac{1}{\kappa}}^2 - \bar{X}_t^2) d\mu_\eta \leq \frac{C+1}{\kappa}$$

for all  $t \in \mathcal{T}_\nu$ , as required, for the first part.

For the second condition, using Proposition 4.4.12 in [1], we can assume that there exists  $C \subset \bar{\Omega}_\eta$ , with  $L(\mu_\eta)(C) = 1$ , such that  $[\bar{X}]$  lifts the standard process  $[X]$  on  $C \times \mathcal{T}_\nu$ . For ease of notation, for  $m \in \mathcal{N}$  and  $t \in \mathcal{T}_\nu$ , let  $[\bar{X}]_{t,m}$  denote the increment  $m([\bar{X}]_{t+\frac{1}{m}} - [\bar{X}]_t)$  and  $[X]_{t,m}$  the corresponding standard increment, for  $t \in [0, 1]$ . We clearly have that  $\circ[\bar{X}]_{t,m} = [X]_{\circ t,m}$  on  $C \times \mathcal{T}_\nu$ . Choose a sequence of  $\{C_m : m \in \mathcal{N}\}$ , with  $C_m \subset C$ , such that each  $C_m \in \mathcal{C}_\eta$  and  $\mu_\eta(C_m) = 1 - \frac{1}{m}$ . As  $C_m$

is internal and  $[\overline{X}]$  lifts  $X$  on  $C_m \times \mathcal{T}_\nu$ , by compactness, we must have that  $[\overline{X}]$  is bounded on  $C_m \times \mathcal{T}_\nu$ ,  $||[\overline{X}]|| \leq D(m)$ , where  $D(m) \in \mathcal{R}$ . Let  $V \subset [0, 1]$  be the set on which the incremental condition (vii) in Definition 3 does not hold. Then  $L(\lambda_\nu)(st^{-1}(V)) = 0$ , and we can choose  $E_m \in \mathcal{C}_\nu$ , with  $\lambda_\nu(E_m) = \frac{1}{m}$ , such that  $E_m \supset st^{-1}(V)$ . Then, we have that, for all  $t \in \mathcal{T}_\nu \setminus E_m$ , for all  $K \leq 2D(m)m$ , that;

$$\begin{aligned} & \circ \int_{[\overline{X}]_{t,m} > K} 1_{C_m} [\overline{X}]_{t,m} d\mu_\eta \\ & = \circ \int_{\overline{\Omega}_\eta} 1_{([\overline{X}]_{t,m} > K) \cap C_m} [\overline{X}]_{t,m} d\mu_\eta \\ & = \int_{\overline{\Omega}_\eta} 1_{([\overline{X}]_{t,m} > K) \cap C_m} [X]_{\circ t, m} dL\mu_\eta \end{aligned}$$

It follows that;

$$\begin{aligned} & \int_{[\overline{X}]_{t,m} > K} 1_{C_m} [\overline{X}]_{t,m} d\mu_\eta \\ & < \int_{[X]_{\circ t, m} > K-1} 1_{C_m} [X]_{\circ t, m} dL(\mu_\eta) + \frac{1}{m} \\ & < \int_{[X]_{\circ t, m} > K-1} [X]_{\circ t, m} dL(\mu_\eta) + \frac{1}{m} \\ & = f^*(K-1) + \frac{1}{m} \end{aligned}$$

where we have used condition (vii) in the definition from footnote 3. The condition (\*) holds trivially when  $K > 2D(m)m$ , as then;

$$\int_{[\overline{X}]_{t,m} > K} 1_{C_m} [\overline{X}]_{t,m} d\mu_\eta = 0$$

It follows that;

$$*\mathcal{R} \models (\forall t \in \mathcal{T}_\nu \setminus E_m)(\forall K,)$$

$$\int_{[\overline{X}]_{t,m} > K} 1_{C_m} [\overline{X}]_{t,m} d\mu_\eta < f^*(K-1) + \frac{1}{m}$$

for all sufficiently large  $m \in \mathcal{N}$ . By overflow, we can satisfy the condition for the same infinite  $\kappa \in *\mathcal{N}$  as above. In particular, we obtain, for infinite  $K$ ,  $t \in \mathcal{T}_\nu \setminus E_\kappa$  that;

$$\int_{[\overline{X}]_{t,\kappa} > K} 1_{C_\kappa} [\overline{X}]_{t,\kappa} d\mu_\eta < f^*(K-1) + \frac{1}{\kappa} \simeq 0$$

It follows, using the criterion in Lemma 3.19 of [5], that  $1_{C_\kappa}[\bar{X}]_{t,\kappa} \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$ , for all  $t \in \mathcal{T}_\nu \setminus E_\kappa$  as required. Letting  $D = C_\kappa E = E_\kappa$  and noting that  $\mu_\eta(D) = 1 - \frac{1}{\kappa} \simeq 1$ ,  $\mu_\eta(E) = \frac{1}{\kappa} \simeq 0$  we obtain the result.

