CONSTRUCTING THE HYPERDEFINABLE GROUP FROM THE GROUP CONFIGURATION

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Abstract. Under $\mathcal{P}(4)^{-}$-amalgamation, we obtain the canonical hyperdefinable group from the group configuration.

The group configuration theorem for stable theories given by Hrushovski [5], which extends Zilber's result for ω-categorical theories [18], plays a central role in producing deep results in *geometric stability theory* (For a complete exposition of it, see [15]). For example, it is pivotal in the proof of the dichotomy theorem for Zariski' structures (See [9]). It is fair to say the group configuration theorem is one of the foundational theorems in geometric stability theory and its applications to algebraic geometry.

The theorem roughly says that one can get the canonical non-trivial type-definable group from the group configuration, a certain geometrical configuration, in stable theories. The complete generalization of the theorem into the context of simple theories seemed unreachable. In their topical paper [1], Ben-Yaacov, Tomasic and Wagner generalize the group configuration theorem by obtaining an invariant group from the group configuration in simple theories. However the group they produce does not completely fit into the first-order context.

On the other hand, Kolesnikov in his important thesis [13], categorizes simple theories by strengthening the type-amalgamation property (the independence theorem [11]), along the lines of early suggestions by Shelah [16] and Hrushovski [6]. These works suggest to us the possibility of using generalized amalgamation for the group configuration problem. This approach proves successful, and in this paper we succeed in getting the canonical hyperdefinable group from the group configuration under stronger type-amalgamation in simple theories. The element of the group is a hyperimaginary, an equivalence class of a type-definable equivalence relation, and the group operation is type-definable, hence the group belongs to the domain of the standard first-order logic.

We assume that the reader is familiar with basics of simplicity theory [17]. Throughout the paper, T is a complete simple theory. We work in a saturated model $\mathcal{M}$ of T with hyperimaginaries, and $a, b, ...$ are (possibly infinitary) hyperimaginaries, M, N are small elementary submodels. (Note that tuples from $\mathcal{M}^{eq}$ are also hyperimaginaries). As usual, $a \equiv_A b$ ($a \equiv^L_A b$) means a, b have the same type (Lascar strong type, resp.) over A. We point out that usually $\text{bdd}(a)$ denotes the *set* of all *countable* hyperimaginaries definable over a [17, 3.1.7]. Here, depending on the context, it can be either a specific sequence which linearly orders the set $\text{bdd}(a)$; or, since a sequence of hyperimaginaries is again a hyperimaginary (of a large arity), a fixed hyperimaginary interdefinable with the sequence.

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1. Around the generalized amalgamation property

The usual amalgamation property (or the independence theorem) for Lascar strong types in simple theories is stated as follows: For $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$ with $A \subseteq B, C$, if p is a Lascar strong type over A and p_B, p_C are nonforking Lascar strong extensions of p over B,C, respectively, then there is $d \models p_A \cup p_B$ such that $d \mathop{\smile\!\!\!\!\!\!\mid}_A BC$. We call it '3-amalgamation' [8] (rather than 2-amalgamation [14]) which shall be compatible with Definition 1.3. Note that we can think of B, C (after naming A) as two vertices of a base edge of a triangle and d a top vertex, and $p_B = \text{Lstp}(d/B), p_C = \text{Lstp}(d/C)$ are the 2 types to be amalgamated. One would expect generalized amalgamation to be a natural generalization of 3-amalgamation, using a tetrahedron and higher dimensional simplices instead of a triangle. Indeed, this is the case, but the following example draws attention to why we need extra care in defining the general n-amalgamation property.

Example 1.1. *In the random graph M in $\mathcal{L} = \{R\}$, choose distinct $a_i, b_i, c_i \in M$ and imaginary elements $d_i = \{a_i, b_i\}$ $(i = 0, 1, 2)$. We can additionally assume that $R(a_0, c_0) \wedge R(b_0, c_1) \wedge \neg R(a_0, c_1) \wedge \neg R(b_0, c_0)$ and $\text{tp}(a_0b_0; c_0c_1) = \text{tp}(a_1b_1; c_1c_2) = \text{tp}(a_2b_2; c_2c_0)$. Now it follows that $\text{Lstp}(d_2/c_0) = \text{Lstp}(d_0/c_0), \text{Lstp}(d_0/c_1) = \text{Lstp}(d_1/c_1)$ and $\text{Lstp}(d_1/c_2) = \text{Lstp}(d_2/c_2)$. However it is easy to see that $\text{Lstp}(d_0/c_0c_1), \text{Lstp}(d_1/c_1c_2), \text{Lstp}(d_2/c_2c_0)$ have no common realization.*

In above example, $\{c_0, c_1, c_2\}$ can be considered as a base triangle, and $\text{Lstp}(d_0/c_0c_1)$, $\text{Lstp}(d_1/c_1c_2)$, $\text{Lstp}(d_2/c_2c_0)$ form other 3 triangles attached to the base triangle. The example shows that even if the edges of the 3 triangles are compatible over the base vertices, there is no common vertex joining the 3 triangles. On the other hand, due to the nature of the random graph if we only work in the home-sort, then any desired 3 types attached on a base triangle with compatible edges will be realized. As we want the notion of generalized amalgamation to be preserved in interpreted theories, Kolesinkov suggests, in his revised works [13] [14], the following as generalized amalgamation which we call here *$K(n)$-amalgamation.* We briefly explain the notation. In this paper, *strong type* indeed means *Lascar strong type.* Likewise, $p \in S_L(A)$ means p is a Lascar strong type over A, and for $B \subseteq A$, $p\lceil_L B$ (or simply $p\lceil B$) denotes $\text{Lstp}(a/B)$ for any (some) $a \models p$. Note that for $q \in S_L(B)$, $q \subseteq p$ means $p\lceil B = q$ or equivalently $p \vdash q$.

Definition 1.2.
- *We say strong types $p_i \in S_L(A_i)$ are* compatible over $A(\subseteq A_i)$ *if each p_i does not fork over A and for i, j, $p_i\lceil_L A_i \cap A_j = p_j\lceil_L A_i \cap A_j$. (Hence $p_i\lceil_L A = p_j\lceil_L A$). We say these A-compatible strong types p_i are* (generically) amalgamated *if there is $q \in S_L(\bigcup_i A_i)$ nonforking over A such that $\cup_i p_i \subseteq q$ (i.e. $q \vdash p_i$).*
- *We say T has* $K(n)$-amalgamation over B *if for B-independent $A = \{a_1, ..., a_n\}$ and any B-compatible $p_i \in S_L(BA_i)$ where $A_i = A \setminus \{a_i\}$ for $i = 1, .., n$, whenever* $\text{bdd}(aB) \subseteq \text{dcl}(aB)$ *for any $a \models p_1\lceil_L B(= p_i\lceil_L B)$, then $p_1 \cup ... \cup p_n$ is generically amalgamated. We say T has* $K(n)$-amalgamation *if it has* $K(n)$-amalgamation *over an arbitrary set.*

The modification is that the realizations of strong types are required to be boundedly closed over the parameter set, i.e. in above $\text{bdd}(aB) \subseteq \text{dcl}(aB)$. Note that $K(2)$-amalgamation is equivalent to 3-amalgamation (usual amalgamation), and due to weak elimination of imaginaries, it can now be seen that the random graph has $K(n)$-amalgamation for all n. Each stable theory has $K(n)$-amalgamation as well, by stationarity.

However when we use inductive arguments for example, we often have to consider not only the bounded closures of vertices of amalgamated types but also those of higher dimensional surfaces as well, since after naming parameters the surface dimension is increasing. Indeed, there exists in the literature another notion of amalgamation, called $\mathcal{P}(n)^-$-amalgamation, introduced by Hrushovski [6] prior to Kolesnikov's work, and which already takes care of this concern. *Moreover, in contrast to $K(n)$-amalgamation (or the statement of the independence theorem), the base simplex is* not *regarded as an embedded parameter,* but *another type to be amalgamated.* We think this is conceptually more correct and we shall take it as our definition of *n-amalgamation*.

Definition 1.3. *Let $I = \mathcal{P}(n)^-(= \mathcal{P}(n)\setminus\{n\})$, ordered by inclusion. Let $(\{A_i\}_{i\in I}, \{\pi^i_j\}_{i\leq j\in I})$ be a directed family: Namely, each $\pi^i_j : A_i \to A_j$ is an elementary map between the two sets, $\pi^i_i = id_{A_i}$, and $\pi^j_k \circ \pi^i_j = \pi^i_k$ for $i \leq j \leq k \in I$. We say T has $\mathcal{P}(n)^-$-*amalgamation*, or simply n-*amalgamation *if whenever*

(1) $\{\pi^{\{i\}}_u(A_{\{i\}}) : i \in u\}$ *is $\pi^{\emptyset}_u(A_\emptyset)$-independent,*

(2) $A_u = \mathrm{bdd}(\bigcup_{i\in u} \pi^{\{i\}}_u(A_{\{i\}}))$,

*for any $u \in I$, then we can extend the direct family to one indexed by $\mathcal{P}(n)$ (by finding A_n and π^j_n) so that (1) and (2) hold for n, too. We say T has $\mathcal{P}(n)^-$-*amalgamation (n-amalgamation) over A*, if $A_\emptyset = \mathrm{bdd}(A)$.*

Since the definition is not transparent to conceptualize with the above notation, we give a rewritten definition as in [2] or [7]. Recall that when we say a hyperimaginary $b = \bar{a}/E$ realizes a type r over $d = \bar{c}/F$, we mean $r = r(\bar{x})$ is a (real) type such that i) $r(\bar{a})$; ii) whenever $E(\bar{e}, \bar{e}')$, then $r(\bar{e})$ iff $r(\bar{e}')$; iii) $r(\bar{a}')$ if $\bar{a}'/E = f(b)$ for some d-automorphism f. If additionally the converse of iii) holds, we call r a complete type of b over d.

Definition 1.4. *We say T has* n-complete amalgamation over a set B *if the following holds: Let W be a collection of subsets of $\{1, ..., n\} = u_n$, closed under subsets. For each $w \in W$, complete type $r_w(x_w)$ over B is given where x_w is possibly an infinite set of variables. Suppose that*

(1) for $w \subseteq w'$, $x_w \subseteq x_{w'}$ and $r_w \subseteq r_{w'}$.

