A NON STANDARD APPROACH TO THE THEORY OF ALGEBRAIC CURVES

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Abstract. This paper is divided into four parts. In the first part, Sections 1-3, using methods developed by Robinson, we find a complete theory suitable for a first order description of specialisations. We use this theory to construct a specialisation having universal properties. In the second part, Sections 4-10, we provide an alternative account of multiplicity for finite morphisms between smooth projective varieties. Traditionally, this has been defined using commutative algebra, rather than geometrically. In model theory, a different approach to defining multiplicity was developed by Boris Zilber, using Zariski structures and infinitesimal neighborhoods. Here, we first formulate Zilber's method in the language of algebraic geometry, using specialisations, and secondly, show that in classical projective situations, for smooth algebraic curves, the 2 definitions essentially coincide. The third part of the paper, Sections 11-15, extends the considerations of the second part, by proving an analogue of Bezout's Theorem for algebraic curves. We achieve this, by showing that in all characteristics, Zilber's notion of Zariski multiplicity coincides with intersection multiplicity, when we consider the full families of projective curves of degree d and e in $P^2(L)$. The result holds even when we consider the intersections at singular points or when the curves contain non-reduced components. In the final part of the paper, Sections 16-18, we give an introduction to Robinson's theory of enlargements in a first order context. In characteristic 0, we construct an enlargement with respect to the complex topology, which is simultaneously a universal specialisation in the sense of the first part of the paper. This shows that Robinson's monads coincide with infinitesimal neighborhoods. We can then re-interpret the results of Parts 2 and 3, in a non-standard context. We obtain, by transfer and compactness, further standard analogues of results in algebraic geometry, reformulated in the complex topology.

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Part 1

1. Specialisations and Valuations

Let L and K be algebraically closed fields with an imbedding $i : L \rightarrow K$. Let $P(K) = \bigcup_{n\geq 1} P^n(K)$ and $P(L) = \bigcup_{n\geq 1} P^n(L)$. By a closed algebraic subvariety of $P^n(K)$, we mean a set $W(K)$ where W is defined by homogeneous polynomial equations with coefficients in K. We say that $W(K)$ is defined over L if we can take the coefficients to lie in L. Let $W_n^m(K)$ denote the $m'th$ Cartesian product of $P^n(K)$. By a closed algebraic subvariety of $W_n^m(K)$, we mean a set $W(K)$ defined by multi-homogeneous polynomial equations with coefficients in K, similarly we can make sense of the notion of being defined over L. Note that by completeness of projective space, the projection maps $pr_{k,m} : W_n^k(K) \rightarrow W_n^m(K)$ are closed.

Definition 1.1. *A specialisation is a map* $\pi = \bigcup_{n\geq 1} \pi_n : P(K) \rightarrow P(K)$, *such that each* $\pi_n : P^n(K) \rightarrow P^n(K)$ *has the following property;*

(i). Let $W_n^m(K)$ *denote the m'th Cartesian product of* $P^n(K)$*. Then, if* $V \subset W_n^m(K)$ *is a closed algebraic subvariety defined over L and* $\bar{a}$ *is an m-tuple of elements from* $W_n(K)$*, such that* $V(\bar{a})$ *holds, then* $V(\pi_n(\bar{a}))$ *holds as well.*

The following compatibility requirement must also hold between the π_n*;*

(ii). Fix the following chain of embeddings i_n *of* $P^n(K)$ *into* $P^{n+1}(K)$*, for* $n \geq 1$.

$i_n : [x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0]$.

Then we require that $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$.

We also require;

(iii). If $j : Im(\pi) \rightarrow P(K)$ *is the inclusion map, then* $\pi \circ j = Id_{Im(\pi)}$

Definition 1.2. *A Krull valuation* v *is a map* $v : K \rightarrow \Gamma \cup \infty$ *where* Γ *is an ordered abelian group with the following properties;*

(i). $v(x) = \infty$ *iff* $x = 0$.
(ii). $v(xy) = v(x) + v(y)$
(iii). $v(x + y) \geq min\{v(x), v(y)\}$
(iv). $v|L = 0$.

Here, we adopt the convention that $\gamma < \infty$ *for* $\gamma \in \Gamma$ *and extend* $+$ *naturally to* $\Gamma \cup \infty$.

We let $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$ *be the valuation ring of* v, $\mathcal{O}_v^* = \{x \in K : v(x) = 0\}$ *the subgroup of units and* $\mathcal{M}_v = \{x \in K : v(x) > 0\}$ *the unique maximal ideal of* $\mathcal{O}v$. *We can identify the algebraically closed residue field* K_{res} *with* $\mathcal{O}_v^* \cup \{0\}$. *Then, by (iv),* $i : L \rightarrow K$ *defines an embedding of* L *into* K_{res}.

Definition 1.3. *We say that two Krull valuations* v_1 *and* v_2 *are equivalent, denoted by* $v_1 \sim v_2$ *if* $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Lemma 1.4. v_1 *and* v_2 *are equivalent iff there exists* $\Theta : \Gamma_1 \rightarrow \Gamma_2$ *such that* $\Theta \circ v_1 = v_2$.

In order to see this, define $\Theta(v_1(x)) = v_2(x)$, this is well defined as if $v_1(x) = v_1(x')$, then $v_1(x/x') = 0$, hence x/x' and x'/x belong to $\mathcal{O}_{v_1}$. If $v_1 \sim v_2$, then x/x' and x'/x belong to $\mathcal{O}_{v_2}$ as well, which gives that $v_2(x) = v_2(x')$. One can easily check that Θ is an isomorphism of ordered abelian groups as required.

Our main result in this section is the following;

Theorem 1.5. *Let* $X := \{\pi : P(K) \rightarrow P(K)\}$ *be the set of specialisations and* $Y := \{v/\sim : v : K \rightarrow \Gamma\}$ *be the set of equivalence classes of Krull valuations. Then there exists a natural bijection between* X *and* Y. *Specifically, there exists maps* Φ *and* Ψ;

$\Phi : X \rightarrow Y$

$\Psi : Y \rightarrow X$

with $\Psi \circ \Phi = Id_X$ *and* $\Phi \circ \Psi = Id_Y$

We first show;

Theorem 1.6. *There exists* $\Psi : Y \rightarrow X$

Proof. Let $[v]$ denote a class of Krull valuations on K. We define a specialisation map $\pi_{[v]}$ as follows;

Let $(x_0 : x_1 : \ldots : x_n)$ denote an element of $P^n(K)$ written in homogeneous coordinates. For some $\lambda \in K$, the elements $\{\lambda x_0, \ldots, \lambda x_n\}$ will lie in $\mathcal{O}_v$ and not all of them will lie in $\mathcal{M}_v$. Let $\pi : \mathcal{O}_v \rightarrow K_{res}$ denote the unique ring morphism such that $\pi \circ i = Id_{K_{res}}$ where i is the inclusion map from K_{res} into $\mathcal{O}_v$, $(*)$. Then $(\pi(\lambda x_0) : \pi(\lambda x_1) : \ldots : \pi(\lambda x_n))$ defines an element of $P^n(K_{res})$. As is easily checked, the mapping is independent of the choice of λ and depends only on $\mathcal{O}_v$, hence we obtain $\pi_{n,[v]} : P^n(K) \rightarrow P^n(K_{res})$. We need to check that each $\pi_{n,[v]}$ satisfies the property (i) required of a specialisation. We will just verify this in the case when $m \leq 2$ for each $n \geq 1$, the other cases are straightforward generalisations;

For $m = 1$, let $V \subset P^n(K)$ be a closed subvariety defined over L, then V is defined by a system of homogeneous equations in the variables $\{x_0, \ldots, x_n\}$ with coefficients in L. Taking a tuple $\bar{a}$ belonging to V, we can assume that the elements $\{a_0, a_1, \ldots, a_n\}$ belong to $\mathcal{O}_v$. Now, using the fact that the residue map π is a ring homomorphism fixing L, the reduced elements $\{\pi(a_0), \pi(a_1), \ldots, \pi(a_n)\}$ also satisfy the same homogeneous equations as required.

For the case when $m = 2$, we use the Segre embedding which is defined by;

$$Segre : P^n(K) \times P^n(K) \rightarrow P^{n(n+2)}(K)$$

$$((x_0 : \ldots : x_n), (y_0 : \ldots : y_n)) \mapsto (x_0y_0 : \ldots : x_0y_n : x_1y_0 : \ldots : x_ny_n)$$

The following diagram is easily checked to commute:

$$\begin{array}{ccc} P^n(K) \times P^n(K) & \xrightarrow{Segre} & P^{n(n+2)}(K) \\ \downarrow{\scriptstyle \pi_{n,[v]} \times \pi_{n,[v]}} & & \downarrow{\scriptstyle \pi_{n(n+2),[v]}} \\ P^n(K_{res}) \times P^n(K_{res}) & \xrightarrow{Segre} & P^{n(n+2)}(K_{res}) \end{array}$$

.

Therefore, it is sufficient to prove that the property holds for $\pi_{n(n+2),[v]} : P^{n(n+2)}(K) \rightarrow P^{n(n+2)}(K_{res})$ when $m = 1$.This is the case covered above.

We need to check the compatibility requirement (ii) for the $\pi_{n,[v]}$, this is a trivial calculation.

By the property $(*)$, $\pi_{[v]}$ fixes $P(K^{res})$, and therefore $(\pi_{[v]} \circ j) = Id_{P(K^{res})}$.

Denote the specialisation map we have obtained by $\pi_{[v]}$ and let $\Psi([v]) = \pi_{[v]}$.

□

We now show;

Theorem 1.7. *There exists* $\Phi : X \rightarrow Y$

Proof. Suppose that we are given a specialisation π. In particular we have a map $\pi_1 : P^1(K) \rightarrow P^1(K)$ satisfying the requirements above. We want to show how to recover a Krull valuation on K.

Let $\gamma : K \rightarrow P^1(K)$ be the map $\gamma : k \mapsto [k : 1]$, so $\pi_1 \circ \gamma : K \rightarrow P^1(K)$. Let $U \subset P^1(L)$ be the open subset defined by $P^1 \setminus [1 : 0]$. Let $\mathcal{O}_K = (\pi_1 \circ \gamma)^{-1}(U)$ and $\mathcal{M}_K = (\pi_1 \circ \gamma)^{-1}([0 : 1])$. We now claim the following;

Lemma 1.8. *$\mathcal{O}_K$ is a subring of K with $Frac(\mathcal{O}_K) = K$ and $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$.*

Proof. Suppose that $\{x, y\} \subset \mathcal{O}_K$, then both $\pi_1([x : 1])$ and $\pi_1([y : 1])$ are in U. Let $C \subset P^1(K) \times P^1(K) \times P^1(K)$ be the closed set defined in coordinates $([u : v], [w : x], [y : z])$ by the equation $uwz = yvx$. As is easily checked, we have that $C([x : 1], [y : 1], [xy : 1])$. By the defining property of π_1, $C(\pi_1([x : 1]), \pi_1([y : 1]), \pi_1([xy : 1]))$ also holds. Therefore, $C([\lambda : 1], [\mu : 1], [\alpha, \beta])$ where $\lambda, \mu, \alpha, \beta$ are in K. By definition of C, we have $\lambda\mu\beta = \alpha$ which forces $\beta \neq 0$. Hence, $\pi_1([xy : 1]) \in U$ and therefore $xy \in \mathcal{O}_K$. Let $D \subset P^1(K) \times P^1(K) \times P^1(K)$ be defined using the same choice of coordinates by the equation $uxz + wvz = yvx$. Then we have that $D([x : 1], [y : 1], [x + y : 1])$ and therefore $D(\pi_1([x : 1]), \pi_1([y : 1]), \pi_1([x + y : 1]))$. Again, we must have $D([\lambda : 1], [\mu, 1], [\delta, \epsilon])$ where $\lambda, \mu, \delta, \epsilon$ are in K. This forces $(\lambda + \mu)\epsilon = \delta$ and therefore $\epsilon \neq 0$, so $x + y \in \mathcal{O}_K$. Clearly, $1 \in \mathcal{O}_K$

which shows that $\mathcal{O}_K$ is a subring of K as required. In order to see that $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$, let $x \in \mathcal{O}_K$ and $y \in \mathcal{M}_K$. We have that $C([\lambda : 1], [0 : 1], [\alpha, \beta])$ where $\pi_1([xy : 1]) = [\alpha, \beta]$. Then $\lambda.0.\beta = 1.1.\alpha$ forcing $\alpha = 0$ and $\beta = 1$, so $xy \in \mathcal{M}_K$. If $x \in \mathcal{M}_K$ and $y \in \mathcal{M}_K$ we obtain $D([0 : 1], [0 : 1], [\delta, \epsilon])$ where $\pi_1([x + y : 1] = [\delta, \epsilon]$. Then $0.1.\beta + 1.0.\epsilon = 1.1.\delta$, so $\delta = 0$ and $\epsilon = 1$, hence $x + y \in \mathcal{M}_K$ as required. Finally, we show that $Frac(\mathcal{O}_K) = K$. Suppose $x \notin \mathcal{O}_K$, then $\pi_1([x : 1]) = [1 : 0]$. We have that $C([x : 1], [1/x : 1], [1 : 1])$, hence $C([1 : 0], [\alpha, \beta], [1 : 1])$ where $\pi_1([1/x : 1]) = [\alpha, \beta]$. This forces $1.\alpha.1 = 0.\beta.1$, hence $\alpha = 0$ and $\beta = 1$. Therefore $1/x \in \mathcal{O}_K$ as required. □

We now further claim the following;

Lemma 1.9. *If π_1 is non-trivial, then $\mathcal{O}_K$ is a proper subring of K.*

Proof. By the same argument as above we have that $\pi_1 \circ \gamma(1/\mathcal{M}_K) = [1 : 0]$, hence *if* $\mathcal{O}_K = K$, using the previous lemma, we must have that $\mathcal{M}_K = 0$. If π_1 is non-trivial, we can find $x \in K$ and $y \in K$ distinct such that $\pi_1([x : 1]) = \pi_1([y : 1])$. By the usual arguments, we then have that $\pi_1([x - y : 1]) = [0 : 1]$, so $x - y \in \mathcal{M}_K$ contradicting the fact that $\mathcal{M}_K = \{0\}$.