□

**Definition 0.11.** Let  $\bar{X}$  be as in Definition 0.7, with  $E_\eta(\bar{X}_0) = 0$ , and let  $\{c_j(t, x) : 1 \leq j \leq \nu\}$  be given as in Lemma 0.8. Then we define;

$\bar{H} : \bar{\mathcal{T}}_\nu \times \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ ,  $\bar{Z} : \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ ,  $\bar{Y} : \bar{\mathcal{T}}_\kappa \times \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ ,  $\bar{W} : \bar{\mathcal{T}}_\kappa \times \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ ,  $\{d_j(t, x) : 1 \leq j \leq \nu\}$ ,  $\bar{S} : \bar{\mathcal{T}}_\nu \times \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ ,  $\bar{Q} : \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$  by;

$$\bar{H}(t, x) = \sqrt{\nu} c_{[\nu t]+1}(s, x)$$

where  $s \geq \frac{[\nu t]+1}{\nu}$ , for  $0 \leq t < 1$  and;

$$\bar{H}(t, x) = 0, \text{ for } t = 1$$

$$\bar{Z}(x) = {}^* \sum_{0 \leq j \leq \nu-1} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2$$

$$\bar{Y}(t, x) = 0, \text{ for } 0 \leq [\nu t] < \frac{\nu}{\kappa} - 1$$

$$\bar{Y}(t, x) = \frac{k}{\nu} (\bar{H}_{\frac{[\nu t]}{\nu}}^2 + \bar{H}_{\frac{[\nu t]-1}{\nu}}^2 + \dots + \bar{H}_{\frac{[\nu t]-\frac{\nu}{\kappa}+1}{\nu}}^2), \text{ for } \frac{\nu}{\kappa} - 1 \leq [\nu t] \leq 1$$

$$\bar{W} = \sqrt{\bar{Y}}$$

$$d_j(s, x) = \frac{1}{\sqrt{\nu}} \bar{W}_{\frac{j-1}{\nu}}(x), \text{ for } 1 \leq j \leq \nu, \text{ and } \frac{j}{\nu} \leq s \leq 1.$$

$$\bar{S}(t, x) = {}^* \sum_{j=1}^{[\nu t]} d_j(1, x) \omega_j$$

$$\bar{Q}(x) = {}^* \sum_{0 \leq j \leq \nu-1} (\bar{S}_{\frac{j+1}{\nu}}(x) - \bar{S}_{\frac{j}{\nu}}(x))^2$$

**Lemma 0.12.** If  $\bar{X}$  is as in Lemma 0.9, and  $X$  is tame, then  $\bar{Y} \in SL^1(\bar{\mathcal{T}}_\nu \times \bar{\Omega}_\eta, \lambda_\nu \times \mu_\eta)$ ,  $\bar{Z} \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$  and  $\bar{S}$  is a nonstandard martingale, with  $\bar{S}_1 \in SL^2(\bar{\Omega}_\eta, \mu_\eta)$ .

*Proof.* The fact that  $\bar{Z} \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$ , (†), follows from Proposition 4.4.3 of [1] and the properties of  $\bar{X}$ . This does not require that  $\bar{X}$  is  $S$ -continuous or has infinitesimal increments.

For the last claim, it is easily seen that the functions  $d_j : [\frac{j}{\nu}, 1] \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$  are  $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$ -measurable, for  $1 \leq j \leq \nu$ . Hence, using Lemma 0.8, we have that  $\overline{S}$  satisfies conditions (i) and (ii) of Definition 0.7. By Proposition 4.4.3 of [1], it is sufficient to show that  $\overline{Q} \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$ , as  $\overline{S}_0 = 0$ . (explain why we can assume this?) We compute;