Moreover for any $a_w \models r_w$,

(2) $\{a_{\{i\}}|i \in w\}$ is B-independent,

(3) a_w is as a set $\mathrm{bdd}(\bigcup_{i\in w} a_{\{i\}}B)$ (and the map $a_w \to x_w$ is a bijection).

Then there is a complete type $r_{u_n}(x_{u_n})$ over B such that (1),(2),(3) hold for all $w \in W\cup\{u_n\}$. We say T has n-complete amalgamation (n-CA) *if it has n-complete amalgamation over any set.*

We leave the reader to show that T has n-CA over B iff T has m-amalgamation over B for all $m \leq n$. The following can be freely used: For B-independent $A = \{a_1, ..., a_n\}$, $\{A_w|w \in \mathcal{P}(u_n)\}$ is a partition of $\mathrm{bdd}(BA)$, where $A_w = \mathrm{bdd}(\cup a_iB|i \in w) \setminus \bigcup_{v\in\mathcal{P}(w)^-} \mathrm{bdd}(\cup a_iB|i \in v)$. For example $n = 2$, $\{\mathrm{bdd}(B), \mathrm{bdd}(a_1B) \setminus \mathrm{bdd}(B), \mathrm{bdd}(a_2B) \setminus \mathrm{bdd}(B), \mathrm{bdd}(Ba_1a_2) \setminus$

$(\operatorname{bdd}(a_1B) \cup \operatorname{bdd}(a_2B))\}$ is a partition of $\operatorname{bdd}(Ba_1a_2)$, since using the fact that $a_1 \mathop{\smile\!\!\!\!\mid}_B a_2$, we have $\operatorname{bdd}(a_1B) \cap \operatorname{bdd}(a_2B) = \operatorname{bdd}(B)$. It also follows in 1.4, for $v, w \in W$, $x_v \cap x_w = x_{v \cap w}$.

Any simple T has $\mathcal{P}(3)^-$-amalgamation due to usual amalgamation (the independence theorem), and we shall see that 4-amalgamation implies $K(3)$-amalgamation (1.8). For each $n > 2$, there is a simple theory having n-CA but not having $(n+1)$-CA over *any set* [14]. (The example also shows n-amalgamation does not necessarily imply k-amalgamation for $k < n$.) All stable theories have n-amalgamation over a model (1.6) for all n. Many important simple structures also have n-CA for all n such as the random graph (1.6), every PAC-structure (over some parameter) [7], and ACFA [2].

As now we know n-CA is the correct notion of generalized amalgamation, in recent work [12], corresponding modifications of terminology in [13][14] in regard to n-CA are made.[1]

The lemma 1.5 and 1.6.1,2 below essentially come from the proof of the generalized independence theorem [2]. We thank Zoe Chatzidakis for her explanation.

Lemma 1.5. *Let T be stable.*

(1) *Suppose that for a set C, whenever $a \mathop{\smile\!\!\!\!\mid}_C b$ with $b = b_1 \cup ... \cup b_n$, then*

$$\operatorname{dcl}(\operatorname{acl}(ab_1C) ... \operatorname{acl}(ab_nC)) \cap \operatorname{acl}(bC) = \operatorname{dcl}(\operatorname{acl}(b_1C) ... \operatorname{acl}(b_nC)). \qquad (\sharp)$$

Then the following are satisfied.

(a) $\operatorname{tp}(\operatorname{acl}(ab_1C) ... \operatorname{acl}(ab_nC) / \operatorname{acl}(b_1C) ... \operatorname{acl}(b_nC))$ *is stationary.*

(b) *Let $A = \{a_1, ... a_n\}$, $B = \{c_1, ..., c_n\}$ be C-independent, respectively. For $1 \le i \le n$, let $v_i = \{1, ..., n\} \setminus \{i\}$. Now given $k \le n$, assume there is a bijective map* $h : \bigcup_{1 \le i \le k} \operatorname{acl}(a_{v_i}C) \to \bigcup_{1 \le i \le k} \operatorname{acl}(c_{v_i}C)$ *where $a_{v_i} = \{a_j | j \in v_i\}$ such that $h(a_i) = c_i$, $h\lceil C = id$ and, for each v_i, $h\lceil \operatorname{acl}(a_{v_i}C)$ is elementary. Then h is C-elementary.*

(2) *In fact, the condition ($\sharp$) holds when the set C is a universe of a model M. (Hence (1)(a),(b) also are true over a model.)*

Proof. We can safely assume a, a_i, b_i, c_i are finite tuples from $\mathcal{M}^{\mathrm{eq}} = \mathcal{M}$.

(1)(a) is immediate from ($\sharp$) using $\operatorname{Cb}(c/d) \subseteq \operatorname{dcl}(cd) \cap \operatorname{acl}(d)$.

(1)(b) For $1 \le i \le k$, let $h_i = h\lceil \operatorname{acl}(a_{v_i}C)$. Then put $h^j = h_1 \cup ... \cup h_j$ (where $h^k = h$) and $D^j_a = \operatorname{dom}(h^j) = \cup_{i=1,...,j} \operatorname{acl}(a_{v_i}C)$ and $D^j_c = \operatorname{ran}(h^j) = \cup_{i=1,...,j} \operatorname{acl}(c_{v_i}C)$. For induction, assume h^{j-1} is elementary for $j > 1$. We shall show that h^j is elementary too. Now for each $i < j$ let $w_i = v_j \cap v_i$. Then $a_{v_i} = \{a_j\} \cup a_{w_i}$. Now since h_j is elementary, there is an automorphism $\hat{h}_j$ extending h_j. Then by induction $h^{j-1} \circ \hat{h}^{-1}$ maps $\hat{h}(D^{j-1}_a)$ to D^{j-1}_c, so they have the same type, in particular, over the set $\bigcup_{i=1,...,j-1} \operatorname{acl}(c_{w_i}C)$ fixed by $h^{j-1} \circ \hat{h}^{-1}$. Note now that $c_j \mathop{\smile\!\!\!\!\mid}_C c_{w_1} ... c_{w_{j-1}}$ and $c_{v_i} = \{c_j\} \cup c_{w_i}$. Hence we can apply (1)(a) to conclude that $\hat{h}(D^{j-1}_a)$ and $D^{j-1}_c = \bigcup_{i=1,...,j-1} \operatorname{acl}(c_{v_i}C)$ also have the same type over $\operatorname{acl}(c_{v_j}C)$, i.e.

[1] For instance, the notion of $K(n)$*-simplicity* is introduced in terms of an *infinite* Morley sequence. This notion is quite natural as showing the equivalence of $K(1)$-simplicity and 3-amalgamation is exactly the way of proving *the independence theorem* [11]. Kolesnikov's ideas in [13] go through to obtain the equivalence of $K(2)$-simplicity and 4-amalgamation. However, counterexamples are constructed witnessing that the equivalence no longer holds for $n > 2$. Then, it is shown that an altered concept of n*-simplicity* (implying $K(n)$-simplicity) defined via a *finite* Morley sequence is now equivalent to $(n+2)$-CA for every n.

there is an elementary map g sending $\hat{h}(D_a^{j-1})$ to D_c^{j-1} fixing $\mathrm{acl}(c_{v_j}C)=\mathrm{ran}(h_j)$. Therefore it follows $h^j(\subseteq g\circ\hat{h})$ is elementary.

(2) It suffices to show that $e\in\mathrm{dcl}(\mathrm{acl}(b_1M)...\,\mathrm{acl}(b_nM))$ for $e\in\mathrm{dcl}(\mathrm{acl}(ab_1M)...\,\mathrm{acl}(ab_nM))\cap\mathrm{acl}(bM)$. Since $e\in\mathrm{dcl}(\mathrm{acl}(ab_1M)...\,\mathrm{acl}(ab_nM))$, there are $e_1...e_n$ and $\mathcal{L}(M)$-formulas $\varphi(x;y_1...y_n)$, $\psi_i(y_i,zw_i)$ with $\models\varphi(e;e_1...e_n)$, $\models\psi_i(e_i,ab_i)$ such that $\models\varphi(u;v)$ implies that u is definable over vM, and $\psi_i(u',v')$ implies that u' is algebraic over $v'M$. Therefore

$$\models\exists y_1...y_n(\varphi(e,y_1...y_n)\wedge\textstyle\bigwedge_i\psi_i(y_i,ab_i)).$$

Now since $e\in\mathrm{acl}(bM)$, $a\downarrow_M eb$ and so $\mathrm{tp}(a/Meb)$ is a coheir extension of $\mathrm{tp}(a/M)$. Thus we have $m\in M$ such that

$$\models\exists y_1...y_n(\varphi(e,y_1...y_n)\wedge\textstyle\bigwedge_i\psi_i(y_i,mb_i)).$$

Hence $e\in\mathrm{dcl}(\mathrm{acl}(b_1M)...\,\mathrm{acl}(b_kM))$. □

Proposition 1.6. (1) *Let T be stable. If a set C satisfies ($\sharp$) in 1.5.1, then for each n, T has n-CA over C.*
(2) *All stable theories have n-CA over a model.*
(3) *The random graph has n-CA over any set.*

Proof. (1) In a stable theory T we can work in $\mathcal{M}^{\mathrm{eq}}$ and substitute algebraic closures for bounded closures. We use the notation in 1.4. It suffices to show the case $W=\mathcal{P}(u_n)^-$ with corresponding types $r_w(x_w)$ (for $w\in W$). Again for $1\le i<k\le n$, let $v_i=\{1,...,n\}\setminus\{i\}$ and $w_i=v_k\cap v_i$. We shall show that $\bigcup_{1\le i\le n}r_{v_i}$ is consistent and realized by $\bigcup_{1\le i\le n}a_{v_i}$ such that $\{a_{\{1\}},...,a_{\{n\}}\}$ is B-independent. (Then the type of its algebraic closure over B extending $\cup_{1\le i\le n}r_{v_i}$ is the desired $r_{u_n}(x_{u_n})$.) Now due to usual amalgamation there is $a_{v_1}a_{v_2}\models r_{v_1}\cup r_{v_2}$ such that $\{a_{\{1\}},...,a_{\{n\}}\}$ is B-independent. Then for induction, assume that there is $a_{v_1}...a_{v_{k-1}}\models r_{v_1}\cup...\cup r_{v_{k-1}}$ (for $2<k$) such that $a_{v_1}...a_{v_{k-1}}$ extends $a_{\{1\}},...,a_{\{n\}}$. Now let $b_{v_k}\models r_{v_k}$. Then there is a map $h:\bigcup_{1\le i<k}b_{w_i}\to\bigcup_{1\le i<k}a_{w_i}$ such that h sends b_{w_i} to a_{w_i}. Hence h is elementary by 1.5.1(b), and extends to an automorphism $\hat{h}$. Then we have $a_{v_k}=\hat{h}(b_{v_k})\models r_{v_k}$. Now $a_{v_1}...a_{v_k}$ realizes $r_{v_1}\cup...\cup r_{v_k}$ if for $y=x_{v_k}\cap(x_{v_1}\cup...\cup x_{v_{k-1}})$, $a_{v_1}...a_{v_{k-1}}\lceil y=a_{v_k}\lceil y$. But this clearly holds since by independence $x_{v_k}\cap x_{v_i}=x_{w_i}$, whence $y=x_{w_1}\cup...\cup x_{w_{k-1}}$. This finishes the proof of (1).