□

We can now construct a Krull valuation on K by a standard method. Let $\Gamma = K^*/\mathcal{O}_K^*$ and define $v : K \rightarrow \Gamma$ by $v(x) = x \ mod \ \mathcal{O}_K^*$ and $v(0) = \infty$. Define an ordering on the abelian group Γ by declaring $v(x) \leq v(y)$ iff $y/x \in \mathcal{O}_K$. This is well defined as if $v(x) = v(x')$ and $v(y) = v(y')$, then y'/y, y/y',x/x' and x'/x are all in $\mathcal{O}_K$. We have that $y'/x' = y/x.y'/y.x/x'$ and $y/x = y'/x'.y/y'.x'/x$, therefore $y'/x' \in \mathcal{O}_K$ iff $y/x \in \mathcal{O}_K$ as required. Transitivity of the ordering follows from the fact that $\mathcal{O}_K$ is a subring of K. $\leq$ is a linear ordering as if $x \in K^*$ and $y \in K^*$ then either x/y or y/x lies in $\mathcal{O}_K$. Finally, we clearly have that if $y/x \in \mathcal{O}_K$ then $yz/xz \in \mathcal{O}_K$, hence $v(x) \leq v(y)$ implies $v(x) + v(z) \leq v(y) + v(z)$. This turns Γ into an ordered abelian group. Properties (i) and (ii) of the axioms for a Krull valuation are trivial to check. Suppose property (iii) fails, then we can find x, y with $v(x + y) < v(x)$ and $v(x + y) < v(y)$. Therefore $(x + y)/x \notin \mathcal{O}_K$ and $(x + y)/y \notin \mathcal{O}_K$. As $1 \in \mathcal{O}_K$, we have that $x/y \notin \mathcal{O}_K$ and $y/x \notin \mathcal{O}_K$ which is a contradiction. Finally, we check property (iv). As π_1, fixes $P^1(L)$, we have that $L^* \subset \mathcal{O}_K^*$, hence $v|L$ is trivial. Denote the valuation we have obtained by v_π and set $\Phi(\pi) = [v_\pi]$. This ends the proof of Theorem 1.7

□

We now complete the proof of Theorem 1.5;

Proof. $\Phi \circ \Psi = Id_Y$.

Let $[v]$ be a class of Krull valuations on K with corresponding specialisation $\pi_{[v]}$ provided by Ψ. Let $\pi_{1,[v]}$ be the restriction to $P^1(K)$. By definition, if $k \in \mathcal{O}_v$ then $\pi_{1,[v]}([k:1]) = [\pi(k), 1]$ where π is the residue map for v. If $k \notin \mathcal{O}_v$, then $\pi_{1,[v]}([k:1]) = [0, 1]$, so we see that $\mathcal{O}_K$ as defined above is exactly $\mathcal{O}_v$. The valuation $v_{\pi_{[v]}}$ constructed from $\pi_{[v]}$ therefore has the same valuation ring $\mathcal{O}_v$, so $v \sim v_{\pi_{[v]}}$ which gives the result.

$\Psi \circ \Phi = Id_X$.

Let π be a given specialisation and $[v_\pi]$ the corresponding class of Krull valuations. Let π_1 be the restriction of π to $P^1(K)$ and π_{1,v_π} the specialisation constructed from v_π restricted to $P^1(K)$. We have;

(i). $\pi_{1,v_\pi}([k:1]) = [0:1]$ iff $v_\pi(k) > 0$ iff $k \in \mathcal{M}_{v_\pi}$ iff $k \in \mathcal{M}_K$ as defined above iff $\pi_1([k:1]) = [0:1]$

(ii). $\pi_{1,v_\pi}([k:1]) = [1:0]$ iff $v_\pi(k) < 0$ iff $k \notin \mathcal{O}_{v_\pi}$ iff $k \notin \mathcal{O}_K$ as defined above iff $\pi_1([k:1]) \notin U$ iff $\pi_1([k:1]) = [1:0]$

(iii). $\pi_{1,v_\pi}([1:0]) = \pi_1([1:0]) = [1:0]$ trivially.

If $k \in \mathcal{O}_{v_\pi}$, then $\pi_{1,v_\pi}([k:1]) = [\alpha(k):1]$ where α is the residue mapping associated to v_π. We also have that $\pi_1([k:1]) \in U$, hence as π_1 is a specialisation that $\pi_1([k:1]) = [\beta(k):1]$ where β is a homomorphism from $\mathcal{O}_{v_\pi}$ to K_{res}. We thus obtain two homomorphisms $\alpha, \beta : \mathcal{O}_{v_\pi} \rightarrow K_{res}$ such that (by (i)) $Ker(\alpha) = Ker(\beta) = \mathcal{M}_{v_\pi}$ and with the property that $\alpha \circ i = \beta \circ i = Id_{K_{res}}$ where i is the natural inclusion of K_{res} in $\mathcal{O}_{v_\pi}$. We thus obtain the splitting $\mathcal{O}_{v_\pi} = K_{res} \oplus Ker(\alpha) = K_{res} \oplus Ker(\beta) = K_{res} \oplus M$ with $Ker(\alpha) = Ker(\beta) = M$. Now, using this fact, we can write any element of $\mathcal{O}_{v_\pi}$ uniquely in terms of K_{res} and M, hence the corresponding projections α and β are the same.

We have shown that $\pi_1 = \pi_{1,v_\pi}$, it remains to check that $\pi_n = \pi_{n,v_\pi}$ for all $n \geq 1$. We prove this by induction on n, the case $n = 1$ having been established.

By the induction hypothesis and the compatibility requirement between the π_n, for $\{k_0, k_1, \ldots, k_n\} \subset \mathcal{O}_{v_\pi}$;

$$\pi_{n+1}([k_0 : k_1 : \ldots : k_n : 0]) = [\pi(k_0) : \pi(k_1) : \ldots : \pi(k_n) : 0] \ (*)$$

where π is the residue map on $\mathcal{O}_{v_\pi}$.

Let $C \subset P^{n+1}(K)$ be the closed subvariety defined using coordinates $[x_0 : x_1 : \ldots : x_{n+1}]$ by the equations $x_0 = x_1 = \ldots = x_{n-1} = 0$. Then by arguments as above and the fact that C is preserved by π_{n+1}, we can find a Krull valuation v' on K with corresponding residue mapping π' such that;

$$\pi_{n+1}([0 : \ldots : 0 : 1 : k_{n+1}]) = [0 : \ldots : 0 : 1 : \pi'(k_{n+1})] \text{ if } v'(k_{n+1}) \geq 0$$

$$= [0 : \ldots : 0 : 0 : 1] \text{ otherwise } (**)$$

Now let D be the closed subvariety of $P^{n+1}(K)$ defined by the equations $x_1 = \ldots = x_n$ and $x_0 = x_{n+1}$. Again, we have that π_{n+1} preserves D, hence there exists a Krull valuation v'' on K with corresponding residue mapping π'' such that;

$$\pi_{n+1}([k : 1 : \ldots : 1 : k]) = [\pi''(k) : 1 \ldots : 1 : \pi''(k)] \text{ if } v''(k) \geq 0$$

$$= [1 : 0 : \ldots : 0 : 1] \text{ otherwise } (***)$$

Let Sum be the closed subvariety of $P^{n+1}(K) \times P^{n+1}(K) \times P^{n+1}(K)$ defined using coordinates $[x_0 : x_1 : \ldots : x_{n+1}]$, $[y_0 : y_1 : \ldots : y_{n+1}]$ and $[z_0 : z_1 : \ldots : z_{n+1}]$ by the equations $x_0 y_1 z_1 + y_0 x_n z_1 = z_0 x_n y_1$ and $x_{n+1} y_1 z_1 + y_{n+1} x_n z_1 = z_{n+1} x_n y_1$. Then, for $k \in K$, we have that $Sum([0 : 0 : \ldots : 1 : k], [k : 1 : \ldots : 0 : 0], [k : 1 : \ldots : 1 : k])$, hence by the properties of a specialisation that $Sum(\pi_{n+1}([0 : 0 : \ldots : 1 : k]), \pi_{n+1}([k : 1 : \ldots : 0 : 0]), \pi_{n+1}([k : 1 : \ldots : 1 : k]))$.

In the generic case when $v_\pi(k), v'(k), v''(k)$ are all non-negative, we obtain $Sum([0 : 0 : \ldots : 1 : \pi'(k)], [\pi(k) : 1 : \ldots : 0 : 0], [\pi''(k) : 1 : \ldots : 1 : \pi''(k)])$ which gives the relations $0.1.1 + \pi(k).1.1 = \pi''(k).1.1$ and

$\pi'(k).1.1 + 0.1.1 = \pi''(k).1.1$, so $\pi(k) = \pi'(k) = \pi''(k)$.

A simple calculation shows that $v_\pi(k) < 0$ iff $v'(k) < 0$ iff $v''(k) < 0$, hence $\mathcal{O}_{v_\pi} = \mathcal{O}_{v'} = \mathcal{O}_{v''}$. We have now shown the following further compatibility between π_1 and π_{n+1}. Namely;

If $\gamma : P^1(K) \to P^{n+1}(K)$ is given by $\gamma : [x_0, x_1] \mapsto [0 : 0 : \ldots : x_0 : x_1]$ then $\pi_{n+1} \circ \gamma = \gamma \circ \pi_1$. (†)

Finally, let Sum' be the closed subvariety of $P^{n+1}(K) \times P^{n+1}(K) \times P^{n+1}(K)$ defined in coordinates $[x_0 : \ldots : x_{n+1}], [y_0 : \ldots : y_{n+1}], [z_0 : \ldots : z_{n+1}]$ by the $(n+1)$ equations $x_j y_1 z_1 + y_j x_n z_1 + z_j x_n y_1$ for $j \neq n$. Let $[k_0 : \ldots : k_{n+1}]$ be an arbitrary element of $P^{n+1}(K)$. Without loss of generality, we may assume that $\{k_0 : \ldots : k_{n+1}\} \subset \mathcal{O}_{v_\pi}$ and that $k_n \in \mathcal{O}^*_{v_\pi}$. Hence, dividing by k_n, the element is of the form $[k_0 : \ldots : k_{n-1} : 1 : k_{n+1}]$ with $\{k_0, \ldots, k_{n-1}, k_{n+1}\} \subset \mathcal{O}_{v_\pi}$. We have that $Sum'([0 : \ldots : 0 : 1 : k_{n+1}], [k_0 : \ldots : k_{n-1} : 1 : 0], [k_0 : \ldots : k_{n-1} : 1 : k_{n+1}])$, hence by specialisation and (†), $Sum'([0 : \ldots : 0 : 1 : \pi(k_{n+1})], [\pi(k_0) : \ldots : \pi(k_{n-1}) : 1 : 0], [\pi_{n+1}(k_0) : \ldots : \pi_{n+1}(k_n) : \pi_{n+1}(k_{n+1})])$. As is easily checked, the case when $\pi_{n+1}(k_n) = 0$ leads to a contradiction, hence we can assume that $\pi_{n+1}(k_n) = 1$ (multiplying by $1/\pi_{n+1}(k_n)$). Now the equations give that $\pi_{n+1}(k_j) = \pi(k_j)$ for $j \neq n$. We have therefore shown that $\pi_{n+1} = \pi_{n+1,v_\pi}$ as required.

□

Theorem 1.5 is now proved.

We can extend Theorem 1.5, using Lemma 1.9, to prove a slightly more general result. Namely, we will denote by $< P(K), P(L), \pi >$ a triple satisfying the properties of a specialisation, Definition 1.1, with the additional requirement that π_1 is non-trivial;

In the rest of this part, we will refer to a specialisation as a triple satisfying this definition.

We restrict the concept of Krull valuation, to mean a non-trivial valuation, satisfying the properties of Definition 1.2.