$$\begin{aligned}
\overline{Q}(x) &= {}^* \sum_{0 \leq j \leq \nu-1} (\overline{S}_{\frac{j+1}{\nu}}(x) - \overline{S}_{\frac{j}{\nu}}(x))^2 \\
&= {}^* \sum_{1 \leq j \leq \nu-1} d_j^2(1, x) \\
&= \frac{1}{\nu} {}^* \sum_{1 \leq j \leq \nu-1} \overline{W}_{\frac{j-1}{\nu}}^2(x) \\
&= \frac{1}{\nu} {}^* \sum_{1 \leq j \leq \nu-1} \overline{Y}_{\frac{j-1}{\nu}}(x) \\
&= \frac{1}{\nu} {}^* \sum_{\frac{\nu-k}{\nu}-1 \leq j \leq \nu-2} (\overline{H}_{\frac{j}{\nu}}^2 + \overline{H}_{\frac{j-1}{\nu}}^2 + \dots + \overline{H}_{\frac{j-\frac{\nu-k}{\nu}+1}{\nu}}^2) \\
&= \frac{1}{\nu} {}^* \sum_{0 \leq j \leq \nu-1} \overline{H}_{\frac{j}{\nu}}^2 d\mu_\eta + r(x) \\
&= {}^* \sum_{1 \leq j \leq \nu} c_j^2(1, x) d\mu_\eta + r(x) \\
&= {}^* \sum_{0 \leq j \leq \nu-1} (\overline{X}_{\frac{j+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2 + r(x) \\
&= \overline{Z}(x) + r(x)
\end{aligned}$$

where  $r(x) \geq 0$  is a remainder term. We have that  $E_\eta(r(x)) \simeq 0$ , and  $r(x) \leq \overline{Z}(x)$ . It follows, easily, that  $r(x) \simeq 0$ , a.e  $L(\mu_\eta)$ ,  $\overline{Q}(x) \simeq \overline{Z}(x)$  a.e  $L(\mu_\eta)$ , and  $\overline{Q}(x) \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$  as required.

For the first part, observe first that  $\overline{H}$  is progressively measurable, that is  $\overline{H}_t$  is measurable with respect to  $\mathcal{C}_\eta^{[\nu t]}$ , hence, so is  $\overline{Y}$ .

By Lemma 3.19 of [5], it is sufficient to prove that;

$$\int_{\overline{Y} > K} \overline{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0, \text{ for } K \text{ infinite}$$

As  $\overline{Y}$  is progressively measurable, the set  $\overline{Y} > K$  is progressively measurable. Moreover, it has infinitesimal measure. This clearly follows from showing that;

$$\int_{\mathcal{T}_\nu \times \overline{\Omega}_\eta} \overline{Y} d\lambda_\nu d\mu_\eta \text{ is finite, } (*)$$

To see (\*), we compute;

$$\begin{aligned}
& \int_{\mathcal{T}_\nu \times \overline{\Omega}_\eta} \overline{Y}(t, x) d\lambda_\nu d\mu_\eta \\
&= \frac{1}{\nu} * \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} \overline{Y}\left(\frac{j}{\nu}, x\right) d\mu_\eta \\
&= \frac{1}{\nu} * \sum_{\frac{\nu}{k}-1 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} \overline{Y}\left(\frac{j}{\nu}, x\right) d\mu_\eta \\
&= \frac{1}{\nu} * \sum_{\frac{\nu}{k}-1 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} \left(\frac{k}{\nu} \overline{H}_{\frac{j}{\nu}}^2 + \overline{H}_{\frac{j-1}{\nu}}^2 + \dots + \overline{H}_{\frac{j-\frac{\nu}{k}+1}{\nu}}^2\right) d\mu_\eta \\
&\leq \frac{1}{\nu} * \sum_{0 \leq j \leq \nu} \int_{\overline{\Omega}_\eta} \overline{H}_{\frac{j}{\nu}}^2 d\mu_\eta \\
&= \frac{1}{\nu} * \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} \nu |c_j(1, x)|^2 d\mu_\eta \quad (\dagger\dagger) \\
&= * \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} |c_j(1, x)|^2 d\mu_\eta \\
&= \int_{\overline{\Omega}_\eta} |\overline{X}_1|^2 d\mu_\eta \quad (\dagger\dagger\dagger)
\end{aligned}$$

where, in ( $\dagger\dagger$ ), we have used Definition 0.11, and, in ( $\dagger\dagger\dagger$ ), we have used the fact that  $\overline{X}_1 = * \sum_{0 \leq j \leq \nu-1} c_j(1, x) \omega_j$ , by Lemma 0.8, and the orthogonality observation (\*) there. Hence, (\*) is shown, by the assumption that  $\overline{X}_1 \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$ . Therefore, it is sufficient to prove that;