(2) It follows from 1.5.2 and (1) above.

(3) Note that for the random graph $\mathcal{M}=(\bar{M},R)$ we can work in $\mathcal{M}^{\mathrm{eq}}$ and substitute algebraic closures for bounded closures. Now since the random graph has weak elimination of imaginaries, for any A there is A' in the home sort $\bar{M}$ such that $\mathrm{acl}(A)=\mathrm{dcl}(A')$. Hence when we check n-CA, we can assume each r_w is a type of a set in $\bar{M}$. Then in $\bar{M}$, since $\mathrm{tp}(A/B)$ is determined by equality and R relations of pairs in $A\cup B$, due to randomness of R we clearly have the generalized amalgamation. □

However, there is a stable theory which does not even have 4-amalgamation over an algebraically closed set (in $\mathcal{M}^{\mathrm{eq}}$). We thank Ehud Hrushovski for supplying us with this example.

Example 1.7. *Let A be an infinite set with $[A]^2=\{\{a,b\}|a,b\in A,a\neq b\}$, and let $B=[A]^2\times\{0,1\}$ where $\{0,1\}=\mathbb{Z}/2\mathbb{Z}$. Also let $E\subseteq A\times[A]^2$ be a membership relation, and let P be a subset of B^3 such that $((w_1,\delta_1)(w_2,\delta_2)(w_3,\delta_3))\in P$ iff there are distinct $a_1,a_2,a_3\in A$*

such that for $\{i,j,k\}=\{1,2,3\}$, $w_i=\{a_j,a_k\}$, *and* $\delta_1+\delta_2+\delta_3=0$. *Now let* M *be a model with the 3-sorted universe* $A, [A]^2, B$ *equipped with relations* E, P *and the projection* $f : B \to [A]^2$. *Then since* M *is a reduct of* $(A, \mathbb{Z}/2\mathbb{Z})^{\mathrm{eq}}$, M *is stable. We work in* M^{eq} *and show that it does not have* $\mathcal{P}(4)^-$*-amalgamation. Note first that* $\mathrm{dcl}(\emptyset)=\mathrm{acl}(\emptyset)$, *and for* $a\in A$, $\mathrm{dcl}(a)=\mathrm{acl}(a)$. *Now choose distinct* $a_1,a_2,a_3,a_4\in A$. *For* $\{i,j,k\}\subseteq\{1,2,3,4\}$, *fix an enumeration* $\overline{a_ia_j}=(b_{ij},...)$ *of* $\mathrm{acl}(a_ia_j)$ *where* $b_{ij}=(\{a_i,a_j\},\delta)\in B=[A]^2\times\{0,1\}$. *Let* $r_{ij}(x_{ij})=\mathrm{tp}(\overline{a_ia_j})$, *and let* x^1_{ij} *be the variable for* b_{ij}. *Note that* $b_{ij}=(\{a_i,a_j\},\delta)$ *and* $b'_{ij}=(\{a_i,a_j\},\delta+1)$ *have the same type over* a_ia_j. *Hence there is* $(\overline{a_ia_j})'=(b'_{ij},...)$ *also realizing* $r_{ij}(x_{ij})$. *Therefore we have complete types* $r_{ijk}(x_{ijk})$, $r'_{ijk}(x'_{ijk})$ *both extending* $r_{ij}(x_{ij})\cup r_{ik}(x_{ik})\cup r_{jk}(x_{jk})$ *realized by some enumerations of* $\mathrm{acl}(a_ia_ja_k)$ *such that* $P(x^1_{ij}x^1_{ik}x^1_{jk})\in r_{ijk}$ *whereas* $\neg P(x^1_{ij}x^1_{ik}x^1_{jk})\in r'_{ijk}$. *Then it is easy to see that* $r_{123}\cup r_{124}\cup r_{134}\cup r'_{234}$ *is inconsistent.*

In the example, $(\{a_2,a_3\},0)\in\mathrm{dcl}(\mathrm{acl}(a_1a_2)\cup\mathrm{acl}(a_1a_3))$, since $(\{a_2,a_3\},0)$ is a unique solution to $P((\{a_1,a_2\},0),(\{a_1,a_3\},0),x)$. But $(\{a_2,a_3\},0)\notin\mathrm{dcl}(\mathrm{acl}(a_2)\cup\mathrm{acl}(a_3))$, i.e. 1.5.1($\sharp$) does not hold over an algebraically closed set. In [8], Hrushovski shows that if a stable T eliminates *generalized finite imaginaries* then T has 4-amalgamation.

Proposition 1.8. *If* T *has 4-amalgamation over* B, *then it has* $K(3)$*-amalgamation over* B.

Proof. Assume T has 4-amalgamation. Now suppose that B-independent $A=\{a_1,a_2,a_3\}$ and B-compatible $p_i\in S_L(BA_i)$ where $A_i=A\smallsetminus\{a_i\}$ $(i=1,2,3)$ are given. Also for $d_i\models p_i\lceil_L B$, assume

$$\mathrm{bdd}(d_iB)\subseteq\mathrm{dcl}(d_iB)\quad(*).$$

Now let $r_\emptyset(x_\emptyset)=\mathrm{tp}(\mathrm{bdd}(B)/B)$. Let $r_i(x_i)=\mathrm{tp}(\mathrm{bdd}(a_iB)/B)$ for $i=1,2,3$, and $r_4(x_4)=\mathrm{tp}(\mathrm{bdd}(d_iB)/B)$ extend $r_\emptyset(x_\emptyset)$. Now we have $r_{14}(x_{14})=\mathrm{tp}(\mathrm{bdd}(a_1d_2B)/B)$ extends $r_1\cup r_4$, since $a_1\perp_B d_2$ implies $x_1\cap x_4=x_\emptyset$. Let $f_1:\mathrm{bdd}(a_1d_2B)\to x_{14}$ be the realization map. Now note that due to compatibility of types p_i, for $i\in\mathbb{Z}/3\mathbb{Z}$, there is an automorphism h_i sending d_{i+2} to d_{i+1} fixing $\mathrm{bdd}(a_iB)$. Then due to (*),

$$h_{i+2}\circ h_i\lceil\mathrm{bdd}(d_{i+2}B)=h^{-1}_{i+1}\lceil\mathrm{bdd}(d_{i+2}B)\quad(**).$$

Now via $f_1\circ h_1:\mathrm{bdd}(a_1d_3B)\to x_{14}$, $\mathrm{bdd}(a_1d_3B)\models r_{14}(x_{14})$. Similarly there is $r_{24}(x_{24})=\mathrm{tp}(\mathrm{bdd}(a_2d_3B)/B)$ extending $r_2(x_2)\cup r_4(x_4)$, since also $x_{14}\cap x_2=x_\emptyset$. Thus by the map $f_2\circ h_2:\mathrm{bdd}(a_2d_1B)\to x_{24}$ where $f_2:\mathrm{bdd}(a_2d_3B)\to x_{24}$, $\mathrm{bdd}(a_2d_1B)\models r_{24}(x_{24})$. Note that $f_2\lceil\mathrm{bdd}(d_3B)=f_1\circ h_1\lceil\mathrm{bdd}(d_3B)$. Moreover, $r_{34}(x_{34})=\mathrm{tp}(\mathrm{bdd}(a_3d_1B)/B)$ extends $r_3(x_3)\cup r_4(x_4)$. Let $f_3:\mathrm{bdd}(a_3d_1B)\to x_{34}$. Note again that $f_3\lceil\mathrm{bdd}(d_1B)=f_2\circ h_2\lceil\mathrm{bdd}(d_1B)$. Now $f_3\circ h_3:\mathrm{bdd}(a_3d_2B)\to x_{34}$ extends $f_1\lceil\mathrm{bdd}(d_2B):\mathrm{bdd}(d_2B)\to x_4$ since from (**), on $\mathrm{bdd}(d_2B)$, $f_3\circ h_3=(f_2\circ h_2)\circ h_3=(f_1\circ h_1\circ h_2)\circ h_3=f_1$. Therefore $f_1\cup f_3\circ h_3:\mathrm{bdd}(a_1d_2B)\cup\mathrm{bdd}(a_3d_2B)\to r_{14}(x_{14})\cup r_{34}(x_{34})$ is a well-defined realization map extending the realizations of $r_j(x_j)$ $(j=1,3,4)$. It now is easy to find additional types r_w so that they satisfy (1),(2),(3) of 1.4 for $n=4$. Therefore by 4-amalgamation we have $d(\equiv^L_B d_i)$ such that $\{a_1,a_2,a_3,d\}$ is B-independent and the type of $\mathrm{bdd}(a_1a_2a_3dB/B)$ extends the types r_w. Obviously, d is the generic realization of $p_1\cup p_2\cup p_3$. ☐

The main object of this paper is a generalization of the group configuration theorem for stable theories to the simple context. We succeed in obtaining a hyperdefinable group from the group configuration under 4-amalgamation. In fact, what we are using is 4-amalgamation over a parameter properly containing a model (see the proof of 2.6). But as indicated even a stable theory need not have such a property. Hence to make it work in a more general context, we introduce the notion of *model-n-CA*, a small variation of n-CA, which is n-CA over sets containing a model such that none of the types concerned fork over the model. The formal definition is as follows.