With these definitions, one has;

Theorem 1.10. *There exists a bijection between the set of specialisations $< P(K), P(L), \pi >$ and equivalence classes of Krull valuations. In particular, any specialisation $< P(K), P(L), \pi >$ as just defined, can be written as $< P(K), P(L'), \pi >$ with $L \subset L' \subset K$ algebraically closed, L' a proper subfield of K, and π mapping $P(K)$ onto $P(L')$, while fixing $P(L')$.*

2. A Model Theoretic Language of Specialisations

We now introduce a model theoretic language which will enable us to describe specialisations in the context of algebraic geometry. In this section, we will assume that K and its residue field have the same characteristic. We will use a many sorted structure $\{\bigcup S_n : n \in \mathcal{N}\}$. Each sort will be the domain of $P^n(K)$ for an algebraically closed field K. We fix an algebraically closed constant field L which we assume to be countable and let K be some non-trivial extension of L, having the same characteristic. In order to describe algebraic geometry, we introduce sets of predicates $\{V_n^m\}$ on the Cartesian powers S_n^m to describe closed algebraic subvarieties of $P^n(K)$ defined over L. In particular, we have constants to denote the individual elements of $P^n(L)$ on each sort S_n. We introduce function symbols $i_n : S_n \rightarrow S_{n+1}$ to describe the imbeddings $P^n(K) \rightarrow P^{n+1}(K)$ defined above. Finally, we will have function symbols $\{\pi_n : n \in \mathcal{N}\}$ to describe the specialisation map $\pi = \cup_{n \geq 1} \pi_n$ from $P(K)$ onto $P(L)$. Observe that $P^n(L)$ is then definable, as $\{x \in P^n(K) : \exists y(\pi_n(y) = x)\}$. We denote the language $< \{V_n^m\}, i_n, \pi_n >$ by $\mathcal{L}_{spec}$ and the theory of the structure $< P(K), P(L), \pi >$ in this language by T_{spec}. We denote the theory of the structure $< P(K), P(L) >$ in the language $\mathcal{L}_{spec} \setminus \{\pi_n\}$ by T_{alg}. Note that the structure $< K, 0, 1, +, . >$ is interpretable in the structure $< P(K), P(L) >$ in the language $\mathcal{L}_{spec} \setminus \{\pi_n\}$ $(*)$. This follows by noting that the points $[1 : 0], [0 : 1]$ and $[1 : 1]$ are named as elements in the sort S_1 and the operations of $+, .$ define algebraic subvarieties in the sorts S_1^3. The structure $< L, 0, 1, +, . >$ is not interpretable in the language $\mathcal{L}_{spec} \setminus \{\pi_n\}$ but any model of T_{alg} will contain an isomorphic copy of $P(L)$ as a substructure. It follows that the models of T_{alg} are exactly of the form $< P(K), P(L) >$ for some algebraically closed field K extending L (use the fact that the axiomatisation of $Th(< K, 0, 1, +, . >)$ can be interpreted in T_{alg} and the field structure can be related to the predicates $\{V_n^m\}$ using the imbeddings i_n). We now claim the following;

Theorem 2.1. *The theory T_{spec} is axiomatised by $T_{axioms} = T_{alg} \cup \Sigma$ where Σ is the set of sentences given by;*

(i). The mappings $\{\pi_n\}$ preserve the predicates $\{V_n^m\}$.

(ii). The compatibility requirement $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$ holds.

(iii). The mappings π_n are trivial on $Im(\pi_n)$.

(iv). The mapping $\{\pi_1\}$ is non-trivial.

(see definition 1.1). In particular, T_{axioms} is complete. Moreover, T_{axioms} is model complete.

The proof of this theorem will be based on Theorem 1.10 and the following result by Robinson, given in [23];

Theorem 2.2. *Let K be an algebraically closed field with a non trivial Krull valuation v and residue field l. Then T_K is model complete in the language $\mathcal{L}_{val}$ and admits quantifier elimination in the language $\mathcal{L}_{rob}$. Moreover, the completions of K are determined by the pair $(char(l), char(K))$, that is $T_K \cup \Sigma$ is complete where Σ is the possibly infinite set of sentences specifying the characteristic of K and l.*

Here, by the language $\mathcal{L}_{rob}$ we mean the language of algebraically closed fields together with a binary predicate $Div(x, y)$ denoting $v(x) \leq v(y)$. By the language $\mathcal{L}_{val}$, we mean a 2-sorted language for the value group and the field, with the usual language for the field sort and the language of ordered groups on the group sort. T_K is the theory which asserts that K is an algebraically closed field, the value group Γ is linearly ordered and abelian, the valuation is non-trivial. For our purposes, we will require a slightly refined version of this result. Namely, we will fix a set of constants for an algebraically closed field L which we can assume to be countable, add to T_K the atomic diagram of L, relativized to the field sort, and the requirement that $v|L$ is trivial. We will denote the corresponding theory by $T_{K,L}$ and the expanded languages by $\mathcal{L}_{rob}$ and $\mathcal{L}_{val}$ again. It is no more difficult to prove that $T_{K,L}$ is model complete, Robinson's original proof in [23] requires the solution of certain valuation equations in the model K given that these equations have a solutions in an extension K', it makes no difference if some of the elements from K are named. In order to show that $T_{K,L}$

is complete, it is sufficient to exhibit a prime model of the theory;

Case 1. $Char(K,L)=(p,p)$, with $p\neq 0$. Take $L(\epsilon)^{alg}$ where ϵ is transcendental over L, define the valuation on L to be zero and extend it to $L(\epsilon)$ non-trivially using say $v_{ord,\epsilon}$, the order valuation in ϵ. Take any extension to $L(\epsilon)^{alg}$.

Case 2. $Char(K,L)=(0,0)$, define a similar valuation on $L(\epsilon)^{alg}$.

We now show the following lemmas;

Lemma 2.1. *Existence of Specialisations*

Let $L\subset L'$ be a pair of algebraically closed fields. Then there exists a specialisation $(P(L'),P(L),\pi)$.

Proof. Let I be a transcendence basis for L' over L. We may assume that I is well ordered by ω and use the method of transfinite induction to construct a tower of algebraically closed fields $L\subset L_i\subset L_{i+1}$ and valuations v_i with residue field L. Clearly, the field L_ω has a valuation v_ω with residue field L. By embedding L' into L_ω over L, so has L'. We can then invoke Theorem 1.5, proving the result. For $i=0$, set $L_0=L$, for i a limit ordinal, set;

$(L_i,L,v_i)=\bigcup_{j<i}(L_j,L,v_j)$

For i not a limit ordinal, let $L_i((t_{i+1}))$ be the field of Laurent series in the variable t_{i+1} over the algebraically closed field L_i. We construct a valuation v_{i+1} on this field, with residue field L_i. Namely, for $f\in L_i((t_{i+1}))$, we set $v_{i+1}(f)=ord_{i+1}(f)$, where $ord_{i+1}(f)$ is the minimum n appearing in the Laurent expansion of f. As is shown in [17], $(L_i((t_{i+1})),v_{i+1})$ is the completion of $(L_i(t),v)$, for the canonical valuation v on the function field $L_i(t)$. It follows that $L_i((t_{i+1}))$ is a Henselian field with respect to v_{i+1}. By Hensel's lemma, $L_i((t_{i+1}))^{alg}$ is a union $\bigcup_{i<\omega}L_i((t_{i+1}^{1/n}))$ of ramified extensions of $L_i((t_{i+1}))$. The valuation v_{i+1} then extends uniquely to the spectral valuation on $L_i((t_{i+1}^{1/n}))$ by the formula;

$\overline{v}_{i+1}(\alpha)=(1/n)v_{i+1}(N_{L_i((t^{1/n}))/L_i((t))}(\alpha))$

Clearly, the residue field L_i is preserved. We set v_{i+1} to be the extension of v_i to L_{i+1}, obtained in this manner. By Theorem 1.5, we obtain a specialisation $(P(L_{i+1}), P(L_i))$, and compose this with the specialisation $(P(L_i), P(L))$ to obtain a specialisation $(P(L_{i+1}), P(L))$. Using Theorem 1.5 again, we obtain a Krull valuation v_{i+1} on L_{i+1} with residue field L. This completes the induction step.

□

We observe that if M is a model of T_{axioms}, then by Theorem 1.10, it can be written as $(P(K), P(K_{res}), \pi)$, with $L \subset K_{res} \subset K$, K_{res} a proper subfield of K, and with π fixing $P(K_{res})$.

Lemma 2.2. *Amalgamation of Specialisations*

Let $(P(K_1), P(L_1), \pi_1)$ and $(P(K_2), P(L_2), \pi_2)$ be models of T_{axioms}, then there exists a further model $(P(K_3), P(L_3), \pi_3)$ such that;

$(P(K_1), P(L_1), \pi_1) \leq (P(K_3), P(L_3), \pi_3)$

and

$(P(K_2), P(L_2), \pi_2) \leq (P(K_3), P(L_3), \pi_3)$

Moreover, if $L = L_1 = L_2$, then we can take $L = L_3$.

Proof. By Theorem 1.10, we can find Krull valuations v_1 and v_2 on K_1 and K_2 such that $\pi_1 = \pi_{v_1}$ and $\pi_2 = \pi_{v_2}$. By condition (iv) of Theorem 2.1, these valuations are non-trivial. Using the refined version of Robinson's completeness result, we can jointly embed (K_1, v_1) and (K_2, v_2) over L into (K_3, v_3) (*). Let L_3 be the residue field of v_3, then as K_3 is algebraically closed, so is L_3 and extends the residue field L_1 of v_1 and L_2 of v_2. Using Theorem 1.5 again, we can construct a specialisation $(P(K_3), P(L_3), \pi_{v_3})$. It remains to see that in fact π_3 extends the specialisations π_1 and π_2. This follows from the fact that if $k \in K_1$, and there exists $l \in L_1$ such $v_1(k - l) > 0$, then this relation is preserved in the embedding $(*)$. Hence the specialisation π_{v_3} already extends the specialisations π_1 and π_2 of $P(K_1)$ and $P(K_2)$ onto $P(L_1)$ and $P(L_2)$ respectively. In order to see the last part, let $(P(K_1), P(L), \pi_1)$ and $(P(K_2), P(L), \pi_2)$ jointly embed in $(P(K_3), P(L_3), \pi')$. By Lemma 2.1, we can construct a specialisation $(P(L_3), P(L), \pi)$. Let $(P(K_3), P(L), \pi_3)$ be the composition of these specialisations. It remains to see that again that π_3 extends the specialisations π_1 and π_2. This follows from the fact that $(*)$ is an L-embedding. $\square$

Lemma 2.3. *Transfer of Formulas*

Let $(P(K), P(L_1), \pi)$ be a specialisation with corresponding (K, v), then there exists a map;

$$\sigma : P(K) \rightarrow K^{eq}$$

$$\sigma : \mathcal{L}_{spec}\text{-formulae} \rightarrow \mathcal{L}_{val}\text{-formulae}$$

such that for any $\phi(x_1, \ldots, x_n)$ which is a $\mathcal{L}_{spec}$-formula and $(k_1, \ldots, k_n) \subset P(K)$;

$$(P(K), P(L_1), \pi) \models \phi(k_1, \ldots, k_n) \text{ iff } (K, v) \models \sigma(\phi)(\sigma(k_1), \ldots, \sigma(k_n))$$

. $(\dagger)$

Moreover, the definition of the map is uniform in K.

Proof. The map σ is defined on the sorts $P^n(K)$ by sending $[k_0, \ldots, k_n]$ to $(k_0, \ldots, k_n)/\sim_n$ where $\sim_n$ is the equivalence relation defined on K^{n+1} from multiplication by K^*. Similarly, σ maps a variable from the sort S_n to the corresponding variable from the sort in K^{eq} defined by $\sim_n$. A closed algebraic subvariety in $\{V_n^m\}$ is defined by a multi-homogeneous equation in the variables $\{(x_{01}, \ldots, x_{n1}), \ldots, (x_{0m}, \ldots, x_{nm})\}$. Let C_n^m be the algebraic variety in $K^{m(n+1)}$ defined by this equation. Then the corresponding formula in K^{eq} is given by;

$$(y_1, \ldots, y_m) \in (\sim_n)^m [\exists x_1 \ldots x_m (C_n^m(x_1, \ldots, x_m) \wedge \textstyle\bigwedge_{i=1}^m x_i/\sim_n = y_i)]$$

For the inclusion maps i_n, let us identify each i_n with its graph, then clearly we can define σ to map the formula $i(x) = y$ to a corresponding formula relating the sorts $\sim_n$ and $\sim_{n+1}$ in K^{eq}.

Note that if $l \in P^n(L)$ is a constant, then $\sigma(l) = (l_0, \ldots, l_n)/\sim_n$ where each l_i is a constant from the atomic diagram of L.

Finally, let $\pi_n : P^n(K) \rightarrow P^n(L_1)$ be a specialisation. Let us assume that we can identify π_n with its graph. We then have that;

$\pi_n([x_0 : \ldots : x_n]) = [y_0 : \ldots : y_n]$

iff

$\exists z \exists z_0 \ldots \exists z_n((\bigwedge_{i=0}^n x_i z = y_i + z_i) \wedge (\bigwedge_{i=0}^n v(z_i) > 0))$.

It is now clear how to define $\sigma(\pi_n)$ as a formula in the sort defined by $\sim_n$.

This completes the definition of σ, it is clear that the definition is uniform in K and a straightforward induction on the length of a formula from $\mathcal{L}_{spec}$ shows that it has the required property (†).

□

Theorem 2.1 is now a fairly straightforward consequence of the above lemmas. We first show model completeness. Suppose that we have models of T_{axioms};

$(P(K_1), P(L_1), \pi_1) \leq (P(K_2), P(L_2), \pi_2)$

By Theorem 1.10, we can find Krull valuations v_1 and v_2 such that $(K_1, v_1) \leq (K_2, v_2)$ and $(K_1, v_1), (K_2, v_2) \models T_{K,L}$. By the refined model completeness result after Theorem 2.2, we have $(K_1, v_1) \prec (K_2, v_2)$, hence using Lemma 2.3, we must have that;

$(P(K_1), P(L_1), \pi_1) \prec (P(K_2), P(L_2), \pi_2)$

as required. Completeness now follows directly from Lemma 2.2 and model completeness. Alternatively, one can exhibit a prime model of the theory, this is clearly possible by taking the specialisations corresponding to the prime models of $T_{K,L}$ above.

3. Constructing a Universal Specialisation

We now construct a specialisation $(P(K_{univ}), P(L), \pi_{univ})$ having the following "universal" property;

If $L \subset L_m$ is an algebraically closed extension of L with transcendence degree m, and $(P(L_m), P(L), \pi_m)$ is a specialisation, then there exists an L-embedding $\alpha_L : L_m \rightarrow K_{univ}$ with the property that $\pi_{univ} \circ \alpha_L = \pi_m$. (*)

This result is required in the subsequent parts of the paper.