$\int_A \overline{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0$ , for a progressively measurable set  $A$  with  $\lambda_\nu \times \mu_\eta(A) \simeq 0$ . (\*\*)

We now verify (\*\*);

Case 1. Let  $A \subset \overline{\Omega}_\eta$ , with  $\mu_\eta(A) \simeq 0$ , then;

$$\begin{aligned}
& \int_{A \times \overline{\mathcal{T}}_\eta} \overline{Y} d\mu_\eta d\lambda_\nu \\
&= \frac{1}{\nu} * \sum_{0 \leq j \leq \nu-1} \int_A \overline{Y} d\mu_\eta \\
&\leq \frac{1}{\nu} * \sum_{0 \leq j \leq \nu-1} \int_A \overline{H}_{\frac{j}{\nu}}^2 d\mu_\eta \quad (\text{as above}) \\
&= \int_A * \sum_{0 \leq j \leq \nu-1} c_j(1, x)^2 d\mu_\eta \\
&= \int_A * \sum_{0 \leq j \leq \nu-1} (\overline{X}_{\frac{j+1}{\nu}} - \overline{X}_{\frac{j}{\nu}})^2 d\mu_\eta = \int_A \overline{Z} \simeq 0
\end{aligned}$$

by (†).

Case 2. Let  $B \subset \overline{T}_\nu$ , with  $B \in \mathcal{D}_\nu$  and  $\lambda_\nu(B) \simeq 0$ . We can write  $B = \bigcup_{1 \leq j \leq s} I_j$ , where  $I_j$  is an interval of the form  $[\frac{i_j}{\nu}, \frac{i_j+1}{\nu})$ , for some  $0 \leq i_j \leq \nu - 1$ , and  $\frac{s}{\nu} \simeq 0$ . We compute, for  $i_j \geq \frac{\nu}{\kappa} - 1$ ;

$$\begin{aligned} & \int_{\overline{\Omega}_\eta \times I_j} \overline{Y}(t, x) d\lambda_\nu d\mu_\eta \\ &= \frac{1}{\nu} \int_{\overline{\Omega}_\eta} \left( \frac{\kappa}{\nu} \overline{H}_{\frac{i_j}{\nu}}^2 + \overline{H}_{\frac{i_j-1}{\nu}}^2 + \dots + \overline{H}_{\frac{i_j-\frac{\nu}{\kappa}+1}{\nu}}^2 \right) d\mu_\eta \\ &= \frac{\kappa}{\nu} \int_{\overline{\Omega}_\eta} (c_{i_j+1}^2 + \dots + c_{i_j-\frac{\nu}{\kappa}+2}^2) d\mu_\eta \end{aligned}$$

We have that;

$$\begin{aligned} \overline{X}(t, x) &= \sum_{0 \leq j \leq t\nu} c_j(1, x) \omega_j(x) \\ \overline{X}(t, x)^2 &= \sum_{0 \leq j, k \leq t\nu} c_j(1, x) c_k(1, x) \omega_j(x) \omega_k(x) \quad (\#) \end{aligned}$$

Then;

$$\begin{aligned} & \int_{\overline{\Omega}_\eta} (\overline{X}_t)^2(x) d\mu_\eta \\ &= \sum_{0 \leq j, k \leq [t\nu]} \int_{\overline{\Omega}_\eta} c_j(1, x) c_k(1, x) \omega_j \omega_k d\mu_\eta \quad (\text{using } (\#)) \\ &= \sum_{0 \leq j \leq [t\nu]} \int_{\overline{\Omega}_\eta} c_j^2(1, x) d\mu_\eta \quad (\text{using Lemma 0.8}) \quad (\#\#) \end{aligned}$$