Definition 1.9. *We say T has* model-n-complete amalgamation *if the following holds: Let $u_n = \{1, ..., n\}$, and $W_n = \mathcal{P}(u_{n+1})^- \setminus \{u_n\}$. Let W be a subset of W_n, closed under subsets. For each $w \in W$, complete type $r_w(x_w)$ over a model M is given where x_w is possibly an infinite set of variables. Suppose that*

(1) for $w \subseteq w'$, $x_w \subseteq x_{w'}$ and $r_w \subseteq r_{w'}$.

Moreover for any $a_w \models r_w$,

(2) $\{a_{\{i\}} | i \in w\}$ is M-independent,

(3) a_w is as a set $\mathrm{bdd}(\cup_{i\in w} a_{\{i\}} M)$ (and the map $a_w \to x_w$ is a bijection).

Then there is a complete type $r_{u_{n+1}}(x_{u_{n+1}})$ over M such that (1),(2),(3) hold for all $w \in W \cup \{u_{n+1}\}$.

For example, if T has model-4-amalgamation, then by choosing $W = \mathcal{P}(\{1, 2, 3, 5\})^-$, we see that T has 4-amalgamation over models. Moreover, for

$$w \in W_4 = \{\{1, 2, 3; 5\}, \{1, 2, 4; 5\}, \{1, 3, 4; 5\}, \{2, 3, 4; 5\}\} \cup \bigcup_{1\leq i<j<k\leq 5} \mathcal{P}(\{i, j, k\}),$$

and the types r_w over M satisfying (1)(2)(3), $r_{\{123;5\}} \cup r_{\{124;5\}} \cup r_{\{134;5\}} \cup r_{\{234;5\}}$ can be generically amalgamated over M. This is 4-amalgamation over a parameter containing the model. Indeed, 4-amalgamation over sets containing models implies model-4-amalgamation. But some stable structure, which must satisfy the latter by 1.6.2, need not satisfy the former.

More generally, each of stability, $(n + 1)$-CA over models, or n-CA implies model-n-CA for every n; and model-n-CA implies n-CA over models. Model-n-CA also holds in aforementioned algebraic examples such as ACFA and PAC-structures. Model-4-CA is the property we shall use, hence covers the case that T is stable.

2. The group configuration

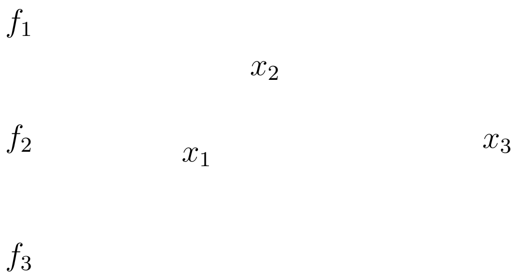
Definition 2.1. *By a* group configuration *we mean a 6-tuple of hyperimaginaries $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ over a hyperimaginary e such that, for $\{i, j, k\} = \{1, 2, 3\}$,*

(1) $f_i \in \mathrm{bdd}(f_j, f_k; e)$,

(2) $x_i \in \mathrm{bdd}(f_j, x_k; e)$,

(3) *all other triples and all pairs from C are independent over e.*

If the group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ over e has the property that $\mathrm{bdd}(f_i; e) = \mathrm{bdd}(\mathrm{Cb}(x_j x_k/f_i e); e)$, *we call such C a* bounded quadrangle. *If additionally x_i, x_j are* inter-definable *over $f_k e$, then we call C a* definable quadrangle *over e.*



Fact 2.2. (1) *If $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is a group configuration/bounded quadrangle over e and* $\mathrm{bdd}(f_i e) = \mathrm{bdd}(f_i' e), \mathrm{bdd}(x_i e) = \mathrm{bdd}(x_i' e)$*, then $C' = (f_1', f_2', f_3', x_1', x_2', x_3')$ is also a group configuration/bounded quadrangle over e. In this case, we say C and C' are* (boundedly) equivalent *over e.*

(2) *For C a group configuration/bounded quadrangle over e and $e' \supseteq e$, if $C \perp_e e'$ then C also is a group configuration/bounded quadrangle over e'.*

(3) *In a group configuration $(f_1, f_2, f_3, x_1, x_2, x_3)$ over e even if we replace f_i by $f_i' =$* $\mathrm{Cb}(x_j x_k / e f_i)$ *for $\{i, j, k\} = \{1, 2, 3\}$, $(f_1', f_2', f_3', x_1, x_2, x_3)$ is still a group configuration (hence a bounded quadrangle) over e.*

Proof. We sketch the proof. (1) Obvious for a group configuration. For a bounded quadrangle notice that in general $\mathrm{Cb}(a_1/a_2)$ and $\mathrm{Cb}(b_1/b_2)$ are interbounded as long as a_i, b_i are interbounded. (2) Easy. (3) Since $x_i x_j \perp_{f_k'} f_k e$ and $x_i \perp_{x_j f_k'} f_k e$, x_i, x_j are interbounded over f_k' (*). On the other hand, $x_j \perp_{f_k e} f_i$ implies $x_i x_j \perp_{f_k e} f_i f_j$ (**), $x_i x_j \perp_{f_k f_i e} f_j$ and thus $x_i x_j x_k \perp_{f_k f_i e} f_j$ and $x_i x_j x_k \perp_{f_k f_i f_j f_j' e} f_j$. Then from (**), it follows $x_i x_k \perp_{f_j'} f_i f_j f_k e$ and from (*) $x_i x_j \perp_{f_i' f_j'} f_i f_j e$. Therefore $f_k' = \mathrm{Cb}(x_i x_j / f_k e) = \mathrm{Cb}(x_i x_j / f_i f_j e) \in \mathrm{bdd}(f_i' f_j')$. The Other independences over e follow easily. □

¿From now on, assume that a group configuration over $\hat{e} = A/\bar{E}$ is given. We shall produce a non-trivial canonical hyperdefinable group from it. By 2.2.3 above, we can replace it by a bounded quadrangle $C = (\hat{f}, \hat{g}, \hat{h}, \hat{a}, \hat{b}, \hat{c})$ over a model M containing A. After naming M, we freely assume that $\emptyset = \mathrm{bdd}(\emptyset)$. We further suppose $\hat{f}, \hat{g}, \hat{h}, \hat{a}, \hat{b}, \hat{c}$ are all boundedly closed (by extending each to its bounded closure, if necessary.) Clearly C still is a bounded quadrangle over $\emptyset$. Let $p = \mathrm{tp}(\hat{f}) (= \mathrm{Lstp}(\hat{f})), q = tp(\hat{g}), r = \mathrm{tp}(\hat{h})$ and let $\Gamma_q(uv) = q(u) \wedge q(v) \wedge u \perp v$. (Later we shall omit q in Γ_q.) Now we can think of $\hat{h}$ as a multi-valued function such that $\mathrm{dom}(\hat{h}) = \mathrm{tp}(\hat{a}/\hat{h}) = \mathrm{Lstp}(\hat{a}/\hat{h})$ and $rag(\hat{h}) = \mathrm{Lstp}(\hat{b}/\hat{h})$. More precisely $b \in k_r(a)$ means $k_r \models r$, $k_r ab \equiv \hat{h}\hat{a}\hat{b}$. Similarly we write $a \in h_q(c)$, $b \in g_p(c)$ for $h_q ca \equiv \hat{g}\hat{a}\hat{c}$, $g_p cb \equiv \hat{f}\hat{b}\hat{c}$, respectively. In the same manner, $b \in \mathrm{dom}(f_p) \equiv \exists c (f_p bc \models \mathrm{tp}(\hat{f}\hat{b}\hat{c}))$, and so on.

We say a set A is n-independent if any subset of A having n elements is independent. Now we define $R = R^q$ to be a symmetric type-definable relation over $\emptyset$ on the set of independent realizations of q such that

$R(fg; f'g')$ iff $f, g, f'g' \models q$, $\{f, g, f', g'\}$ 3-independent, and there are b and $a \perp fgf'g'$ such that $f(a) \cap g(b) \neq \emptyset$, $f'(a) \cap g'(b) \neq \emptyset$.

It is easy to see that $a \perp fgf'g'$ above can be replaced by $b \perp fgf'g'$. Similarly, one can define R^p, R^{pq} by replacing $f, g, f'g' \models q$ by $f, g, f'g' \models p$ or $f, f' \models p$, $g, g' \models q$, respectively.

Lemma 2.3. (1) *If $fg \models \Gamma_q$, and $c \in g(a) \cap f(b)$ with $c \perp fg$, then*

(a) *f, g are interbounded over $e := \mathrm{Cb}(ba/fg)$,*

(b) $e \mathop{\smile\!\!\!\!\!\!\;|}\, f$, $e \mathop{\smile\!\!\!\!\!\!\;|}\, g$, *and*

(c) (f, g, e, a, b, c) *forms a bounded quadrangle.*

(2) *(1) still holds when we replace* $f, g \models q$ *by* $f, g \models p$*, or* $f \models p, g \models q$.

(3) *If* $(fg, f'g') \models R$ *(or* R^p, R^{pq}*), then any element in* $\{f, g, f', g'\}$ *is in the bounded closure of the other 3 elements.*

Proof. (1)(a) Note that from $ab \mathop{\smile\!\!\!\!\!\!\;|}_e fg$ and that a, b are interbounded over fg, it follows that a, b are interbounded over e, too (*). Now from $c \mathop{\smile\!\!\!\!\!\!\;|}_g f$, we have $ca \mathop{\smile\!\!\!\!\!\!\;|}_g fe$. Moreover from $c \mathop{\smile\!\!\!\!\!\!\;|}_f g$, it follows $cb \mathop{\smile\!\!\!\!\!\!\;|}_{fe} g$, and then by (*), $ca \mathop{\smile\!\!\!\!\!\!\;|}_{fe} g$. Hence $g \in \mathrm{bdd}(\mathrm{Cb}(ca/g)) \subseteq \mathrm{bdd}(fe)$. By a similar argument, $f \in \mathrm{bdd}(ge)$ can be shown too.