Model theoretically, using theorem 2.1, it is easy to show the existence of such a structure. Namely, let $(P(K_{univ}), P(L_1), \pi_1)$ be a 2^ω saturated model of the theory T_{axioms}. Using Lemma 2.1, we can construct a specialisation $(P(L_1), P(L), \pi)$. Let $(P(K_{univ}), P(L), \pi_{univ})$ be the composition of the specialisations $(P(K_{univ}), P(L_1), \pi_1)$ and $(P(L_1), P(L), \pi)$. Then, if $L \subset L_m$ is an algebraically closed extension of L of transcendence degree m, clearly $\bigcup_{n\geq 1} Card(S^n(Th(\mathcal{M}))) \leq 2^\omega$, where $\mathcal{M} = (P(L_m), P(L), \pi_m)$. This follows as L was assumed to be countable. Hence, by elementary model theory, there exists an L-embedding α_L of $(P(L_m), P(L), \pi_m)$ into $(P(K_{univ}), P(L_1), \pi_1)$. Clearly, α_L also defines an L-embedding of $(P(L_m), P(L), \pi_m)$ into $(P(K_{univ}), P(L), \pi_{univ})$ with the required properties. For the non-model theorist, we give a more algebraic construction, replacing the use of types by an explicit amalgamation of the possible valuations;

Proof. Suppose, inductively, we have already constructed a specialisation $(P(K_n), P(L), \pi_n)$ which has the property $(*)$ for all extensions $L \subset L_m$ with L_m algebraically closed of transcendence degree $m \leq n$. We will construct K_{n+1} having this property for $m \leq n+1$. By Theorem 1.5, we can find a Krull valuation v_n on K_n corresponding to the specialisation π_n. Let t be a new transcendental element. The extensions of v_n to $K_n(t)$ are completely classifiable. In fact, we have the following result in [12] (Theorem 3.9), we refer the reader to the paper for the definition of each family of valuations;

The extensions of v_n are of the form;

(i). $v_{n,a,\gamma}$ where $a \in K_n$ and γ is an element of some ordered group extension of $v(K_n)$.

(ii). $v_{n,A}$ where A is a pseudo Cauchy sequence in (K_n, v_n) of transcendental type.

Let I be a fixed enumeration of these valuations. Inductively, we assume that $Card(K_n) \leq 2^\omega$ in which case the dimension of $v(K_n)$ as a vector space over $\mathcal{Q}$ has dimension at most 2^ω as well. Clearly then the number of non-isomorphic (over K_n) valuations from (ii) is at most 2^ω and the same holds for the valuations obtained from (i) by noting that the number of order types of γ is at most 2^ω (it is easily checked that 2 new elements of the value group, γ_1 and γ_2, having the same order type, define isomorphic valuations in the case of (i)). Hence, we can assume that I is well ordered and apply the method of transfinite induction to construct a series of specialisations $(P(K_{n,i}), P(L), \pi_{n,i})$ as follows;

For $i = 0$, set $(P(K_{n,0}), P(L), \pi_{n,0}) = (P(K_n), P(L), \pi_n)$

Given $i \in I$ with i not a limit ordinal, let v_{i+1} be the next valuation in the enumeration. Let $(K_n\{t\}, \overline{v_{i+1}})$ be the completion of $(K_n(t), v_{i+1})$ and let $\overline{v_{i+1}}$ also denote the unique extension of this valuation to the algebraic closure $K_n\{t\}^{alg}$. This defines a Krull valuation and hence a specialisation $(P(K_n\{t\}^{alg}), P(L'), \pi_{n,i+1})$ where L' is the algebraic closure of the residue field of v_{i+1}, having transcendence degree at most 1 over L. Using arguments as above, we can construct a specialisation $(P(L'), P(L), \pi)$. Composing these specialisations, we obtain a specialisation $(P(K_n\{t\}^{alg}), P(L), \pi_{n,i+1})$. (One can omit this step by enumerating in I only those valuations which preserve the residue field L) Now, using Lemma 2.2 and Theorem 2.1, amalgamate the specialisations $(P(K_n\{t\}^{alg}), P(L), \pi_{n,i+1})$ and $(P(K_{n,i}), P(L), \pi_{n,i})$ to form a specialisation;

$$(P(K_{n,i}), P(L), \pi_{n,i}) \prec (P(K_{n,i+1}), P(L), \pi_{n,i+1}).$$

For i a limit ordinal, we set;

$$(P(K_{n,i}), P(L), \pi_{n,i}) = \bigcup_{j<i}(P(K_{n,j}), P(L), \pi_{n,j})$$

By the usual union of chains arguments we have that;

$$(P(K_{n,j}), P(L), \pi_{n,j}) \prec (P(K_{n,i}), P(L), \pi_{n,i}) \text{ for } j < i.$$

Repeating this process, we obtain a structure $(P(K_{n+1}), P(L), \pi_{n+1})$ such that;

$$(P(K_n), P(L), \pi_n) \prec (P(K_{n+1}), P(L), \pi_{n+1}).$$

It remains to check that this structure has the universal property $(*)$ for $m = n+1$. Let L_{n+1} be an algebraically closed extension of L with transcendence degree $n+1$ and specialisation $(P(L_{n+1}), P(L), \pi)$. Let v_π be the corresponding valuation and its restriction to $L \subset L_n \subset L_{n+1}$, a subfield of transcendence degree n. The corresponding specialisation

$(P(L_n), P(L), \pi)$ already factors through $(P(K_n), P(L), \pi_n)$ (†) and the valuation v_π appears as v_i in the enumeration I when restricted to $L_n(t)$. By a standard result in valuation theory, see [17], there exists an $L_n(t)$-embedding $\tau : L_n(t)^{alg} \rightarrow L_n\{t\}^{alg}$ such that $v_\pi = \overline{v_i} \circ \tau$ (††) (see notation above). Combining (†) and (††), we obtain an embedding $\alpha : (P(L_{n+1}), P(L)) \rightarrow (P(K_{n,i}), P(L))$ such that $\pi = \pi_{n,i} \circ \alpha$. This proves the result. It is now clear that the structure

$$(P(K_{univ}), P(L), \pi_{univ}) = \bigcup_{i>0} (P(K_i), P(L), \pi_i)$$

has the required universal property, is a model of T_{axioms} and;

$$(P(K_i), P(L), \pi_i) \prec (P(K_{univ}), P(L), \pi_{univ}) \text{ for } i > 0.$$

□

Part 2

4. Some Algebraic Notation

We will work mainly in the language of Weil's Foundations, namely using varieties instead of schemes. K will denote a big algebraically closed field. $L \subset K$ will denote a small algebraically closed field. By an affine variety V, we mean a closed subset of K^n in the Zariski topology. If V is irreducible, we denote the ring of regular functions on V by $K[V]$ and the function field by $K(V)$. If $k \subset K$ is perfect, we say that V is defined over k if $I(V)$, the radical ideal of functions vanishing on V is generated by polynomials with coefficients in k. Any irreducible affine variety V has a minimal field of definition k_V with the property that any automorphism fixes V setwise iff it fixes k_V pointwise. This is a classical result due to Weil, but is in fact a special case of a more general construction due to model theorists of canonical bases, see [8]. By a variety, we will mean a set V, a covering of subsets $V_1, \ldots V_m$ and for each i a bijection $f_i : V_i \rightarrow U_i$ with U_i an affine variety and such that for each $1 \leq i, j \leq m$, $U_{ij} = f_i(V_i \cap V_j)$ is an open subset of U_i and $f_{ij} = f_j f_i^{-1}$ is an isomorphism between the affine varieties U_{ij} and U_{ji}. A variety V then inherits a natural Zariski topology by declaring $U \subset V$ open if for each i, $f_i(U \cap V_i)$ is open in U_i. For $k \subset K$, we will say that V is defined over k, if the data (U_i, U_{ij}, f_{ij}) is defined over k in the sense of affine varieties. We let $P^n(K)$ denote n-dimensional projective space over K, that is $K^{n+1}/\sim$, where $\sim$ is the equivalence relation on $K^{n+1} \setminus \{\bar{0}\}$ given by $(x_0, \ldots, x_n) \sim (y_0, \ldots y_n)$ iff $\lambda(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$ for some $\lambda \in K$. Writing elements of

$P^n(K)$ in homogenous coordinates, $(x_0 : x_1 : \ldots : x_n)$, we have natural bijections f_i between K^n and $P^n(K)_i = \{\bar{x} : x_i \neq 0\}$. This gives $P^n(K)$ the structure of a variety defined over the prime subfield and an induced Zariski toplogy. By a projective variety V, we mean a closed subset of $P^n(K)$, using the coordinate charts f_i, V automatically is a variety in the sense defined above. Equivalently, a projective variety V is defined by a set of homogenous polynomials in $K[x_0, \ldots x_n]$ and is defined over k if the ideal $I(V)$ is generated by homogenous polynomials with coefficients in k. If a variety is V defined over k and $k \subset L \subset K$ with L algebraically closed then we will use the notation $V(L)$ to denote V considered as a variety over L. In this case, we will require that a subvariety of $V(L)$ is defined over L.
We will use the notation $X \times_Y Z$ to denote the fibre product of two varieties X and Z over Y. Given a variety V defined over k and a tuple of elements $\bar{a} \in V^n$, we will use $k(\bar{a})$ to denote the field of definition of $\bar{a}$. In the case when $X = Spec(L)$, corresponding to an L rational point $j : Spec(L) \rightarrow Z$;

$$\begin{array}{ccc} L \times_Z Y & \xrightarrow{i} & X \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{f} & Z \end{array}$$

we will often use the notation $L \times_Z Y$ to denote the geometric fibre $f^{-1}(y)$ of a point $y \in Z$, considered as a variety over L. Similar notation will be used in the case of sheaves. Given varieties Y,Z and a morphism $g : Y \rightarrow Z$, we define the pullback of a coherent sheaf F on Z to be the sheafification of

$$g^*F = O_Y \otimes_{g^{-1}O_Z} g^{-1}F$$

where $g^{-1}F(U) = lim_{\rightarrow, g(U) \subset V} F(V)$. Again, in the case when $j : Spec(L) \rightarrow Z$ is an L rational point and F is a coherent sheaf on Z, $j^{-1}F = F_z$,the localised sheaf at z, and $L \otimes_{O_{z,Z}} F_z$ is a vector space over L which, by slight abuse of notation, corresponds to the fibre of the sheaf F at z. Given a morphism $f : X \rightarrow Y$, we let $\Omega_{X/Y}$ denote the sheaf of relative differentials on X. We will use the geometric construction of $\Omega_{X/Y}$ as $\Delta^* J/J^2$ where $\Delta : X \rightarrow X \times_Y X$ is the diagonal embedding and J/J^2 is the normal bundle of $\Delta(X)$ in $X \times_Y X$. In the case when $Y = Spec(L)$ for $k \subset L \subset K$ and k the field of definition of X, we use the notation $\Omega_{X/L}$ to denote the sheaf of meromorphic differentials on X and $\Omega^*_{X/L}$ the sheaf of meromorphic vector fields. There is a canonical isomorphism;

$$d : m_z/m_z^2 \rightarrow (\Omega_{X/L})_z \otimes L$$

$$d(f + m_z^2) = df$$

relating the sheaf of differentials to the cotangent space at a point. Using this isomorphism and Nakayama's Lemma, one has that for an algebraic variety X of dimension n over $k \subset L$, $\Omega_{X/L}$ is a locally free module of rank n on the nonsingular locus U of X, see [16] for details.

5. Zariski Structures

Definition 5.1. *Let $(\mathcal{M}, \tau)$ be a topological space and let $\{C\}$ denote the collection of closed sets on $(\mathcal{M})$. We call $(\mathcal{M}, \tau)$ a Zariski structure if the following axioms hold;*

(L) Language: Basic relations are closed;

The diagonals $\Delta_i \subset \mathcal{M}^i \times \mathcal{M}^i$ are closed.

Any singleton in $\mathcal{M}$ is closed.

Cartesian products of closed sets are closed

(P) Properness: The projection maps $pr : \mathcal{M}^{n+1} \rightarrow \mathcal{M}^n$ are proper and continuous, that is the images and inverse images of closed sets under pr are closed

(DCC) Descending Chain Condition: The topology given by the closed sets on $\mathcal{M}^n$ is Noetherian for all $n \geq 1$. The condition (DCC) implies that every closed set C can be written uniquely (up to permutation) as a union of irreducible closed sets;

$$C = C_1 \cup \ldots \cup C_n$$

(DIM) Dimension: The following notion of dimension for closed sets $C \subset \mathcal{M}^n$ is well defined;

For irreducible C, $dim(C)$ is the maximum m for which there exists a chain of irreducible closed sets $C_0 \subset C_1 \subset \ldots C_m = C$.

For arbitrary closed C, $dim(C) = max_{1\leq i\leq m}\{dim(C_i)\}$ for C_i the irreducible components of C

(PS) Pre-Smoothness: For all closed irreducible sets $C_1, C_2 \subset \mathcal{M}^n$, with $C_1 \cap C_2 \neq \emptyset$,

$dim_{comp}(C_1 \cap C_2) \geq dim(C_1) + dim(C_2) - dim(\mathcal{M}^n)$

(DF) Definability of fibres: If $C \subset \mathcal{M}^{n+m}$ is closed, then

$F(C,k) = \{\bar{a} \in \mathcal{M}^n : dim(C(\bar{a})) \geq k\}$

is closed.

(GF) Generic fibres: If $C \subset \mathcal{M}^{n+m}$ is closed and irreducible, then

$dim(C) = dim(pr(C)) + min_{\bar{a}\in pr(C)} dim C(\bar{a})$

Remarks 5.2. *The definition of dimension easily implies the following properties;*

(DU) Dimension of unions: For C_1, C_2 closed, then

$dim(C_1 \cup C_2) = max\{dim(C_1), dim(C_2)\}$

(DP) The dimension of a point is 0.