It follows that;

$$\begin{aligned} & \int_{\overline{\Omega}_\eta} (c_{i_j+1}^2 + \dots + c_{i_j-\frac{\nu}{\kappa}+2}^2) d\mu_\eta \\ &= \int_{\overline{\Omega}_\eta} \left( \overline{X}_{\frac{i_j+1}{\nu}}^2 - \overline{X}_{\frac{i_j+1-\frac{\nu}{\kappa}}{\nu}}^2 \right) d\mu_\eta \end{aligned}$$

and, therefore, that;

$$\begin{aligned} & \int_{\overline{\Omega}_\eta \times I_j} \overline{Y}(t, x) d\lambda_\nu d\mu_\eta \\ & \frac{\kappa}{\nu} \int_{\overline{\Omega}_\eta} \left( \overline{X}_{\frac{i_j+1}{\nu}}^2 - \overline{X}_{\frac{i_j+1-\frac{\nu}{\kappa}}{\nu}}^2 \right) d\mu_\eta \leq \frac{C+1}{\nu} \end{aligned}$$

using Lemma 0.10. We then have that;

$$\begin{aligned} & \int_{\bar{\Omega}_\eta \times B} \bar{Y}(t, x) d\lambda_\nu d\mu_\eta \\ &= {}^* \sum_{1 \leq j \leq s} \int_{\bar{\Omega}_\eta \times I_j} \bar{Y}(t, x) d\lambda_\nu d\mu_\eta \leq \frac{s(C+1)}{\nu} \simeq 0 \end{aligned}$$

as required.

Case 3. Let  $B \in \mathcal{D}_\nu \times \mathcal{C}_\eta$ , with  $(\lambda_\nu \times \mu_\eta)(B) = \delta \simeq 0$ . Let;

$$I = \{i : 0 \leq i \leq \nu, \mu_\eta(pr_\eta(B \cap pr_\nu^{-1}(\frac{i}{\nu}))) > \delta^{\frac{1}{2}}\}$$

Let  $C = \bigcup_{i \in I} [\frac{i}{\nu}, \frac{i+1}{\nu})$ , so  $C \in \mathcal{D}_\nu$ , and let  $B_1 = B \cap pr_\nu^{-1}(C)$ . As  $B_1 \subset B$ , and by construction of  $C$ , we have that;

$$\delta \geq (\lambda_\nu \times \mu_\eta)(B_1) > \delta^{\frac{1}{2}} \lambda_\nu(C)$$

It follows that  $\lambda_\nu(C) < \delta^{\frac{1}{2}} \simeq 0$ . By Case 2, we have that;

$$\begin{aligned} & \int_{B_1} \bar{Y} d(\lambda_\nu \times \mu_\eta) \\ & \leq \int_{\bar{\Omega}_\eta \times C} \bar{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0 \end{aligned}$$

Let  $B_2 = B \cap B_1^c$ , then  $B_2 \in \mathcal{D}_\nu \times \mathcal{C}_\eta$  and  $(\lambda_\nu \times \mu_\eta)(B_2) \simeq 0$ , and to show Case 3, it is sufficient to prove that;

$$\int_{B_2} \bar{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0$$

We say that  $B \in \mathcal{D}_\nu \times \mathcal{C}_\eta$  is wide, if there exists  $\epsilon \simeq 0$ , with  $\mu_\eta(pr_\eta(B \cap pr_\nu^{-1}(t))) \leq \epsilon$ , for  $t \in \mathcal{T}_\nu$ , and note that  $B_2$  is wide. We are thus reduced to;

Case 4. Suppose  $B$  is progressively measurable and wide, and let;

$$I_j = \{i \in {}^*\mathcal{N} : 0 \leq i \leq \nu - 1, \text{rem}(2, i) = j, B \cap pr_\nu^{-1}(\frac{i}{\nu}) \neq \emptyset\}, \text{ for } 0 \leq j \leq 1$$

$$S_j = \bigcup_{i \in I_j} [\frac{i}{\nu}, \frac{i+1}{\nu}), 0 \leq j \leq 1$$

$$B_j = B \cap pr_\nu^{-1}(S_j), 0 \leq j \leq 1$$

Then  $B = \bigcup_{0 \leq j \leq 1} B_j$ , and each  $B_j$  is progressively measurable and wide. Let;

$$V_j = \{(i, s) \in {}^* \mathcal{N}^2 : 1 \leq i \leq \nu - 1, 0 \leq s < 2^i, \text{rem}(2, s) = j, B \cap pr_\nu^{-1}(\frac{i}{\nu}) \neq \emptyset, B \cap pr_\eta^{-1}(\frac{s}{\eta}) \neq \emptyset\}, \text{ for } 0 \leq j \leq 1$$

$$W_j = \bigcup_{(i,s) \in V_j} [\frac{i}{\nu}, \frac{i+1}{\nu}) \times [\frac{s}{2^i}, \frac{s+1}{2^i}), 0 \leq j \leq 1$$