(1)(b) There are h_1u_1, h_2u_2 such that $h_1fbu_1c, h_2gau_2c \models \hat{f}\hat{g}\hat{a}\hat{b}\hat{c}$. Then since c is boundedly closed, by amalgamation we have

$$hu \models \mathrm{tp}(h_1u_1/cbf) \cup \mathrm{tp}(h_2u_2/cag).$$

such that $\{u, c, f, g\}$ is independent. Then we have k, k' such that $hgkauc, hfk'buc \models \hat{f}\hat{g}\hat{h}\hat{a}\hat{b}\hat{c}$. From $c \mathop{\smile\!\!\!\!\!\!\;|}\, fgh$, we have $ba \mathop{\smile\!\!\!\!\!\!\;|}_{fg} kk'$. From $b \mathop{\smile\!\!\!\!\!\!\;|}\, fgh$ and $u, a \in \mathrm{bdd}(kk'b)$, it follows that $ba \mathop{\smile\!\!\!\!\!\!\;|}_{kk'} fg$ and thus $e \in \mathrm{bdd}(kk')$ (†). On the other hand, $f \mathop{\smile\!\!\!\!\!\!\;|}_h g$ implies $k' \mathop{\smile\!\!\!\!\!\!\;|}_h gk$ and $k' \mathop{\smile\!\!\!\!\!\!\;|}\, gk$. Hence $\{g, k, k'\}$ is independent. Similarly $\{f, k, k'\}$ is independent. Then from (†), $e \mathop{\smile\!\!\!\!\!\!\;|}\, f$, $e \mathop{\smile\!\!\!\!\!\!\;|}\, g$.

(1)(c) This easily follows from (a),(b), and (*).

(2) Similar to (1).

(3) There are c, c', b and $a \mathop{\smile\!\!\!\!\!\!\;|}\, fgf'g'$ such that $c \in f(a) \cap g(b)$, $c' \in f'(a) \cap g'(b)$. Hence $ab \mathop{\smile\!\!\!\!\!\!\;|}_{fg} f'g'$ and $ab \mathop{\smile\!\!\!\!\!\!\;|}_{f'g'} fg$. Therefore $e = \mathrm{Cb}(ab/fg)$ and $e' = \mathrm{Cb}(ab/f'g')$ are interbounded (**). From (1)(a), f, g are interbounded over e, and so are f', g' over e'. Hence it follows from (**), $g' \in \mathrm{bdd}(f'e') = \mathrm{bdd}(f'e) \subseteq \mathrm{bdd}(f'fg)$ and similarly for the other relations. □

The proof of 2.3.1(b) above is essentially due to Frank O. Wagner.

Lemma 2.4. (1) *For* $fg, f'g' \models \Gamma_q$, $R(fg, f'g')$ *iff there are* b *and* $a \mathop{\smile\!\!\!\!\!\!\;|}\, fgf'g'$ *such that* $f(a) \cap g(b) \neq \emptyset$, $f'(a) \cap g'(b) \neq \emptyset$ *and* $fg \mathop{\smile\!\!\!\!\!\!\;|}_e f'g'$ *where* $e = \mathrm{Cb}(ba/fg)$.

(2) *Given independent* $f, g \models q$*, there are* f', g' *such that* $R(fg, f'g')$.

(3) *Above (1)(2) hold for* R^p, R^{pq}*, as well.*

Proof. (1) (⇒) Note that since $a \mathop{\smile\!\!\!\!\!\!\;|}\, fgf'g'$, $ba \mathop{\smile\!\!\!\!\!\!\;|}_{fg} f'g'$, $ba \mathop{\smile\!\!\!\!\!\!\;|}_{f'g'} fg$. Hence $e, e' = \mathrm{Cb}(ba/f'g')$ are interbounded. Then due to 2.3.1(a), f', g' are interbounded over e. Now from $fg \mathop{\smile\!\!\!\!\!\!\;|}\, f'$, $fg \mathop{\smile\!\!\!\!\!\!\;|}_e f'$, thus $fg \mathop{\smile\!\!\!\!\!\!\;|}_e f'g'$.

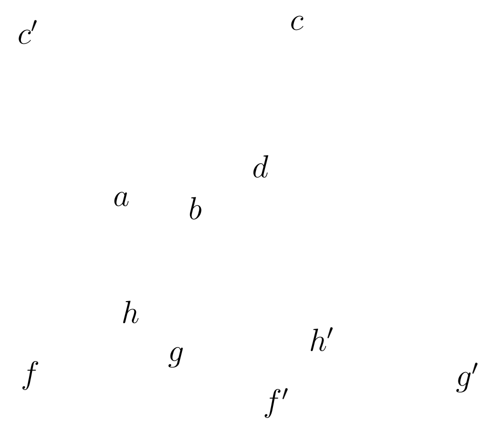
(⇐) Again since $a \mathop{\smile\!\!\!\!\!\!\;|}\, fgf'g'$, e, e' are interbounded. Then from $fg \mathop{\smile\!\!\!\!\!\!\;|}_e f'g'$, equivalently $fg \mathop{\smile\!\!\!\!\!\!\;|}_{e'} f'g'$, and 2.3.1(b), $\{f, g, f', g'\}$ is 3-independent.

(2) By amalgamation there is $c \in \mathrm{ran}(f) \cap \mathrm{ran}(g)$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}\, fg$. Choose $a \in g^{-1}(c), b \in f^{-1}(c)$. Then, by the extension axiom, we have $f'g'$ such that $\{ab, fg, f'g'\}$ is e-independent where $e = \mathrm{Cb}(ba/fg)$ and $f'g' \equiv_{abe} fg$. Then the right hand side of (1) follows easily.

(3) Clear. □

The following lemma is crucial to our argument.

Lemma 2.5. *Let* $R(fg, f'g')$ *hold. Namely,* $\{f, g, f', g'\}$ *is 3-independent, and we can find* c, c', d *and* $a \mathop{\smile\!\!\!\!\!\!\;|}\, fgf'g'$ *such that* $c \in f(a) \cap g(d)$, $c' \in f'(a) \cap g'(d)$. *Then there are* $h, h' \models p$ *and* b *such that* $c \in h(b), c' \in h'(b)$, $b \mathop{\smile\!\!\!\!\!\!\;|}\, hh'ff'gg'$, $\{f, h, f', h'\}$ *is 3-independent, and* $hh' \mathop{\smile\!\!\!\!\!\!\;|}_{ff'} gg'$. *It follows that* $R^{pq}(hf, h'f')$ *and* $R^{pq}(hg, h'g')$.



Proof. Note since $fac, f'ac' \models \hat{g}\hat{a}\hat{c}$, there are $k_0, b_0; k_1, b_1$ such that $f'k_0ab_0c', fk_1ab_1c \models \hat{g}\hat{h}\hat{a}\hat{b}\hat{c}$. Then by amalgamation, we have kb, hh' such that

$$kb \models \mathrm{tp}(k_0b_0/af') \cup \mathrm{tp}(k_1b_1/af),\ kb \mathop{\smile\!\!\!\!\!\!\mid}_a ff' \text{ and } h'f'kabc', hfkabc \models \hat{f}\hat{g}\hat{h}\hat{a}\hat{b}\hat{c}.$$

We can further assume that $hh' \mathop{\smile\!\!\!\!\!\!\mid}_{ff'a} gg'$ (*). Now from $kb \mathop{\smile\!\!\!\!\!\!\mid}_a ff'$ and $a \mathop{\smile\!\!\!\!\!\!\mid} fgf'g'$, it follows $ab \mathop{\smile\!\!\!\!\!\!\mid}_k ff'$, $b \mathop{\smile\!\!\!\!\!\!\mid}_k ff'hh'$, and $b \mathop{\smile\!\!\!\!\!\!\mid} ff'hh'$. Then from (*), we have $a \mathop{\smile\!\!\!\!\!\!\mid}_{hh'ff'} gg'$, $ab \mathop{\smile\!\!\!\!\!\!\mid}_{hh'ff'} gg'$, and $b \mathop{\smile\!\!\!\!\!\!\mid} ff'gg'hh'$. Moreover from $f \mathop{\smile\!\!\!\!\!\!\mid}_k f'$, we have that $hf \mathop{\smile\!\!\!\!\!\!\mid}_k h'f'$, and thus $\{f, h, f', h'\}$ is 3-independent. Then since $hh' \mathop{\smile\!\!\!\!\!\!\mid}_{ff'} gg'$, $\{g, h, g', h'\}$ is 3-independent as well. Therefore $R^{pq}(hf, h'f')$ and $R^{pq}(hg, h'g')$. □

Now define $R' = (R')^q$ by

$R'(fg; f'g')$ iff $f, g, f'g' \models q$, $\{f, g, f', g'\}$ 3-independent, and for any a, b such that $f(a) \cap g(b) \neq \emptyset$ and $a \mathop{\smile\!\!\!\!\!\!\mid} fg$, there are $a'b' \equiv^L_{fg} ab$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid} fgf'g'$, $f'(a') \cap g'(b') \neq \emptyset$.

Again we also define $(R')^p$, $(R')^{pq}$ by substituting $f, g, f'g' \models p$ or $f, f \models p, gg' \models q$, respectively.

Assumption For the rest of this paper, we assume that T has 4-CA, or more weakly, model-4-CA. Recall that both implies 4-CA over models.

Notation For bounded closed sequences $a_1, a_2, \ldots$, we use $\overline{a_1a_2\ldots}$ to denote some sequence of $\mathrm{bdd}(a_1a_2\ldots)$ extending the orderings of $a_1, a_2, \ldots$. More precisely, if $x_i \mapsto a_i$ is a bijective indexing of a_i by the variable x_i, then $\overline{a_1a_2\ldots}$ is an indexing $x(\supseteq x_1 \cup x_2 \cup \ldots) \mapsto \mathrm{bdd}(a_1a_2\ldots)$ extending each $x_i \mapsto a_i$.

We shall prove that R and R' are equivalent. Note that clearly R' implies R.