(DI) Dimension of irreducible sets: If $C_1 \subsetneq C_2$ and C_2 is irreducible, then $dim(C_1) < dim(C_2)$.

We now show the following;

Theorem 5.3. *Let V be a smooth projective variety of dimension m defined over k and $k \subset L$ with L algebraically closed, then $V(L)$ considered as a toplogical space with closed sets given by the algebraic subvarieties defined over L is an irreducible Zariski structure of dimension m.*

Proof. We will verify the axioms;

(L) We need only verify that the diagonals $\Delta_i \subset V^i \times V^i$ are closed.

(P) An algebraic variety V is complete if for all varieties Y, the projection morphism

$pr : V \times Y \rightarrow Y$

is closed. Taking Y to be V^n in the above definition, complete varieties have the property that the projection maps

$pr : V^{n+1} \rightarrow V^n$

are closed. If $W \subset V$ is a closed subvariety of a complete variety V, then, as is easily checked, W is also complete. By assumption V is a closed subvariety of $P^N(L)$ for some N. Now it is a classical fact that $P^N(L)$ is complete, see for example [10].

(DCC) Let $\{W_i : i < \omega\}$ be an infinite descending chain of closed subvarieties of V^n. Let $\{U_1, \ldots U_n\}$ be an affine open cover of V^n. Then $\{U_j \cap W_i : i < \omega\}$ defines a descending chain of closed subvarieties of each U_j. By the Nullstellensatz, each such chain stabilises inside U_j. Then clearly the chain stabilises inside V^n.

(DIM) For W an irreducible subvariety of V^n, we let $dim_{geom}(W) = t.deg(L(W)/L)$. Then dim_{geom} corresponds to dim as defined above. To see this, suppose that $dim(W) \geq n+1$, and W is irreducible, then by definition one can find an irreducible closed subvariety $W' \subset W$ with $dim(W') \geq n$ and so inductively $dim(W') \geq n$. Now take any affine open subset of V^n intersecting W', so we may assume that W and W' are affine as the function fields are unchanged. Let $L[W]$ denote the coordinate domain of W, p the proper prime ideal corresponding to W' and dim_{Krull} the Krull dimension of an integral domain. By Krull's theorem, $height(p) + dim_{Krull}(L[W]/p) = dim_{Krull}(L[W])$, $dim_{Krull}(L[W]) = t.deg(L(W))$ and $dim_{Krull}(L[W]/p) = t.deg(L(W'))$, hence $t.deg(L(W'))) < t.deg(L(W))$. It follows that $dim_{geom}(W') < dim_{geom}(W)$ and so $dim_{geom}(W) \geq n+1$. Conversely, if $dim_{geom}(W) \geq n+1$, then again assuming W is irreducible and affine, if we take $f \in L[W]$ to be a non-unit, then each irreducible component of $V(f) \subset W$ has codimension 1 in X, see [10]. Therefore, $dim_{geom}(V(f)) \geq n$ and inductively $dim(V(f)) \geq n$. As each component of $V(f)$ is a proper closed subset of X, $dim(W) \geq n+1$. Now clearly we have that dim_{geom} corresponds to dim and so in particular we know that $dim(V^n) = mn$ and the notion of dim on V^n is well defined.

(PS) A simple calculation shows that for $(x_1 \ldots x_n) \in V^n$, $m_{\bar{x}} \cong \Sigma_{i=1}^n O_{x_1 \ldots \hat{x_i} \ldots x_n} \otimes m_{x_i}$. Hence,

$Tan_{\bar{x}}(V^n) = (m_{\bar{x}}/m_{\bar{x}}^2)^* \cong \Sigma_{i=1}^n (m_{x_i}/m_{x_i}^2)^* = \Sigma_{i=1}^n Tan_{x_i}(V)$.
Therefore, V^n is smooth.

Now we use the following lemma;

Lemma 5.4. *If X is a non-singular algebraic variety of dimension n, and Y, Z are irreducible closed subsets. Then if W is a component of $Y \cap Z$, we have,*

$$dim(W) \geq dim(Y) + dim(Z) - n$$

or equivalently

$$codim(W) \leq codim(Y) + codim(Z)$$

Proof. We have that $Y \cap Z \cong Y \times Z \cap \Delta(X)$ inside $X \times_L X$. Let $g_1, \ldots, g_n$ be uniformisers on an open subset U inside X. Then we saw above that $\Omega_{X/L}$ is just the pullback of the conormal sheaf J/J^2 for the inclusion of $\Delta(X)$ inside $X \times_L X$. As $\Omega_{X/L}$ is locally free, so is J/J^2, and in particular generated freely on $\Delta(U)$ by the functions $g_1 \otimes 1 - 1 \otimes g_1, \ldots, g_n \otimes 1 - 1 \otimes g_n$. At a point $x \in \Delta(U)$, we have that $g_1 \otimes 1 - 1 \otimes g_1, \ldots, g_n \otimes 1 - 1 \otimes g_n$ generate J_x/J_x^2 and therefore form a basis for the vector space $J_x/m_x J_x$ as clearly any function belonging to J_x lies in m_x the ideal of functions in $O_{X \times X, x}$ vanishing at x. Then, as $J_x/m_x J_x$ is just the base change $J \otimes k(x)$ of the ideal sheaf J at the point x, it follows by Nakayama's lemma that these functions generate J on an open neighborhood U containing x (not freely!). It follows that $Y \times Z \cap \Delta(X)$ is cut out by exactly n equations inside $Y \times Z$, so by standard dimension theory we have the result.

□

It follows immediately that V^n satisfies (PS).

In order to check the final 2 axioms we introduce the following definitions;

Definition 5.5. *If $\bar{a}, \bar{b} \in V^n$ are tuples of elements, we define $locus(\bar{a}/\bar{b})$ to be the intersection of all closed subvarieties defined over $k(\bar{b})$ containing $\bar{a}$ and $locus_{irr}(\bar{a}/\bar{b})$ to be the intersection of all closed subvarieties defined over $k(\bar{b})^{alg}$. We define $dim(\bar{a}/k)$ to be $t.deg(k(\bar{a})/k)$ and $dim(\bar{a}/k\bar{b})$ to be $t.deg(k(\bar{a})/k(\bar{b}))$; if the underlying field k is clear from context, we will abbreviate this to $dim(\bar{a}/\bar{b})$*

By the condition DCC, it is clear that $locus$ and $locus_{irr}$ are well defined. $locus_{irr}$ is an irreducible subvariety of V^n containing $\bar{a}$, as if V is an irreducible components of $locus_{irr}$ containing $\bar{a}$ and k_V is the minimal field of definition, then k_V has only finitely many congugates under an automorphism fixing $k(\bar{b})^{alg}$, hence $k_V \subset k(\bar{b})^{alg}$.

Definition 5.6. *If $W \subset V^n$ is an irreducible closed subvariety, $\bar{a} \in W$, and $\bar{b}$ is a tuple of elements such that $k(\bar{b})$ contains a field of definition for W, then we say that $\bar{a}$ is generic in W over $\bar{b}$ if $locus(\bar{a}/\bar{b}) = W$.*

Lemma 5.7. *Let $W \subset V^n$ be an irreducible closed subvariety defined over k and $\bar{a}$ generic in W over k. Then $dim(W) = t.deg(k(\bar{a})/k) = dim(\bar{a}/k)$.*

Proof. By the above, $dim(W) = dim_{geom}(W) = t.deg(k(W)/k)$. By choosing an open affine subvariety of W containing $\bar{a}$ and defined over k, we can assume that W is affine. Now define a map $ev : k[W] \rightarrow k(\bar{a})$ by setting $ev(f) = f(\bar{a})$. ev is injective as if $f(\bar{a}) = 0$, then as f has coefficients in k and $\bar{a}$ is generic in W over k, $f|W = 0$. Clearly ev extends to a map on $k(W)$ which is an isomorphism.

□

(DF) Let $W \subset V^{n+m}$ be a closed subvariety and pr the projection onto n factors. We can cover $(P^N(L))^{(n+m)}$ with finitely many affines of the form $A^{N(n+m)}$, hence we may assume that W is a closed subvariety of $A^{N(n+m)}$ and show that $\Gamma(\bar{y}) = \{\bar{a} : dim(W(\bar{a})) \geq k+1\}$ is closed in $pr(W)$. By additivity of $t.deg$ and the lemma, this occurs iff we can find algebraically independent elements $b_1 \ldots b_k b_{k+1} \subset \bar{b} \subset L$ such that $W(\bar{b}\bar{a})$ holds iff

$\exists_{\sigma(k+2)} \ldots \exists_{\sigma(Nm)} W(x_1, \ldots, x_{Nm}, \bar{a})$

has maximal dimension for some permutation $\sigma \in S_{Nm-(k+1)}$. We may write each projection W_σ in the form

$\bigcap_i F_i(x_1 \ldots x_{k+1}, \bar{y}) = 0 \cap \bigcap_j Q_j(x_1 \ldots x_{k+1}, \bar{y}) \neq 0$

where F_i and Q_j are polynomials in the variables $\bar{x}\bar{y}$. Let $\theta_\sigma(\bar{y})$ define the closed set given by the vanishing of all coefficients in the F_i. Then an easy calculation shows that $\Gamma_\sigma(\bar{y}) = \{\bar{y} \in pr(W) : \theta_\sigma(\bar{y})\}$, which is closed.

(GF) We first show the following;

Lemma 5.8. *Let $W \subset V^{n+m}$ be closed and irreducible, defined over k. Then $\bar{a}\bar{b}$ is generic in W over k iff $\bar{a}$ is generic in $pr(W)$ over k and $\bar{b}$ is generic in $W(\bar{a})$ over $k(\bar{a})$.*

Proof. One direction is straightforward, if $\bar{a}$ is not generic in $pr(W)$, then $\bar{a} \in E \subsetneq pr(W)$ and $\bar{a}\bar{b} \in pr^{-1}(E) \subsetneq W$. If $\bar{b}$ is not generic in $W(\bar{a})$, then we can find $X \subsetneq W(\bar{a})$ containing $\bar{b}$ defined over $k(\bar{a})$. As we are working in a product of $P^m(L)$, we can define X by a series of n-homogeneous equations with coefficients in $k(\bar{a})$. Applying Frobenius to these equations, we can in fact assume that the coefficients lie in $k < \bar{a} >$. Now a straightforward exercise in clearing denominators and writing affine equations in homogeneous form shows that we can write X as the fibre $Y(\bar{a})$ for some closed subvariety Y of V^{n+m}. Intersecting with W if necessary gives a proper closed $Y \subsetneq W$ with $\bar{a}\bar{b} \in W$ and defined over k.

For the other direction, suppose that $\bar{a}\bar{b}$ is not generic in W over k, then there exists X defined over k such that $\bar{a}\bar{b} \in X \subsetneq W$. Then $\bar{a} \in pr(X)$ which is also closed and defined over k. Hence, $pr(X) = pr(W)$. As $\bar{b} \in X(\bar{a})$, we have that $dim(X(\bar{a})) = dim(W(\bar{a})) = m$. By (DF),

$X_m = \{\bar{a} \in pr(X) : dim(X(\bar{a})) = dim(W(\bar{a})) = m\}$

is constructible and, by automorphism, can be seen to be defined over k. Hence, as $\bar{a}$ was assumed to be generic, X_m is open inside

$pr(X)$. Now, using Lemma 5.7 and the hypotheses on $\bar{a}$,$\bar{b}$, $dim(X) \geq dim(\bar{a}\bar{b}/k) = dim(\bar{b}/\bar{a}k) + dim(\bar{a}/k) = m + dim(pr(W))$. However, choosing $\bar{a}'\bar{b}'$ generic in W over k, we have that $dim(W) = dim(\bar{a}'\bar{b}'/k) = m + dim(pr(W))$ by the properties of X_k. Hence, $dim(X) \geq dim(W)$ contradicting the fact that $X \subsetneq W$ and W was assumed to be irreducible.

□

Using the lemma, we can give an easy proof of (GF);

Let $W \subset V^{n+m}$ be closed, irreducible and defined over k. Choose $\bar{a}\bar{b}$ generic in W over k. Then

$dim(W) = dim(\bar{a}\bar{b}/k) = dim(\bar{b}/\bar{a}k) + dim(\bar{a}/k) = dim(pr(W)) + min_{\bar{a}\in pr(W)}W(\bar{a})$.

The last equality follows from the previous lemma and (DF).

We have therefore checked all the axioms.