By the progressive measurability of  $B$ ,  $B = \bigcup_{0 \leq j \leq 1} W_j$  and each  $W_j$  is progressively measurable and wide. Let  $B_{ij} = B_i \cap W_j$ ,  $0 \leq i \leq 1$ ,  $0 \leq j \leq 1$ . Then  $B = \bigcup_{0 \leq i,j \leq 1} B_{ij}$  and each  $B_{ij}$  is progressively measurable and wide. We say that  $B \in \mathcal{D}_\nu \times \mathcal{C}_\eta$  is separated if, for all  $(t, x) \in B$ ,  $(t + \frac{1}{\nu}) \notin pr_\nu(B)$ , and  $(t, \frac{[x2^{[t\nu]}]+1}{2^{[t\nu]}}) \notin B$ , for  $[t\nu] \geq 1$  and  $0 \leq [x2^{[t\nu]}] \leq 2^{[t\nu]} - 2$ . . By construction, each  $B_{ij}$  is separated, for  $0 \leq i, j \leq 1$ . We are thus reduced to;

Case 5. Suppose  $B$  is progressively measurable, wide and separated.

Observe that;

$$\begin{aligned} & \kappa([\bar{X}]_t - [\bar{X}]_{t-\frac{1}{\kappa}}) \\ &= \kappa(\sum_{j=0}^{[t\nu]-1} (\bar{X}_{\frac{j+1}{\nu}} - \bar{X}_{\frac{j}{\nu}})^2 - \sum_{j=0}^{[\nu(t-\frac{1}{\kappa})]-1} (\bar{X}_{\frac{j+1}{\nu}} - \bar{X}_{\frac{j}{\nu}})^2) \\ &= \kappa(\sum_{j=[t\nu]-\frac{\nu}{\kappa}}^{[t\nu]-1} (c_{j+1})^2) \\ &= \frac{\kappa}{\nu} (\sum_{j=[t\nu]-\frac{\nu}{\kappa}}^{[t\nu]-1} (\bar{H}_{\frac{j}{\nu}})^2) \\ &= \bar{Y}_{t-\frac{1}{\nu}} \end{aligned}$$

It follows from Lemma 0.10, that there exists  $E'$  with  $\mu_\eta(E') = 0$ , such that  $1_D \bar{Y}_t \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$ , ( $\dagger\dagger$ ) for all  $t \in \mathcal{T}_\nu \setminus E'$ . We now compute;

$$\begin{aligned} & \int_B \bar{Y} d(\lambda_\nu \times \mu_\eta) \\ & \leq \int_{B \cap (D^c \times \mathcal{T}_\nu)} \bar{Y} d(\lambda_\nu \times \mu_\eta) + \int_{B \cap (\bar{\Omega}_\eta \times E')} \bar{Y} d(\lambda_\nu \times \mu_\eta) \\ & + \int_{B \cap (D \times \mathcal{T}_\nu \setminus E')} \bar{Y} d(\lambda_\nu \times \mu_\eta) \end{aligned}$$

$$\begin{aligned}
&\simeq \int_{B \cap (D \times \mathcal{T}_\nu \setminus E')} \bar{Y} d(\lambda_\nu \times \mu_\eta) \text{ (by Cases 1,2)} \\
&= \int_{\mathcal{T}_\nu \setminus E'} \int_{\bar{\Omega}_\eta} 1_D \bar{Y}_t d\mu_\eta d\lambda_\nu \\
&= \int_{\mathcal{T}_\nu \setminus E'} g(t) d\lambda_\nu \text{ (where } g \simeq 0 \text{ on } \mathcal{T}_\nu \setminus E') \\
&\simeq 0
\end{aligned}$$

where we have used the assumption ( $\dagger\dagger$ ) and the fact that  $B$  is wide in the penultimate line. It follows that  $\bar{Y} \in SL^1(\bar{\Omega}_\eta \times \mathcal{T}_\nu)$  as required.

□

**Theorem 0.13.** *Any tame martingale  $X$  is representable as a stochastic integral;*

$$X(t, x) = \int_0^t F(s, x) d\beta_s$$

where  $F : [0, 1] \times \bar{\Omega}_\eta \rightarrow \mathcal{R} \in L^2([0, 1] \times \bar{\Omega}_\eta, L(\mu_\eta))$ , and  $\beta_s$  is a Brownian motion.