Theorem 2.6. *R and R' are equivalent.*

Proof. We shall show that R implies R'. Let $R(fg, f'g')$ hold. We use Lemma 2.5 with the same notation. We then have cc', abd, hh' such that $c \in f(a) \cap h(b) \cap g(d)$; $c' \in f'(a) \cap h'(b) \cap g'(d)$; $b \mathop{\smile\!\!\!\!\!\!\mid} fghf'g'h'$; $f' \in \mathrm{bdd}(fhh'), g' \in \mathrm{bdd}(hgh')$; and $\{f, g, h, h'\}$ is independent. Now let $\overline{hf}$, $\overline{hg}$, $\overline{fg}$, $\overline{hbc} = \mathrm{bdd}(hb) = \mathrm{bdd}(hc)$, $\overline{fac} = \mathrm{bdd}(fc) = \mathrm{bdd}(fa)$, $\overline{gdc} = \mathrm{bdd}(gc) = \mathrm{bdd}(gd)$ be sequences of boundedly closed sets extending the boundedly closed sequences f, g, h, c, a, b, d (See Notation above 2.6). Since $hbc \equiv h'bc'$ we also have $\overline{h'bc'} \equiv \overline{hbc}$.

Now, to show $R'(fg, f'g')$, assume there are a_1, d_1, c_1 such that $c_1 \in f(a_1) \cap g(d_1)$ and $c_1 \mathop{\smile\!\!\!\!\!\!\mid}\, fg$. We also have $\overline{fa_1c_1} \equiv \overline{fac} \equiv \overline{gd_1c_1} \equiv \overline{gdc}$. Then by 4-amalgamation (over a model which is named), we can find c_2 such that

$$\mathrm{tp}(\overline{fgh}) \cup \mathrm{tp}(\overline{fhc}) \cup \mathrm{tp}(\overline{ghc}) \cup \mathrm{tp}(\overline{fgc_1}) \subseteq \mathrm{tp}(\overline{fghc_2}),$$

where $\overline{fa_1c_1}, \overline{gd_1c_1} \subset \overline{fgc_1}$; and $\overline{fac}, \overline{hbc} \subset \overline{fhc}$. Hence there are b_2, a_2, d_2 such that

$$c_2 \in f(a_2) \cap g(d_2) \cap h(c_2).$$

In particular,

$$\overline{fa_2c_2}\ \overline{gd_2c_2}\ \overline{fg} \equiv \overline{fa_1c_1}\ \overline{gd_1c_1}\ \overline{fg};\ \overline{fh}\ \overline{hb_2c_2} \equiv \overline{fh}\ \overline{hbc};\text{ and } \overline{gh}\ \overline{hb_2c_2} \equiv \overline{gh}\ \overline{hbc}\ (*).$$

What we are going to amalgamate next are the following strong types:

$$\mathrm{Lstp}(\overline{hb_2c_2}/fg; h),\ \mathrm{Lstp}(\overline{hbc}/h'f; h) \text{ and } \mathrm{Lstp}(\overline{hbc}/h'g; h).$$

(Equivalently $\mathrm{tp}(\overline{c_2fg;h})$, $\mathrm{tp}(\overline{ch'f;h})$, $\mathrm{tp}(\overline{ch'g;h})$ and $\mathrm{tp}(\overline{h'fg;h})$, where $h, b_2, c_2 \subset \overline{c_2h}$, and $h, b, c \subset \overline{ch}$.) The base parameter is h (here is the place where we need model-4-CA, since indeed the parameter is Mh; also see the example after 1.9), and each realization is boundedly closed over the parameter. Each type does not fork over h. Additionally due to (*), it can be seen that the 3 strong types are h-compatible. Hence we have $\overline{hb_3c_3}$, a generic solution of the types.

Now we analyze consequences obtained from the generic solution. Note first that we have $\overline{fhabc} = \mathrm{bdd}(f, bc; h)$ extending $\overline{fac}$, $\overline{hbc}$. Also since $c' \in \mathrm{bdd}(h', bc; h)$, we have $\overline{hh'bcc'} = \mathrm{bdd}(h', bc; h)$ extending $\overline{hbc}$, $\overline{h'bc'}$. Similarly there is $\overline{hgbdc} = \mathrm{bdd}(g, bc; h)$ extending $\overline{hbc}$, $\overline{gdc}$. Note here that the existence of the generic solution indeed says that there exist elementary maps $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ with

$\mathrm{dom}(\tilde{h}_1) = \mathrm{bdd}(f, bc; h)\,\mathrm{bdd}(h', bc; h)\,\mathrm{bdd}(fh'; h)$,
$\mathrm{dom}(\tilde{h}_2) = \mathrm{bdd}(f, b_2c_2; h)\,\mathrm{bdd}(g, b_2c_2; h)\,\mathrm{bdd}(fg; h)$,
$\mathrm{dom}(\tilde{h}_3) = \mathrm{bdd}(h', bc; h)\,\mathrm{bdd}(g, bc; h)\,\mathrm{bdd}(h'g; h)$,

fixing $\mathrm{bdd}(fh'; h)$, $\mathrm{bdd}(fg; h)$ $\mathrm{bdd}(gh'; h)$, respectively, such that $\tilde{h}_1(\overline{hbc}) = \tilde{h}_2(\overline{hb_2c_2}) = \tilde{h}_3(\overline{hbc}) = \overline{hb_3c_3}$, the generic solution. Moreover they are compatible with elementary maps sending $\overline{fhabc} \to \overline{fha_2b_2c_2}$, $\overline{hgbdc} \to \overline{hgb_2d_2c_2}$ and $\overline{hh'bcc'} \xrightarrow{id} \overline{hh'bcc'}$. (In particular, maps $\tilde{h}_1\lceil \mathrm{bdd}(h', bc; h) = \tilde{h}_3\lceil \mathrm{bdd}(h', bc; h)$.) Hence there are a_3, c'_3, d_3 such that $a_3 = \tilde{h}_1(a) = \tilde{h}_2(a_2)$, $c'_3 = \tilde{h}_1(c') = \tilde{h}_3(c')$, $d_3 = \tilde{h}_2(d_2) = \tilde{h}_3(d)$ and

(1) $\overline{fha_3b_3c_3}\ \overline{hh'b_3c_3c'_3} \equiv_{\mathrm{bdd}(fh';h)} \overline{fhabc}\ \overline{hh'bcc'}$;
(2) $\overline{fha_3b_3c_3}\ \overline{hgb_3d_3c_3} \equiv_{\mathrm{bdd}(fg;h)} \overline{fha_2b_2c_2}\ \overline{hgb_2d_2c_2}$;
(3) $\overline{hh'b_3c_3c'_3}\ \overline{hgb_3d_3c_3} \equiv_{\mathrm{bdd}(gh';h)} \overline{hh'bcc'}\ \overline{hgbdc}$.

Then by (*) and (2), $a_1d_1 \equiv^L_{fg} a_2d_2 \equiv^L_{fg} a_3d_3$. Also since $f' \in \mathrm{bdd}(fh'h)$, from (1), $c'_3 \in f'(a_3) \cap h'(b_3)$ and $c_3 \in f(a_3)$. Note that $g' \in \mathrm{bdd}(gh'h)$, since $R(hg, h'g')$ holds. Then, similarly from (3), $c'_3 \in g'(d_3)$ and $c_3 \in g(d_3)$. Therefore $R'(fg, f'g')$. □

3. Type-definability of the transitive closure of R

In this section, we use $R \equiv R'$ (Theorem 2.6) to prove that, the transitive closure of R is type-definable. The proof is similar to the proof in [4] that the transitive closure of the relation $\sim_1$ forms a hyperimaginary canonical base, (or the improvement of this proof in [17, 3.3.1]).

Let $\tilde{R}$ be the transitive closure R. We remark that if both $\{a,b,c,d\}$, $\{a',b',c,d\}$ are 3-independent and $ab \downarrow_{cd} a'b'$, then $\{a,b,a',b'\}$ is also 3-independent.

Lemma 3.1. *Suppose that $R(fg,hk)$, $R(hk,f'g')$ and $fg \downarrow_{hk} f'g'$. Then $R(fg,f'g')$ and $fg \downarrow_{f'g'} hk$.*

Proof. By the previous remark, $\{f,g,f',g'\}$ is 3-independent (*). Now since $R(fg,hk)$, there are b and $a \downarrow fghk$ such that $f(a)\cap g(b) \neq \emptyset$, $h(a)\cap k(b)\neq\emptyset$ (so $ab \downarrow_{hk} fg$). Then since $R'(hk,f'g')$, there are $a'b' \equiv^L_{hk} ab$ such that $a' \downarrow hkf'g'$, $f'(a')\cap g'(b')\neq\emptyset$. Hence by amalgamation, we have

$$a''b'' \models \mathrm{Lstp}(ab/hk,fg)\cup \mathrm{Lstp}(a'b'/hk,f'g'), \text{ and } a''b'' \downarrow_{hk} fgf'g'.$$

It follows then $a'' \downarrow fgf'g'$ and $f(a'')\cap g(b'')\neq\emptyset$, $f'(a'')\cap g'(b'')\neq\emptyset$. This with (*) says $R(fg,f'g')$. It remains to show $fg \downarrow_{f'g'} hk$. Since $g \downarrow hkf'g'$, $g \downarrow_{f'g'} hk$. Now by 2.3.3, $f \in \mathrm{bdd}(gf'g')$, and therefore $fg \downarrow_{f'g'} hk$. The proof is finished. □

Theorem 3.2. *The following are equivalent.*

1. $\tilde{R}(\bar f,\bar g)$.
2. *For some $\bar h$, $R(\bar h,\bar f)$ and $R(\bar h,\bar g)$.*
3. *For some $\bar h$ with $\bar h \downarrow_{\bar f} \bar g$ and $\bar h \downarrow_{\bar g} \bar f$, $R(\bar h,\bar f)$ and $R(\bar h,\bar g)$.*