□

Definition 5.9. *Given a closed subvariety W of $V^m(L)$ and a closed $F \subset W \times V^m$, all defined over k, we say that F is a cover of W if $pr(F) = W$ and that $\bar{a} \in W$ is regular for the cover if $dimF(\bar{a}) = dimF(\bar{a}')$ for $\bar{a}'$ generic in W over k.*

6. Specialisations and Dimension in the Context of Zariski Structures

We recall the construction of a universal specialisation $(P(K_{univ}), P(L), \Pi_{univ})$, $(*)$, from Part 1, Section 3. We replace the subscript *univ* by ω. Observe that by the results of the previous section, if V is a smooth projective variety defined over L, then $V(L)$ is a Zariski structure, in the language consisting of closed subvarieties defined over L. In the same language we have that $V(L) \prec V(K_\omega)$, this is an easy consequence of quantifier elimination for algebraically closed fields. We define a closed set of $V(K_\omega)$ to be of the form $C(\bar{x}, \bar{a})$, where C is closed in $V(L)$, and $\bar{a}$ is a tuple from $V(K_\omega)$. One can easily verify that $V(K_\omega)$ is then a Zariski structure, either by verifying the axioms as we did for $V(L)$, or by using a general result for Zariski structures, shown in [28], or in [22]. For a tuple $\bar{a}$ from $V(K_\omega)$, and a set of parameters $\bar{b}$, possibly infinite, we define $dim(\bar{a}/\bar{b})$ to be the transcendence degree of $L(\bar{b}\bar{a})/L(\bar{b})$ As before, one can easily verify that this is the same as the dimension of the irreducible locus of $\bar{a}$ over $L(\bar{b})$. It is also a simple exercise to see that this also coincides with the model theoretic dimension, in this case Morley rank. We clearly have, from $(*)$, that $\Pi_\omega : V(K_\omega) \rightarrow V(L)$ defines a homomorphism of Zariski structures, in the sense that for all closed $W \subset V^m$ defined over L and $\bar{a} \in W(K_\omega)$, we have that $\Pi_\omega(\bar{a}) \in W(L)$. It is simple to recast the result of Section 3, in the context of Zariski structures;

Let $L \subset L_m$ be a field extension of finite transcendence degree m, V a smooth projective variety defined over L, and suppose the map $\pi : V(L_m) \rightarrow V(L)$ is a homomorphism of Zariski structures, then there exists an L-embedding $\alpha : L_m \rightarrow K_\omega$ with the property that $\Pi_\omega \circ \alpha = \pi$.

7. Infinitesimal Neighborhoods

From now on, we fix a pair of Zariski structures and the specialisation map Π_ω, to give a triple $((V(L), V(K_\omega), \Pi_\omega)$ where V is a smooth projective variety defined over L.

Definition 7.1. *For $\bar{a} \in V(L)^n$, we define the infinitesimal neighborhood of $\bar{a}$ to be;*

$$\mathcal{V}_{\bar{a}} = \Pi_\omega^{-1}(\bar{a})$$

The first property of infinitesimal neighborhoods is that we can move inside closed sets.

Lemma 7.2. *If $W(\bar{y})$ is an irreducible closed set defined in $V(L)$, $\bar{b} \in W$ and $dim(W) = r$, then there exists a $\bar{b}' \in \mathcal{V}_{\bar{b}} \cap W(K_\omega)$ such that $dim(\bar{b}'/L) = r$*

Proof. Consider the collection of constructible sets inside $V(L)^n$

$$W(\bar{y}) \cup \{\neg C(\bar{y}) : C\ closed,\ definable\ over\ \mathrm{L}, dim(W(\bar{y}) \cap C(\bar{y}) < r)\}$$

As W is irreducible of dimension r, any finite subcollection has a realisation in $V(L)^n$. By compactness, we can find a realisation $\bar{b}'$ in $W(K)$ for $L \subset K$ such that $dim(\bar{b}'/L) = r$. It then follows that we can define a partial specialisation $\pi : V(K) \rightarrow V(L)$ by setting $\pi(\bar{b}') = \bar{b}$, for if $C(\bar{y})$ is a closed set defined over L such that $\neg C(\bar{b})$, then we must have that $dim(W(\bar{y}) \cap C(\bar{y}) < d$ otherwise, W being irreducible,

$W(\bar{y}) \subset C(\bar{y})$, so by construction $\neg C(\bar{b}')$ also holds. Now, using the result of Section 6, applied to the field $L(\bar{b}')$ which has transcendence degree r over L, we may assume that $L(\bar{b}') \subset K_r \subset K_\omega$ and the specialisation π is given by the restriction of Π_ω.

□

We now come to the critical theorem, a more general version of which was originally proved by Zilber in the context of abstract Zariski structures, see [28] or [29];

Theorem 7.3. *Suppose that $F \subset D \times V^k$ is an irreducible finite cover of D with D a smooth subvariety of V^m, and F, D defined over L, such that $F(a, b)$. If $a' \in \mathcal{V}_a \cap D(K_\omega)$ is generic in D over L, then we can find $b' \in \mathcal{V}_b$ such that $(a', b') \in F(K_\omega)$.*

We here only sketch the proof, full details can also be found in [18]. We first consider the following collection of constructible sets defined over K_ω, with $a' \in \mathcal{V}_a \cap D(K_\omega)$ generic over L;

$$\{F(a', y)\} \cup \{\neg C(d, y) : d \in V(K_\omega), \neg C(\Pi_\omega(d), b)\}$$

As F is a finite cover and K_ω is algebraically closed, a realisation b' of this collection lies in $V(K_\omega)$ and $F(a', b')$ holds. Moreover, $\Pi_\omega(b') = b$, otherwise, as the diagonal $x = y$ is closed, we have that $b' \neq y$ is in the collection which is ridiculous.

If the collection is inconsistent, we find a closed set $Q \subset V^{n+k}$ such that $F(a', y) \subseteq Q(d, y)$ whereas $\neg Q(\pi(d), b)$.

The point of the smoothness assumption is to show that the parameter space

$$L(x, z) \subset D \times V^n \ = \{(x, z) : F(x, y) \subset Q(z, y)\}$$

which in general is not relatively closed in $D \times V^n$ at least corresponds to a closed set over a dense open subset of D. More precisely, there is a closed subvariety $P(x, z) \subset D \times V^n$ and $D' \subset D$, $dim(D') < dim(D)$, all defined over L, such that

1. $P(x, z) \subset L(x, z)$.

2. $L(x,z) \subset P(x,z) \cup (D' \times V^n)$ $(*)$

We have by assumption that $L(a',d)$ holds. As a' was chosen to be generic over L and $a' \notin D'$, $P(a',d)$ holds. Applying the specialisation Π_ω gives that $P(a,\Pi_\omega(d))$, hence $F(a,y) \subset Q(\Pi_\omega(d),y)$, hence $Q(\Pi_\omega(d),b)$ holds as well, contradicting the assumption.

Remarks 7.4. *In fact the theorem can be improved to give the following more general result;*

Suppose that $F \subset D \times V^k$ is an irreducible generically finite cover of D with D a subvariety of V^m. Then, if $a \in D$ is a regular point for the cover and contained in the non-singular locus of D, $a' \in \mathcal{V}_a \cap D(K_\omega)$ is generic in D over L, then we can find $b' \in \mathcal{V}_b$ such that $(a',b') \in F(K_\omega)$.

8. Zariski Unramified Maps and Multiplicity

The purpose of introducing infinitesimal neighborhoods is to define an abstract notion of Zariski multiplicity.

Definition 8.1. *Zariski multiplicity*

Let hypotheses be as in Theorem 7.3

Given $(a,b) \in F$, set

$mult_{ab}(F/D) = Card(F(a',K_\omega)) \cap \mathcal{V}_b)$ *for* $a' \in \mathcal{V}_a \cap D$ *generic over* L

We want to show this is well defined.

Proof. Suppose $a'' \in \mathcal{V}_a \cap D$ with $Card(F(a'',K_\omega) \cap \mathcal{V}_b) = n$. Consider the relation $N(x,y_1,\ldots,y_n) \subset D \times V^{nk}$, given by

$N(x,y_1,\ldots,y_n) = F(x,y_1) \wedge \ldots \wedge F(x,y_n)$

Then we have that N is a finite cover of D and, moreover, by smoothness of D, each irreducible component of N has dimension at least

$n(dim(F)+(n-1)k)-(n-1)(dim(D)+nk) = dim(D)+n(n-1)k-n(n-1)k = dim(D)$

so clearly each component is a finite cover of D. Now, choose an irreducible component N_i containing $(a'', b_1'', \ldots, b_n'')$, so by specialisation also contains $(a, b, \ldots, b)$ and consider the open set $U \subset N_i$ given by

$$U(x, y_1, \ldots, y_n) = N_i(x, y_1, \ldots, y_n) \wedge y_1 \neq y_2 \neq \ldots \neq y_n$$

Then, for $a' \in \mathcal{V}_a$ generic in D, it follows we can find a tuple $(b_1', \ldots, b_n')$ such that $N_i(a', b_1', \ldots, b_n')$, and $(b_1', \ldots, b_n') \in \mathcal{V}_{(b,\ldots,b)}$. As is easily checked, the tuple $(a', b_1', \ldots, b_n')$ is generic inside N_i, hence must lie inside U. This proves that the $b_1', \ldots, b_n'$ are distinct, hence $Card(F(a', K_\omega) \cap \mathcal{V}_b) \geq n$.

□

Definition 8.2. *We say that a point* $(ab) \in F$ *is Zariski ramified if* $mult_{ab}(F/D) \geq 2$. *Otherwise, we call such a point Zariski unramified.*

Now suppose $F \subset D \times V^n$ is an irreducible finite cover of D with D smooth, then we have the following easily checked lemma

Lemma 8.3. $mult_a(F/D) =_{def} \Sigma_{b \in F(a,L)} mult_{ab}(F/D)$ *does not depend on the choice of* $a \in D$, *and is equal to the size of a generic fibre over* D

A simple consequence is the following:

Lemma 8.4. *If* $\bar{a}' \in D(L)$, *then* $F(\bar{a}')$ *contains a point of ramification in the sense of Zariski structures iff* $|F(\bar{a}')| < |F(\bar{a})|$ *where* $\bar{a}$ *is generic in* D.

Proof. We have seen that $|F(\bar{a})| = \Sigma_{\bar{b} \in F(\bar{a}',L)} mult_{\bar{a}',\bar{b}}(F/D)$. If $|F(\bar{a}')| < |F(\bar{a})|$, then there must exist $\bar{b} \in F(\bar{a}')$ with $mult_{(\bar{a}',\bar{b})}(F/D) \geq 2$ so the result follows by the definition of ramification in Zariski structures. The converse is similar.

□

We will also require the following results, that Zariski multiplicity is multiplicative over composition and preserved by open maps.

Lemma 8.5. *Suppose that* F_1, F_2 *and* F_3 *are smooth, irreducible, with* $F_2 \subset F_1 \times V^k$ *and* $F_3 \subset F_2 \times V^l$ *finite covers. Let* $(abc) \in F_3 \subset F_1 \times V^k \times V^l$. *Then* $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1) mult_{bc}(F_3/F_2)$.

Proof. To see this, let $m = mult_{ab}(F_2/F_1)$ and $n = mult_{bc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1(K_\omega)$ generic over L. By definition, we can find distinct $b_1 \ldots b_m$ in $V^k(K_\omega) \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1, we have that $dim(a'b_i/L) = dim(a'/L) = dim(F_1) = dim(F_2)$, so each $(a'b_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over L. Again by definition, we can find distinct $c_{i1} \ldots c_{in}$ in $V^l(K_\omega) \cap \mathcal{V}_c$ such that $F_3(a'b_ic_{ij})$ holds. Then the mn distinct elements $(a'b_ic_{ij})$ are in $\mathcal{V}_{abc}$, so by definition of multiplicity $mult_{abc}(F_3/F_1) = mn$ as required.

□

Lemma 8.6. *Let $\pi_1 : X \to D$ and $\pi_2 : Y \to D$ be covers with assumptions as in remarks following Theorem 7.3. Suppose moreover that there exist open smooth subvarieties $U \subset X$ and $V \subset Y$ and an isomorphism $f : U \to V$ such that $\pi_2 \circ f = \pi_1$ on U. Then if $a \in D$ is a regular point for the cover π_1 and $(ab) \in U$, $mult_{ab}(X/D) = mult_{af(b)}(Y/D)$.*

Proof. We may assume that the open set U is maximal with the property that $(ab) \in U$ and there exists an isomorphism with $V \subset Y$. Suppose $mult_{ab}(X/D) = m$. Then we can find $a' \in \mathcal{V}_a \cap D(K_\omega)$ generic in D over L and $b_1, \ldots, b_m$ distinct such that $X(a'b_i)$ holds for $1 \leq i \leq m$. It will be sufficient to show that $Y(a'f(b_i))$ holds and $f(b_i) \in \mathcal{V}_{f(b)}$, for $1 \leq i \leq m$, then, as f is injective, $mult_{a,f(b)}(Y/D) \geq m$ and the result follows by symmetry. By the fact that $\pi_2 \circ f = \pi_1$ on U we clearly have that $Y(a'f(b_i))$ holds. Let $\overline{graph(f)}$ be the projective closure of the graph of f in the projective variety $X \times Y$ and π_X, π_Y the projections onto the coordinates X and Y. Then π_X satisfies the conditions of the remarks after Theorem 7.3, and moreover by assumption the point $(ab) \in X$ is regular for the cover π and contained in the non-singular locus of X. Hence, we can find $(cd) \in \mathcal{V}_{af(b)}$ such that $\overline{graph(f)}(a'b_i, cd)$ holds. As $graph(f)$ is a $1-1$ correspondence between U and V, if $(a'b_i, cd) \in \overline{graph(f)} \setminus graph(f)$ then $(a'b_i, cd) \in F_X \cup F_Y$ where F_X, F_Y consist of the infinite fibres of the projections π_X and π_Y respectively. By (DF), both of these are defined over L and have dimension strictly less than $\overline{graph(f)}$. This contradicts the fact that $(a'b_i, cd)$ is generic inside $\overline{graph(f)}$ over L, hence $(a'b_i, cd) \in graph(f)$ and as f is a bijection $(cd) = (a'f(b_i))$. This shows that $f(b_i) \in \mathcal{V}_{f(b)}$ as required.

□

9. Etale Morphisms and Algebraic Multiplicity

We review here the algebraic notions which will be required in the following section.