*Proof.* By Lemma 0.9, there exists a nonstandard martingale  $\bar{X}$ , with  ${}^\circ(\bar{X}_t) = X_{\circ t}$ , for  $t \in \bar{\mathcal{T}}_\nu$ , a.e  $L(\mu_\eta)$ . Let notation be as in Definition 0.11. Then by Lemma 0.12, we have shown that  $\bar{Y} \in SL^1(\bar{\mathcal{T}}_\nu \times \bar{\Omega}_\eta)$ . We have that  $\bar{S} = \int \bar{W} d\chi$ , where  $\chi$  is Anderson's random walk, and, therefore, the quadratic variation;

$$[\bar{S}] = \bar{Q} = \int \bar{W}^2 dt.$$

We claim that;

$${}^\circ[\bar{S}](x, t) = \int_0^t f(x, s) ds \text{ a.e } dL(\mu_\eta) \text{ (*)}$$

where  $f \in L^1(\bar{\Omega}_\eta \times [0, 1])$ . To see this, we first claim that  $\bar{W}_x^2 \in SL^1(\bar{\mathcal{T}}_\nu)$  a.e  $dL(\mu_\eta)$ , (\*\*). Suppose not, then, using Theorem 9 of [2], there exists  $A$  with  $L(\mu_\eta)(A) > 0$ , such that;

$${}^\circ \int_0^1 \bar{W}_x^2 d\lambda_\nu > \int_0^1 {}^\circ \bar{W}_x^2 dL(\lambda_\nu).$$

But then;

$${}^\circ \int_A \int_0^1 \bar{W}^2 d\lambda_\nu d\mu_\eta$$

$$\begin{aligned} &\geq \int_A \circ \int_0^1 \overline{W}^2 d\lambda_\nu dL(\mu_\eta) \\ &> \int_A \int_0^1 \circ \overline{W}^2 dL(\lambda_\nu) dL(\mu_\eta) \end{aligned}$$

contradicting the fact that  $\overline{W} \in SL^2(\overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta, \lambda_\nu \times \mu_\eta)$ . Hence, (\*\*)  
is shown. Let  $V_x(t) = \int_0^t \overline{W}_x^2 d\lambda_\nu$ , for  $t \in [0, 1]$ . By (\*\*), we have that;

$$\circ V_x(t) = \int_0^t \circ \overline{W}_x^2 dL(\lambda_\nu)$$

We claim that  $\circ V_x$  is absolutely continuous, (\*\*\*)). Suppose not, then there exist internal  $B_n \subset \mathcal{T}_\nu$ , with each  $B_n$  a finite union of intervals with real endpoints, such that  $\lambda(B_n \cap [0, 1]) < \frac{1}{n}$ , where  $\lambda$  is Lebesgue measure, and  $\epsilon \in \mathcal{R}_{>0}$ , such that;

$$\int_{B_n} \circ \overline{W}_x^2 dL(\lambda_\nu) > \epsilon$$

$$\text{Then } \circ \int_{B_n} \overline{W}_x^2 d\lambda_\nu \geq \int_{B_n} \circ \overline{W}_x^2 dL(\lambda_\nu) > \epsilon$$

$$\text{and } \lambda_\nu(B_n) \simeq \lambda(B_n \cap [0, 1]) < \frac{1}{n}$$

as each  $B_n$  is a finite union of intervals. We can extend the sequence  $(B_n)_{n \in \mathcal{N}}$  to an internal sequence indexed by  ${}^*\mathcal{N}$ . By overflow, we can find an infinite  $\rho \in {}^*\mathcal{N} \setminus \mathcal{N}$ , with  $B_\rho \in \mathcal{D}_\nu$ , such that  $\lambda_\nu(B_\rho) < \frac{1}{\rho} \simeq 0$  and;

$$\int_{B_\rho} \overline{W}_x^2 d\lambda_\nu > \epsilon$$

This contradicts (\*\*). Hence, (\*\*\*) is shown. By real analysis, see [6] Theorem 7.18, the derivative  $f_x = (\circ V_x)'$  exists a.e  $d\lambda$ ,  $f_x \in L^1([0, 1])$  and;

$$\circ[\overline{S}](x, t) = \int_0^t f(x, s) ds \text{ a.e } dL(\mu_\eta)$$

We compute;

$$\begin{aligned} &\int_{\overline{\Omega}_\eta} \int_0^1 f(x, s) ds \\ &= \int_{\overline{\Omega}_\eta} \int_{\overline{\mathcal{T}}_\nu} \circ \overline{W}^2 dL(\lambda_\nu) dL(\mu_\eta) \\ &= \circ \int_{\overline{\Omega}_\eta} \int_{\overline{\mathcal{T}}_\nu} \overline{W}^2 d\lambda_\nu d\mu_\eta \end{aligned}$$

which is finite, as  $\bar{W} \in SL^2(\bar{\mathcal{T}}_\nu \times \bar{\Omega}_\eta, \lambda_\nu \times \mu_\eta)$ , hence  $f \in L^1(\bar{\Omega}_\eta \times [0, 1])$ , thus (\*) is shown.