Proof. It suffices to show (1) implies (3). We prove this by induction on the length of an R-chain. Note that 2.4.2 gives the induction step for length 0. Now assume that there are $\bar f,\bar f_n,\bar h'$ such that $\tilde R(\bar f,\bar f_n)$ with the R-chain length n and $R(\bar f_n,\bar h')$. By the induction hypothesis for n, there is $\bar h$ such $\bar h \downarrow_{\bar f} \bar f_n$ and $\bar h \downarrow_{\bar f_n} \bar f$ (*), $R(\bar h,\bar f)$ and $R(\bar h,\bar f_n)$. By extension, we can assume $\bar h \downarrow_{\bar f\bar f_n} \bar h'$ (**). Then by (*), $\bar h \downarrow_{\bar f_n} \bar h'\bar f$ (***). In particular, $\bar h \downarrow_{\bar f_n} \bar h'$. Hence from the lemma 3.1, $R(\bar h,\bar h')$ and $\bar h \downarrow_{\bar h'} \bar f_n$. Then it follows from (***), $\bar h \downarrow_{\bar h'} \bar f$. Moreover again by (*)(**), we have $\bar h \downarrow_{\bar f} \bar h'$. Hence the induction step for $n+1$ is shown. □

4. The generic group operation on $\Gamma/\tilde R$

Recall that in section 2, we define $\Gamma(xy)=\Gamma_q(xy)=q(x)\wedge q(y)\wedge x \downarrow y$. Now since R is symmetric, clearly $\tilde R$ is an equivalence relation on Γ. By putting $(\tilde R(\bar x,\bar y)\wedge\Gamma(\bar x)\wedge\Gamma(\bar y))\vee \bar x=\bar y$, we can extend $\tilde R$ to a type-definable equivalence relation on the whole universe. We shall find the canonical hyperdefinable group from the hyperdefinable generic group operation on $\Gamma/\tilde R$. First we state some more properties of R and $\tilde R$.

Lemma 4.1. *Let $\bar f = f_1f_2,\bar g \models \Gamma$, and let $e=\bar f/\tilde R$.*

1. *$\tilde R(\bar f,\bar g)$ and $\bar f \downarrow_e \bar g$ iff $R(\bar f,\bar g)$.*
2. *For a,b such that $f_1(a)\cap f_2(b)\neq\emptyset$ and $a \downarrow \bar f$, e is interbounded with $\mathrm{Cb}(ab/\bar f)$.*
3. *$f_i \downarrow e$, and f_1,f_2 are interbounded over e.*

Proof. (1) ($\Rightarrow$) By 3.2, there is $\bar{h}$ such that $\bar{h} \perp_{\bar{f}} \bar{g}$, $\bar{h} \perp_{\bar{g}} \bar{f}$, $R(\bar{h}, \bar{f})$ and $R(\bar{h}, \bar{g})$. Then since $\bar{f} \perp_e \bar{g}$, $\bar{f} \perp_{\bar{h}} \bar{g}$, and then by 3.1, $R(\bar{f}, \bar{g})$.

($\Leftarrow$) $R(\bar{f}, \bar{g})$ implies $\tilde{R}(\bar{f}, \bar{g})$. By the extension axiom, there is $\bar{g}' \models \operatorname{tp}(\bar{g}/e)$ such that $\bar{g}' \perp_e \bar{g}$ (*). Hence $\tilde{R}(\bar{g}, \bar{g}')$ and by the proof of ($\Rightarrow$), $R(\bar{g}, \bar{g}')$. Again, by extension, we can assume that $\bar{f} \perp_{e\bar{g}} \bar{g}'$. Hence $\bar{f} \perp_{\bar{g}} \bar{g}'$ and then by 3.1, $\bar{f} \perp_{\bar{g}'} \bar{g}$. Therefore by (*), $\bar{f} \perp_e \bar{g}$.

(2) Let $e_1 = \operatorname{Cb}(ab/\bar{f})$. By extension, there is $\bar{f}' \models \operatorname{tp}(\bar{f}/abe_1)$ such that $\bar{f} \perp_{abe_1} \bar{f}'$. Hence $\{ab, \bar{f}, \bar{f}'\}$ is e_1-independent and from 2.4.1, $R(\bar{f}, \bar{f}')$ and $e_1 = \operatorname{Cb}(ab/\bar{f}')$. Then by (1), $\bar{f} \perp_e \bar{f}'$ (**). Let $e_2 = \operatorname{Cb}(\bar{f}/\bar{f}')$. Then due to (**), $e_2 \in \operatorname{bdd}(e)$. Moreover since $\bar{f} \perp_{e_2} \bar{f}'$, $e \perp_{e_2} \bar{f}'$, and $e \in \operatorname{dcl}(\bar{f}')$, $e_2 \in \operatorname{bdd}(\bar{f}')$, we have $e \in \operatorname{bdd}(e_2)$. Thus $\operatorname{bdd}(e) = \operatorname{bdd}(e_2)$. Similarly since $\bar{f} \perp_{e_1} \bar{f}'$, it can be too seen $\operatorname{bdd}(e_1) = \operatorname{bdd}(e_2)$. Therefore $\operatorname{bdd}(e_1) = \operatorname{bdd}(e)$.

(3) follows from (2) and 2.3.1. $\square$

The proof of the following lemma only uses 4-CA over a model.

Lemma 4.2. *Let $R(gh, vw)$. Then for any $c \in g(a), d \in v(a)$ with $a \perp gv$, there are $c'a'd' \equiv^L_{gv} cad$ and b' such that $a' \perp ghvw$, $c' \in h(b'), d' \in w(b')$.*

Proof. By 2.5, there are a_0, b_0, c_0, d_0, t_0 and f, u such that $c_0 \in g(a_0) \cap f(t_0) \cap h(b_0)$, $d_0 \in v(a_0) \cap u(t_0) \cap w(b_0)$, $t_0 \perp fghuvw$, $\{f, g, u, v\}$ is 3-independent, and $fu \perp_{gv} hw$. It follows $R^{pq}(fh, uw)$, $R^{pq}(fg, uv)$ and $gv \perp_{fu} hw$ (*). Let $e = \operatorname{bdd}(\operatorname{Cb}(t_0a_0/fg))$. Then from 2.4.3, $e = \operatorname{bdd}(\operatorname{Cb}(t_0a_0/uv))$ and $fg \perp_e uv$. Similarly for $k = \operatorname{bdd}(\operatorname{Cb}(t_0b_0/fh)) = \operatorname{bdd}(\operatorname{Cb}(t_0b_0/uw))$, $fh \perp_k uw$. We now let $\overline{ea_0t_0} = \operatorname{bdd}(ea_0t_0) = \operatorname{bdd}(ea_0)$, $\overline{evu} = \operatorname{bdd}(ev)$, $\overline{egf} = \operatorname{bdd}(eg)$ $\overline{ga_0c_0} = \operatorname{bdd}(ga_0)$, $\overline{va_0d_0} = \operatorname{bdd}(va_0)$, $\overline{gv}$ be sequences of bounded closed sets (see Notation above 2.6). Note that there are sequences $\overline{gac}$, $\overline{vad}$ such that

$$\overline{ga_0c_0} \equiv \overline{gac} \text{ and } \overline{va_0d_0} \equiv \overline{vad}.$$

Now by 4-amalgamation over the named model, there is a_1 such that

$$\operatorname{tp}(\overline{egv}) \cup \operatorname{tp}(\overline{ega_0}) \cup \operatorname{tp}(\overline{eva_0}) \cup \operatorname{tp}(\overline{gva}) \subseteq \operatorname{tp}(\overline{egva_1}),$$

and $\{e, g, v, a_1\}$ is independent, where

$$\overline{efg}, \overline{ga_0c_0}, \overline{ea_0t_0} \subset \overline{ega_0};\ \overline{evu}, \overline{va_0d_0}, \overline{ea_0t_0} \subset \overline{eva_0}; \text{ and } \overline{gac}, \overline{vad} \subset \overline{gva}.$$

In particular, there are $c_1(\subset \overline{ga_1}), d_1(\subset \overline{va_1}), t_1(\subset \overline{ea_1})$ (all are in $\overline{egva_1}$) such that

(1) $\overline{ga_1c_1}\ \overline{ea_1t_1} \equiv_{\overline{egf}} \overline{ga_0c_0}\ \overline{ea_0t_0}$;
(2) $\overline{va_1d_1}\ \overline{ea_1t_1} \equiv_{\overline{evu}} \overline{va_0d_0}\ \overline{ea_0t_0}$;
(3) $\overline{ga_1c_1}\ \overline{va_1d_1} \equiv_{\overline{gv}} \overline{gac}\ \overline{vad}$;

and, from 2.3.1, $\{t_1, f, u, e\}$ is independent (**). Due to (1)(2), $ft_0c_0 \equiv ft_1c_1$ and $ut_0d_0 \equiv ut_1d_1$. Hence there are enumerations such that

$$\overline{ft_0c_0} \equiv \overline{ft_1c_1} \text{ and } \overline{ut_0d_0} \equiv \overline{ut_1d_1}.$$

Similarly by 4-CA, we have $t_2; c_2, d_2, b_2$ such that

$$\operatorname{tp}(\overline{kfu}) \cup \operatorname{tp}(\overline{kft_0}) \cup \operatorname{tp}(\overline{kut_0}) \cup \operatorname{tp}(\overline{fut_1}) \subseteq \operatorname{tp}(\overline{kfut_2}),$$

(4) $\overline{ft_2c_2}\ \overline{kb_2t_2} \equiv_{\overline{khf}} \overline{ft_0c_0}\ \overline{kb_0t_0}(\subset \overline{kft_0})$;
(5) $\overline{ut_2d_2}\ \overline{kb_2t_2} \equiv_{\overline{kwu}} \overline{ut_0d_0}\ \overline{kb_0t_0}(\subset \overline{kut_0})$;
(6) $\overline{ft_2c_2}\ \overline{ut_2d_2} \equiv_{\overline{fu}} \overline{ft_1c_1}\ \overline{ut_1d_1}(\subset \overline{fut_1})$;

and $\{t_2, f, u, k\}$ is independent. Hence due to (*),(**),2.3.1(a) and (6), we can apply amalgamation to have

$$d'c't' \models \mathrm{Lstp}(d_1c_1t_1/fu; gv) \cup \mathrm{Lstp}(d_2c_2t_2/fu; hw)$$

such that $t' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{fu} ghvw$ (†). Then there are the desired a', b' such that

(7) $d'c't'a' \equiv^L_{fguv} d_1c_1t_1a_1$, and $d'c't'b' \equiv^L_{fhuw} d_2c_2t_2b_2$.