Definition 9.1. *A morphism f of finite type between varieties X and Y is said to be etale if for all $x \in X$ there are open affine neighborhoods U of x and V of $f(x)$ with $f(U) \subset V$ such that restricted to these neighborhoods the pull back on functions is given by the inclusion;*

$f^*: L[V] \rightarrow L[V]\frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n}$

and $det(\frac{\partial f_i}{\partial x_j})(x) \neq 0$, $(*)$

The coordinate free definition of etale is that f should be flat and unramified, where a morphism f is unramified if the sheaf of relative differentials $\Omega_{X/Y} = 0$, clearly this last confition is satisfied using the condition $(*)$. If we tensor the exact sequence,

$f^*\Omega_Y \rightarrow \Omega_X \rightarrow \Omega_{X/Y} \rightarrow 0$

with $L(x)$ the residue field of x, we obtain an isomorphism

$f^*\Omega_Y \otimes L(x) \rightarrow \Omega_X \otimes L(x)$.

Identifying $\Omega_X \otimes L(x)$ with $T^*_{x,X}$ gives that

$df : (m_x/m_x^2)^* \rightarrow (m_{f(x)}/m^x_{f(x)})^*$

is an isomorphism of tangent spaces or dually $f^*(m_{f(x)}) = m_x$. Call this property of etale morphisms $(**)$.

The notion of an etale morphism simplifies considerably when we assume that X and Y are smooth algebraic varieties over L, see [16];

Theorem 9.2. *If X and Y are non-singular algebraic varieties over L and $f: X \rightarrow Y$ is a morphism, then f is etale iff $df : (m_x/m_x^2)^* \rightarrow (m_{f(x)}/m^2_{f(x)})^*$ is an isomorphism everywhere.*

Remarks 9.3. *This gives us a convenient test for etaleness given an arbitrary morphism of finite type between smooth varieties X and Y. If we take local uniformisers $g_1, \ldots g_n$ at $x \in X$, the dg_i generate Ω_X freely on an open U' of x. If we pull back a set of uniformisers $f^*f_1, \ldots, f^*f_n$ on Y to X, we can locally define the Jacobian $Jac^{\bar{f}}_{\bar{g}}$ to be;*

$det(\frac{\partial f^* f_i}{\partial g_j})$

which means write the 1*-forms* $f^*df_i = \Sigma_j a_{ij} dg_j$ *and take* $det(a_{ij})$*. If* f *is etale in a neighborhood of* x*, the* f^*df_i *also generate* Ω_X *freely on an open* U'' *of* x*. Taking the intersection* $U'' = U \cap U'$*, gives us that the Jacobian* $Jac^{\bar{f}}_{\bar{g}}|U'' \neq 0$*. Conversely, if* $Jac^{\bar{f}}_{\bar{g}}(x) \neq 0$*, then it is non zero on an open neighborhood* U'' *of* x *and by the above theorem we have that* f *is etale on this neighborhood.*

We will also require some facts about the etale topology on an algebraic variety Y, see [15] for more details. We consider a category Y_{et} whose objects are etale morphisms $U \rightarrow Y$ and whose arrows are Y-morphisms from $U \rightarrow V$. This category has the following 2 desirable properties. First given $y \in Y$, the set of objects of the form $(U,x) \rightarrow (Y,y)$ form a directed system, namely $(U,x) \subset (U',x')$ if there exists a morphism $U \rightarrow U'$ taking x to x'. Secondly, we can take "intersections" of open sets U_i and U_j by considering $U_{ij} = U_i \times_Y U_j$; the projection maps are easily show to be etale and the composition of etale maps is etale, so $U_{ij} \rightarrow Y$ still lies in Y_{et}. If Y is an irreducible variety over K, then all etale morphisms into Y must come from reduced schemes of finite type over K, though they may well fail to be irreducible considered as algebraic varieties. Now we can define the local ring of Y in the etale toplogy to be;

$$O^{\wedge}_{y,Y} = lim_{\rightarrow, y \in U} O_U(U)$$

As any open set U of Y clearly induces an etale morphism $U \rightarrow_i Y$ of inclusion, we have that $O_{y,Y} \subset O^{\wedge}_{y,Y}$. We want to prove that $O^{\wedge}_{y,Y}$ is a Henselian ring and in fact the smallest Henselian ring containing $O_{y,Y}$. We need the following lemma about Henselian rings;

Lemma 9.4. *Let* R *be a local ring with residue field* k*. Suppose that* R *satisfies the following condition;*

If $f_1, \ldots f_n \in R[x_1, \ldots x_n]$ *and* $\bar{f}_1 \ldots \bar{f}_n$ *have a common root* $\bar{a}$ *in* k^n*, for which* $Jac(\bar{f})(\bar{a}) = (\frac{\partial \bar{f}_i}{\partial x_j})_{ij}(\bar{a}) \neq 0$*, then* $\bar{a}$ *lifts to a common root in* R^n *(*).*

Then R *is Henselian.*

It remains to show that $O^{\wedge}_{y,Y}$ satisfies $(*)$.

Proof. Given $f_{1,} \ldots f_n$ satisfying the condition of $(*)$, we can assume the coefficients of the f_i belong to $O_{U_i}(U_i)$ for covers $U_i \rightarrow Y$; taking the intersection $U_{1\ldots i\ldots n}$ we may even assume the coefficients define functions on a single etale cover U of Y. By the remarks above we can consider U as an algebraic variety over K, and even an affine algebraic variety after taking the corresponding inclusion. We then consider the variety $V \subset U \times A^n$ defined by $Spec(\frac{R(U)[x_1,\ldots,x_n]}{f_1,\ldots f_n})$. Letting $u \in U$ denote the point in U lying over $y \in Y$, the residue of the coefficients of the f_i at u corresponds to the residue in the local ring R, which tells us exactly that the point $(u, \bar{a})$ lies in V. By the Jacobian condition, we have that the projection $\pi : V \rightarrow U$ is etale at the point $(u, \bar{a})$, and hence on some open neighborhood of $(u, \bar{a})$, using Nakayama's Lemma applied to $\Omega_{V/U}$. Therefore, replacing V by the open subset $U' \subset V$ gives an etale cover of U and therefore of Y, lying over y. Now clearly the coordinate functions $x_1, \ldots x_n$ restricted to U' lie in $O^{\wedge}_{y,Y}$ and lift the root $\bar{a}$ to a root in $O^{\wedge}_{y,Y}$

□

We define the Henselisation of a local ring R to be the smallest Henselian ring $R' \supset R$, with $R' \subset Frac(R)^{alg}$. We have in fact, see [15], that;

Theorem 9.5. *Given an algebraic variety Y, $O^{\wedge}_{y,Y}$ is the Henselisation of $O_{y,Y}$*

Definition 9.6. *Given smooth projective curves C_1, C_2 and a finite morphism $f : C_1 \rightarrow C_2$, the algebraic multiplicity of f at a is $ord_a(f^*h)$ where h is a local uniformiser for C_2 at $f(a)$.*

Remarks 9.7. *This is independent of the choice of h, as the quotient of 2 uniformisers h/h' is a unit in $\mathcal{O}_{f(a)}$. Given finite morphisms $f : C_3 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$, if $ord_{a,f(a)}(C_3/C_2) = m$ and $ord_{f(a),gf(a)}(C_2/C_1) = n$, then taking a local uniformiser h at $gf(a)$, we have that $g^*h = h_1^n u$ locally at $f(a)$ for a unit u and uniformiser h_1 in $\mathcal{O}_{f(a)}$. Similarily $f^*g^*h = h_2^{mn}u'$ for a unit u' and uniformiserh_2 in $\mathcal{O}_a$. This shows that $ord_{a,gf(a)}(C_3/C_1) = mn$, so the branching number is also multiplicative for smooth projective curves.*

Definition 9.8. *Given smooth projective varieties X_1, X_2 and a finite morphism $f : X_1 \rightarrow X_2$, the algebraic multiplicity $mult^{alg}_{af(a)}(X_1/X_2)$ of f at $a \in X_1$ is $length(O_{a,X_1}/f^*m_{f(a)})$ where $m_{f(a)}$ is the maximal ideal of the local ring $O_{f(a)}$.*

Remarks 9.9. *Note that this is finite, by the fact that finite morphisms have finite fibres and the ring* $O_{a,X_1}/f^*m_{f(a)}$ *is a localisation of the fibre* $f^{-1}(f(a)) \cong R(f^{-1}(U)) \otimes_{R(U)} L \cong R(f^{-1}(U))/m_{f(a)}$ *where* U *is an affine subset of* X_2 *containing* $f(a)$.

We now have the following, which generalises the result for curves;

Theorem 9.10. *Algebraic multiplicity is multiplicative;*

Given finite morphisms $f : X_3 \to X_2$ *and* $g : X_2 \to X_1$ *between smooth projective varieties, for* $a \in X_3$ *we have that*

$$mult_{af(a)}(X_3/X_2)mult_{f(a)gf(a)}(X_2/X_1) = mult_{agf(a)}(X_3/X_1).$$

Proof. The proof is an exercise in algebra, which we give for want of a convenient reference. First, the morphisms f and g are flat. This requires the following lemma, given as an exercise in [9], and the fact that smooth varieties are regular and Cohen-Macauley;

Lemma 9.11. *Let* $f : X \to Y$ *be a morphism of varieties over* L. *Assume that* Y *is regular,* X *is Cohen-Macauley and that every fibre of* f *has dimension equal to* $dim(X) - dim(Y)$. *Then* f *is flat.*

Now we have a tower of local rings $(R, m) \subset (S, n) \subset (T, o)$ with algebraically closed residue field L. Each extension is free by the flatness result and finiteness. For a finite free extension $(R, m) \subset (S, n)$ of local rings, we also have the easily checked result that;

$$[S : R] = dim_{Fr(R)} S \otimes_R Fr(R) = dim_L S \otimes_{R/mR} R/mR = dim_L(S/mS)(*).$$

For an extension $(R, m) \subset (S, n)$ of local rings, we have that $length(S/mS) = dim_L(S/mS)$, hence, by (*), the theorem reduces to checking that $[T : R] = [T : S][S : R]$ which is standard, see [27].

□

10. Equivalence of the Notions

This section is devoted to the main proofs of this part of the paper. We show that, for morphisms between smooth projective varieties, the notions of etaleness and a Zariski unramified map essentially coincide (*). We then show that, in the case of smooth algebraic curves (smooth projective varieties of dimension 1), algebraic multiplicity essentially coincides with Zariski multiplicity. Finally, we give a local version of $(*)$ for, possibly singular, algebraic curves. For the rest of this part of the paper, we will often abbreviate smooth algebraic curves to just algebraic curves, as is done in [9], reserving the terminology singular algebraic curves to denote irreducible projective varieties of dimension 1.

Theorem 10.1. *Let hypotheses be as in Theorem 7.3, with the additional assumptions that $charL = 0$ and F is smooth. Then F is a Zariski unramified cover of D iff F is an etale cover of D.*

Let pr be the projection map of F onto D, then pr is a projective morphism. By Zariski's Main Theorem, pr factors as a composition $F \rightarrow_{pr_1} F' \rightarrow_{pr_2} D$ with pr_1 having connected fibres, $pr_{1*}O_F = O_{F'}$and pr_2 a finite morphism. The formal inverse pr_1^{-1} from F' to F is a morphism corresponding to the identification of $pr_{1*}O_F$ and $O_{F'}$, hence pr_1 is in fact an isomorphism. We may therefore assume that pr is a finite morphism.

Now suppose that pr is etale, then, pr is flat, see [16] for how this follows from Definition 9.1. As D is irreducible,

$$dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y))$$

is independent of $y \in D$. As pr is etale, $pr_* : T_{x,F} \rightarrow T_{pr(x),D}$ is an isomorphism, hence, by a simple calculation;

$$dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y)) = |F(y)| \text{ for } y \in D.$$

This shows that $|F(y)|$ is independent of $y \in D$. By Lemma 8.4, this shows that pr is a Zariski unramified cover.

Conversely, suppose that pr is Zariski unramified. We first show that for generic $\bar{a} \in D$, $|F(\bar{a})| = deg(pr) = deg[k(F) : k(D)]$. As $char(k(F)) = 0$, the extension is seperable so we can find a primitive element $g \in k(F)$ such that $k(F) = k(D)(g)$. Clearly the minimum polynomial p of g over $k(D)$ has degree $n = deg[k(F) : k(D)]$. Let $h_1, \ldots h_{n-1} \in k(D)$ be the coefficients of p, then $R(D)(h_1 \ldots h_{n-1})$ determines the function ring of a Zariski open subset U of D. Clearly $R(U)[g]$ is an integral extension of $R(U)$ and corresponds to the projection restricted to $U' = pr^{-1}(U) \cap g \neq 0$. By dimension theory, the zero set $Z(g) \subset D$ cannot intersect with a generic fibre of the original map $pr : F \rightarrow D$. Now we consider the discriminant $D(p)$ of the polynomial p as a regular function on U and we have that for generic $\bar{a} \in U$ that $D(p)(\bar{a}) \neq 0$. This implies that for generic $\bar{a} \in U$ $|pr^{-1}(\bar{a})| = n = deg[k(F) : k(D)]$. Now we are in a position to apply Theorem 5, p145, of [26] which requires that D should be smooth, namely that $pr_* : T_{x,F} \rightarrow T_{pr(x),D}$ is an isomorphism for $x \in F$. As both F and D were assumed to be nonsingular, this is sufficient to show that pr is etale by Theorem 9.2.