We have that;

$$[\bar{S}]_t \simeq [\bar{X}]_t = \bar{Z}_t \text{ a.e. } dL(\mu_\eta)$$

This follows by computing the remainder term  $r(x)$  in the proof of Lemma 0.12 and using the fact that  $\bar{Z}$  is  $S$ -continuous. This last is a consequence of the fact that  $\bar{X}$  is  $S$ -continuous and  $\bar{X}_1 \in SL^2(\bar{\Omega}_\eta)$ , using Theorem 4.2.16 of [1]. Hence, we have;

$${}^\circ[\bar{X}](x, t) = \int_0^t f(x, s) ds \text{ a.e. } dL(\mu_\eta) \text{ (***)}$$

Define a new adapted process  $g$  by;

$$g(x, t) = f^{\frac{-1}{2}}(x, t) \text{ if } f(x, t) \neq 0, \text{ and } g(x, t) = 0 \text{ otherwise.}$$

Let  $1_g$  be the characteristic function of the set  $\{(x, t) : g(x, t) = 0\}$ . We have that;

$$E(\int_0^1 g(x, s)^2 d{}^\circ[\bar{X}]) = E(\int_0^1 g(x, s)^2 f(x, s) ds) \leq 1$$

hence,  $g \in L^2(\nu_{\circ[\bar{X}]})$ . Let  $G \in SL^2(\bar{X})$  be a 2-lifting of  $g$ , and  $1_G$  a 2-lifting of  $1_g$ . We can assume that  $G \cdot 1_G = 0$ . Define;

$$\beta(x, t) = {}^\circ(\int_0^t G(x, s) d\bar{X}(x, s) + \int_0^t 1_G(x, s) d\chi(x, s))$$

Since,  $G$  and  $1_G$  have disjoint supports;

$$\begin{aligned} [\beta](x, t) &= {}^\circ[\int G d\bar{X}](x, t) + {}^\circ[\int 1_G d\chi](x, t) \\ &= {}^\circ(\int G^2 d\bar{X})(x, t) + {}^\circ[\int 1_G dt](x, t) \\ &= \int_0^t g^2 f ds + \int_0^t 1_g^2 ds = \int_0^t 1 ds = t \end{aligned}$$

It follows, using Proposition 4.4.13 and 4.4.18 of [1], this requires that  $\bar{X}$  has infinitesimal increments, that  $\beta$  is a Brownian motion, adapted to the filtration  $(\bar{\Omega}_\eta, \mathcal{D}_t, L(\mu_\eta))$ . We have that  $f^{\frac{1}{2}} \in L^2(\nu_\beta)$  and;

$$\int f^{\frac{1}{2}} d\beta = \int f^{\frac{1}{2}} g d^{\circ}\bar{X} + \int f^{\frac{1}{2}} 1_g d^{\circ}\chi = \int f^{\frac{1}{2}} g d^{\circ}\bar{X}$$

since  $f^{\frac{1}{2}} 1_g = 0$ . It remains to show that  ${}^{\circ}\bar{X} = \int f^{\frac{1}{2}} g d^{\circ}\bar{X}$ , since, we then get the result by setting  $F = f^{\frac{1}{2}}$ . Using Doob's inequality;

$$\begin{aligned} & E(\sup_{q \leq 1, q \in \mathcal{Q}} ({}^{\circ}\bar{X}(q) - \int_0^q f^{\frac{1}{2}} g d^{\circ}\bar{X})^2) \\ & \leq 4E(({}^{\circ}\bar{X}(1) - \int_0^1 f^{\frac{1}{2}} g d^{\circ}\bar{X})^2) \\ & = 4E(\int_0^1 (1 - f^{\frac{1}{2}} g)^2 d^{\circ}\bar{X}) \\ & = 4E(\int_0^1 (1 - f^{\frac{1}{2}} g)^2 dt) = 0 \end{aligned}$$

as  $f^{\frac{1}{2}} g = 1$ , whenever  $f \neq 0$ .

□

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