Hence it follows from (†), $a' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} ghvw$. Moreover by (3)(7), $c'a'd' \equiv^L_{gv} cad$; by (4)(7), $c' \in h(b')$; and by (5)(7), $d' \in w(b')$. The proof is finished. □

We are ready to define the promised generic operation on $\Gamma/\tilde{R}$. Let

$$\centerdot(x_1y_1, x_2y_2; x_3y_3) := \exists xyz(\tilde{R}(x_1y_1; xy) \wedge \tilde{R}(x_2y_2; yz) \wedge \tilde{R}(xz; x_3y_3) \wedge \bigwedge_{i=1,2,3}\Gamma(x_iy_i) \wedge x_1y_1/\tilde{R} \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} x_2y_2/\tilde{R} \wedge \{x, y, z\} \text{ is independent}).$$

Note that for $x_1y_1, x_2y_2 \models \Gamma$, $x_1y_1/\tilde{R} \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} x_2y_2/\tilde{R}$ iff $\exists x'_1y'_1x'_2y'_2\tilde{R}(x_1y_1; x'_1y'_1) \wedge \tilde{R}(x_2y_2; x'_2y'_2) \wedge \{x'_1, y'_1, x'_2, y'_2\}$ independent. Hence $\centerdot$ is a partial type over $\emptyset$.

Claim 1. The relation $\centerdot$ is a hyperdefinable partial type over $\emptyset$ such that, for any independent $e_1 = f_1g_1/\tilde{R}$, $e_2 = f_2g_2/\tilde{R} \in \Gamma/\tilde{R}$, there is $e_3 = fh/\tilde{R} \in \Gamma/\tilde{R}$ such that $(e_1, e_2; e_3)$ realizes $\centerdot(x_1y_1, x_2y_2; x_3y_3)$. (See the explanation above 1.4): It suffices to show there exist f, g, h such that $e_1 = fg/\tilde{R}$, $e_2 = gh/\tilde{R}$ and $\{f, g, h\}$ is independent. Now by 4.1.3, $e_i \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} g_i$, $e_i \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} f_i$ and f_i, g_i are interbounded over e_i. Then, by amalgamation, there exists $g \models \mathrm{tp}(f_2/e_1) \cup \mathrm{tp}(g_1/e_2)$ and $\{g, e_1, e_2\}$ independent. We also have f, h such that $fg \equiv_{e_1} f_1g_1$, $gh \equiv_{e_1} f_2g_2$. Then, $\{f, g, h\}$ is independent too.

Claim 2. $e_1 \centerdot e_2 = e_3 = fh/\tilde{R}$ does not depend on the choice of f, g, h, i.e. $e_1 \centerdot e_2$ is unique: Suppose there are f', g', h' such that $e_1 = f'g'/\tilde{R}$, $e_2 = g'h'/\tilde{R}$ and $\{f', g', h'\}$ is independent. Then we can also find independent $\{u, v, w\}$ such that $u \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{e_1e_2} fghf'g'h'$ and $e_1 = uv/\tilde{R}$, $e_2 = vw/\tilde{R}$. Hence, from 4.1.2, $uvw \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{e_1e_2} fghf'g'h'$, and $\{f, u, e_1, e_2\}$ is independent (‡). We shall prove that $R(fh, uw)$. (Then the by the same proof, $R(f'h', uw)$ and thus $\tilde{R}(fh, f'h')$.) Note from (‡) and 4.1.1, $R(fg, uv)$ and $R(gh, vw)$. Hence there are b, $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} fguv$ and $c \in g(a) \cap f(b)$, $d \in v(a) \cap u(b)$. Moreover, by 4.2, we have $a'b'c'd'$ such that $c'a'd' \equiv^L_{gv} cad$ and $a' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} ghvw$, $c' \in g(a') \cap h(b')$, $d' \in v(a') \cap w(b')$. Now again due to (‡) and 2.3.1, we have $e_1 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} e_2$, $fu \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} hw$ $(*)$, and $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} fu$, $cad \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} fu$, $c'a'd' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} hw$. Hence, by amalgamation, we have

$$c_1a_1d_1 \models \mathrm{Lstp}(cad/gv; fu) \cup \mathrm{Lstp}(c'a'd'/gv; hw),$$

such that $a_1 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{gv} fhuw$ $(**)$. Then we have b_1, b'_1 such that

$$c_1a_1d_1b_1 \equiv_{fguv} cadb,\ c_1a_1d_1b'_1 \equiv_{ghvw} c'a'd'b'.$$

Thus, $c_1 \in f(b_1) \cap h(b'_1)$, $d_1 \in u(b_1) \cap w(b'_1)$ and from $(**)$, $b_1 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} fhuw$. Moreover from $(*)$ and the remark above 3.1, $\{f, h, u, w\}$ is 3-independent. Therefore $R(fh, uw)$, as desired.

Claim 3. This generically given group satisfies the genericity properties in [17, 4.7.1]: Note that since $\{f, g, h\}$ independent, it follows for $i = 1, 2$, $e_i \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}} e_1 \centerdot e_2$. For generic associativity, let $\{k_1, k_2, k_3\}$ be independent realizations of $\Gamma/\tilde{R}$.

Subclaim. There exists independent $\{h_1, h_2, h_3, h_4\}$ such that $h_1h_2/\tilde{R} = k_1, h_2h_3/\tilde{R} = k_2$ and $h_3h_4/\tilde{R} = k_3$: As in the proof of Claim 1, we can find h_2, h_3, h_4 independent such that

$h_2h_3/\tilde{R} = k_2$ and $h_3h_4/\tilde{R} = k_3$. Now, for $h_1'h_2'/\tilde{R} = k_1$, amalgamation of $\mathrm{Lstp}(h_2'/k_1)$ and $\mathrm{Lstp}(h_2/k_2k_3)$ gives the subclaim.

Now, by the subclaim, $k_1 \centerdot k_2 = h_1h_3/\tilde{R}$ and $k_2 \centerdot k_3 = h_2h_4/\tilde{R}$. Then $k_1 \centerdot (k_2 \centerdot k_3) = h_1h_4/\tilde{R}$ and $(k_1 \centerdot k_2) \centerdot k_3 = h_1h_4/\tilde{R}$ as well. Finally it can be easily seen that $\centerdot$ is generically surjective. Hence Claim 3 is verified.

Therefore we have the following;

Theorem 4.3. *Given the group configuration, there exists a canonical hyperdefinable group and a definable bijection mapping $\Gamma/\tilde{R}$ to the generic types of the group such that $\centerdot$ is mapped to the group multiplication generically.*

In recent work [10], the completion of the group configuration theorem is achieved. Namely, a hyperdefinable group action of the group on a hyperdefinable set is recovered, so that the induced bounded quadrangle from the hyperdefinable homogeneous space is equivalence to the original bounded quadrangle.

5. 1-based theories

One application of 4.3 is the following result. This extends the theorem [3, 3.23] that, in any 1-based non-trivial ω-categorical simple T, an infinite vector space over some finite field is definably recovered in $\mathcal{M}^{eq}$. Recall that T is *non-trivial* if there are hyperimaginaries a_1, a_2, a_3 and A such that for $1 \leq i < j \leq 3$, a_i, a_j are independent over A whereas $\{a_1, a_2, a_3\}$ is dependent over A.

Theorem 5.1. *Suppose that T is 1-based, non-trivial, having model-4-CA. Then there is a hyperdefinable infinite bounded-by-Abelian group V over a model M of SU-rank 1 generic types. Moreover for the bounded subgroup $V_0 = V \cap \mathrm{bdd}(M)$, V/V_0 forms a vector space over the division ring R of $\mathrm{bdd}(M)$-endomorphisms of V such that for $b, a_1, ..., a_n \in V$, $b \in \mathrm{bdd}(a_1...a_n)$ iff $b + V_0 = \alpha_1(a_1 + V_0) + ... + \alpha_n(a_n + V_0)$ for some $\alpha_i \in R$.*

Proof. By the proof of Lemma 3.22 in [3], there exists a non-trivial rank-1 Lstp p over some model M. For convenience, let $M = \emptyset$ after naming the model. As p is non-trivial, there exists $\{a, b, c\}$ realizing p such that b, c is independent and $a \in \mathrm{bdd}(b, c) \setminus \mathrm{bdd}(b) \cup \mathrm{bdd}(c)$. Let yx realize $\mathrm{tp}(ab/c)$ with $yx \perp_c ab$. Then $dim(ay/bx) = 1$ as $y \in \mathrm{bdd}(abx)$ and $a \perp bx$. Let $z = \mathrm{Cb}(\mathrm{Lstp}(ay/bx))$, then by 1-basedness, $z \in \mathrm{bdd}(ay) \cap \mathrm{bdd}(bx)$. Moreover, by a straightforward rank calculation, $SU(z) = 1$. This gives a bounded quadrangle (a, b, c, x, y, z). Now by Theorem 4.3, we obtain a hyperdefinable group G over $\emptyset$ such that the generic types all have SU-rank 1. The group G is 1-based since the underlying theory is 1-based. Now we use the following fact [17, 4.8.4],

Fact 5.2. *Suppose G is an 1-based group hyperdefinable over $\emptyset$ in a simple theory. Then for the normal subgroup $G^0_\emptyset$, the smallest $\emptyset$-hyperdefinable subgroup of bounded index, the commutator subgroup $(G^0_\emptyset)'$ of $G^0_\emptyset$ has boundedly many elements and contained in the center of $G^0_\emptyset$.*

Therefore, if we set $G^0 = V$, then V is the desired bounded-by-Abelian hyperdefinable group. Note that by above V' is contained in the normal subgroup $V_0 = V \cap \mathrm{bdd}(\emptyset)$. Indeed again from [17, 4.8.18], the Abelian group V/V_0 forms a vector space over a division ring R of bdd($\emptyset$)-endomorphisms of V, and dependence in V/V_0 is given by linear dependence of the vector space. □

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