Remarks 10.2. *When $char(L) = p$, the analogy fails. If we consider the Frobenius map $Fr : P^1 \rightarrow P^1$, then $Graph(Fr) \subset P^1 \times P^1$ is a finite cover of P^1, which is smooth. The projection map pr onto the second coordinate is unramified in the sense of Zariski structures as pr is a bijection. However pr fails to be etale in the sense of algebraic geometry as $pr_* : T_{x,Graph(Fr)} \rightarrow T_{pr(x),P^1}$ is zero everywhere. However the following theorem shows that this is the only bad example and highlights one advantage of the Zariski method, namely that it is insensitive* only *to Frobenius.*

Theorem 10.3. *Let hypotheses be as in Theorem 7.3, with the additional assumptions that $char(L) = p \neq 0$ and F is smooth. Then, if F is an etale cover of D, F is a Zariski unramified cover. Conversely, if F is a Zariski unramified cover, then pr factors as a composition $F \rightarrow_{pr_1} F' \rightarrow_{pr_2} D$ in $Proj$ with pr_1 a purely inseperable connected cover and pr_2 an etale cover.*

Proof. As in the previous theorem, we may assume that pr is a finite morphism. Suppose first that $F \rightarrow D$ is a finite morphism with F and D affine. We first find a field L such that $k(F)/L$ is a purely inseperable extension and $L/k(D)$ is seperable. Let R' be the integral closure of $R(D)$ in L and R'' the integral closure of $R(D)$ in $k(F)$. As $R(F)$ is integral over $R(D)$ we have that $R(F) \subset R''$, but F was assumed to be smooth so $R(F)$ is integrally closed in $k(F)$ and therefore $R'' = R(F)$. As the extensions $k(D) \subset L \subset k(F)$ are finite algebraic, by [27], both $R(F)$ and R' are finite R' and $R(D)$ modules respectively. Therefore, corresponding to the ring inclusions

$$R(D) \rightarrow R' \rightarrow R(F)$$

we have the sequence of finite morphisms

$F \rightarrow_{pr_1} Spec(R') \rightarrow_{pr_2} D$

We first consider the cover $F \rightarrow_{pr_1} Spec(R')$. Let $g_1, \ldots g_m$ generate $R(F)$ over R'. As the extension $k(F)/L$ is purely inseperable, we can write the minimum polynomials p_i of g_i in the form $r_{i,0}g^{p^{n_i}} - r_{i,1} = 0$ where $r_{i,0}$ and $r_{i,1}$ are in R'. As $R(F)/R'$ is finite, we can also find monic polynomials q_i with coefficients in R' satisfied by g_i. Choose polynomials $t_i = s_{i,0}x^{m_i} + s_{i,1}x_{m_i-1} + \ldots s_{i,m_i}$ such that $p_i t_i = q_i$. By equating coefficients, we have that $r_{i,0} = s_{i,0}^{-1}$ and $r_{i,1}/r_{i,0} \in R'$. Hence, we can take the p_i to be monic with coefficients in R'. As the p_i are minimal monic polynomials, we conclude that that $R(F)$ is an extension of the form $R'[g_1, \ldots, g_m]/(g_1^{p^{n_1}} - \lambda_1, \ldots, g_m^{p^{n_m}} - \lambda_m)$ with $\lambda_i \in R'$. This is easily checked to be a connected cover of $Spec(R')$. In fact if we let $\theta = (Fr^{-n_1}, \ldots, Fr^{-n_m}) \circ (\lambda_1 \ldots \lambda_m)$, where the λ_i are considered as regular functions on $Spec(R')$ and Fr^{-n_i} is the formal inverse Frobenius map, then the cover corresponds to the projection of $Graph(\theta) \subset Spec(R') \times A^m$ onto $Spec(R')$. As F was assumed to be smooth, $Spec(R')$ is a smooth seperable Zariski unramified cover of D. Applying the previous theorem, we conclude that $Spec(R')$ is an etale cover of D. Now, for the case when F and D are projective varieties, let U_i be an affine cover of D and $R'(U_i)$ the corresponding normalisations. By uniqueness of integral closure, the $R'(U_i)$ patch to form a cover F' of D. In fact, by a classical result, see [16], we may assume that F' is a smooth projective variety. As etaleness is a local condition for smooth varieties, the cover F' is etale. Finally, check that the local maps $pr_1 : F_i \rightarrow R'(U_i)$ patch on overlaps to give a morphism $pr_1 : F \rightarrow F'$. Clearly, this is an insperable connected cover, in fact if F' is defined by the homogenous equations $< f_1, \ldots f_n >$ inside P^N, then F is isomorphic to the closed subvariety of $P^N \times P^m$ defined by the extra equations $< Y_i^{p^{n_i}} X_N^{j(i)} - \lambda_i(X_0, \ldots, X_N) Y_0^{p^{n_i}} >$ where $1 \leq i \leq m$ and $j(i)$ is the degree of the polynomial λ_i in the affine coordinates $P^N(L)_i$.

□

Remarks 10.4. *We now show that the notions of Zariski multiplicity and algebraic multiplicity coincide for smooth algebraic curves when* $char(L) = 0$*, as usual with assumptions being as in Theorem 7.3., and find an analogous result when* $char(L) = p$*. Unfortunately, it does not seem possible to achieve this by counting points in the fibres, as in the previous theorems, so we need to find a local method. This will be the subject of the remainder of this section.*

Remarks 10.5. *We consider the case when F and D are smooth curves. Given a curve C, defined over a field of characteristic p, with function field $L(C)$, we let $L(C)^{1/p}$ be the field obtained by extracting p^{th} roots of $L(C)$ in some fixed algebraic closure. We denote by C_p the unique (up to isomorphism) curve, having function field $L(C)^{1/p}$. Corresponding to the inclusion $i : L(C) \rightarrow L(C)^{1/p}$, we obtain a morphism $Frob : C_p \rightarrow C$, which, by some abuse of the standard terminology, (the standard terminology is L-linear Frobenius), we will refer to as Frobenius. Although $L(C)$ and $L(C)^{1/p}$ are clearly isomorphic as fields, they may not be isomorphic over L. Hence, C and C_p are not necessarily isomorphic curves. The Frobenius morphism may be explicitly realised as follows;*

Let C be embedded in P^n, for some n, defined by the homogeneous polynomials $\{f_1, \ldots, f_m\}$. Let C' be the variety defined by $\{\overline{f_1}, \ldots, \overline{f_m}\}$, where, for $1 \leq j \leq m$, $\overline{f_j}$ is the homogeneous polynomial obtained by applying inverse Frobenius to the coefficients. Then, by a straightforward calculation using Jacobians, C' defines a projective curve. The morphism Frobenius;

$$Fr : P^n \rightarrow P^n$$

$$Fr([X_0 : \ldots : X_n]) = [X_0^p : \ldots : X_n^p]$$

restricts to define a morphism $Fr : C' \rightarrow C$. Let Rat_k denote the rational functions of degree k on P^n. Then Fr induces a map;

$$Fr^* : Rat_k \rightarrow Rat_{kp}$$

by the formula;

$$(Fr^* F)(X_0, \ldots, X_n) = F(X_0^p, \ldots, X_n^p)$$

For a homogeneous polynomial f_j defining C, we have that;

$$Fr^*(f_j) = (\overline{f}_j)^p$$

Hence, Fr^* restricts to define an L-linear map;

$$Fr^* : L(C) \rightarrow L(C')$$

One can also define a map;

$Fr^{-1*} : L[X_0, \ldots, X_n] \rightarrow L[X_0^{1/p}, \ldots, X_n^{1/p}]$

by the formula;

$(Fr^{-1*}F)(X_0, \ldots, X_n) = F(X_0^{1/p}, \ldots, X_n^{1/p})$

For a homogeneous polynomial $\overline{f_j}$ *defining* C'*, we have that;*

$Fr^{-1*}(\overline{f_j}) = (f_j)^{1/p}$

Hence, Fr^{-1*} *restricts to define an* L*-linear isomorphism;*

$Fr^{-1*} : L(C') \rightarrow L(C)^{1/p}$ (†)

We have that $Fr^{-1*} \circ Fr^* = Id$*, restricted to* Rat_k*, hence;*

$Fr^{-1*} \circ Fr^* : L(C) \rightarrow L(C') \rightarrow L(C)^{1/p}$ (††)

is the inclusion map. Using the fact that C_p *and* C' *are nonsingular projective curves, by* (†) *we obtain an isomorphism* $\theta : C_p \rightarrow C'$*. By* (††)*, we have that;*

$Fr \circ \theta = Frob : C_p \rightarrow C$

Hence, without loss of generality, we can identify the morphisms Fr *and the more abstractly defined morphism* $Frob$*.*

Theorem 10.6. *Let hypotheses be as in Theorem 7.3, with the additional assumption that* $char(L) = 0$ *and* F*,* D *are smooth curves. Then the notions of Zariski multiplicity and algebraic multiplicity coincide.*

Proof. As D has a non-constant meromorphic function, we can write D as a finite cover of $P^1(L)$. As we have checked both algebraic multiplicity and Zariski multiplicity are multiplicative over composition, a straightforward calculation shows that we need only check the notions agree for the branched finite cover $\pi : F \rightarrow P^1(L)$. (1)

Now consider this cover restricted to A^1, let x be the canonical coordinate with $ord_a(\pi^*(x)) = m$, so we have that $\pi^*x = h^m u$, for u a unit in $\mathcal{O}_a$ and h a uniformiser at a. (2)

As u is a unit and $char(L)=0$, the equation $z^m=u$ splits in the residue field of $\mathcal{O}_a^\wedge$. By Hensel's Lemma and Theorem 9.5, it is solvable in $\mathcal{O}_a^\wedge$. By the definition of $\mathcal{O}_a^\wedge$, we can find an etale morphism $\pi:(U,b)\rightarrow(F,a)$ containing such a solution in the local ring $\mathcal{O}_b$. We may assume that U is irreducible and moreover, as π is etale, that U is smooth. (3)

Now we can embed U in a projective smooth curve F' and, as F' is smooth, extend the morphism π to a projective morphism from F' to F. (4)

We claim that $(ba)\in graph(\pi)\subset F'\times F$ is unramified in the sense of Zariski structures. For this we need the following fact whose algebraic proof relies on the fact that etale morphisms are flat, see [14];

Fact 10.7. *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & Spec((A[T]/f(T))_d) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ U & \xrightarrow{h} & Spec(A) \end{array}$$

where $f(T)$ is a monic polynomial in $A[T]$, $f'(T)$ is invertible in $(A[T]/f(T))_d$ and g,h are isomorphisms. (5)

Using Lemma 8.6 and the fact that the open set V is smooth, we may safely replace $graph(\pi)$ by $\overline{graph(\pi')}\subset F''\times F$ where F'' is the projective closure of $Spec((A[T]/f(T))$, F is the projective closure of $Spec(A)$ and $\overline{graph(\pi')}$ is the projective closure of $graph(\pi')$ and show that $(g(b)a)$ is Zariski unramified. Note that over the open subset $U=Spec(A)\subset F$, $\overline{graph(\pi')}=Spec((A[T]/f(T)$ as this is closed in $U\times F''$. For ease of notation, we replace $(g(b)a)$ by (ba). (6)

Suppose that f has degree n. Let $\sigma_1\ldots\sigma_n$ be the elementary symmetric functions in n variables $T_1,\ldots T_n$. Consider the equations

$\sigma_1(T_1,\ldots,T_n)=a_1$

$\ldots$

$\sigma_n(T_1, \ldots, T_n) = a_n$ (*)

where $a_1, \ldots a_n$ are the coefficients of f with appropriate sign. These cut out a closed subscheme $C \subset Spec(A[T_1 \ldots T_N])$. Suppose $(ba) \in graph(\pi') = Spec(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1)$,$(a'b_2) \in Spec(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n-tuple $(b_1b_2c'_1 \ldots c'_{n-2})$ corresponding to the roots of f over a'. The tuple $(a'b_1b_2c'_1 \ldots c'_{n-2})$ satisfies C, hence so does the specialisation $(abbc_1 \ldots c_{n-2})$. Then the tuple $(bbc_1 \ldots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a. However such a solution is unique up to permutation and corresponds to the roots of f over a. This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$. As (ab) lies inside $Spec(A[T]/f(T))_d$, this contradicts the fact that f' is invertible in $A[T]/f(T))_d$. (7)

In (2) we may therefore assume that $\pi^* x = h^m$ for h a local uniformiser at a. Now we have the sequence of ring inclusions given by

$$L[x] \rightarrow L[x, y]/(y^m - x) \rightarrow R$$

$$x \mapsto \pi^* x, y \mapsto h$$

where R is the coordinate ring of F in some affine neighborhood of a. It follows that we can factor our original map such that F is etale near a over the projective closure of $y^m - x = 0$. (8)

Again, repeating the argument from (4) to (7), we just need to check that the projective closure of $y^m - x$ has multiplicity m at 0 considered as a cover of $P^1(\bar{k})$. This is trival, let $\epsilon \in \mathcal{V}_0$ be generic over

$\mathcal{M}$,then as we are working in characteristic 0 we can find distinct $\epsilon_1, \ldots \epsilon_m$ in $\mathcal{M}_*$ solving $y^m = \epsilon$. By specialisation, each $\epsilon_i \in \mathcal{V}_0$. (9) □

Remarks 10.8. *Given the hypotheses of Theorem 10.6, with the modification that* $char(L) = p \neq 0$, *we define a point* $(ab) \in F$ *to be wildly ramified if* $mult^{alg}_{(ab)}(F/D)$ *is divisible by* p. *Theorem 10.6 holds excluding wildly ramified points. In order to see this, we first replace the argument* (1), *by showing that, for any* given *point* $a \in D$, *we can find a finite morphism* f *from* D *to* $P^1(L)$, *such that* f *is etale in an open neighborhood of* a;

As a is a non-singular, we can find a uniformising element t in the local ring $O_{a,D}$ of D. Considering t as an element of the function field $L(D)$, we obtain an embedding $L(t) \subset L(D)$, which, as D is non-singular, determines a unique morphism f from D to $P^1(L)$. Restricting the morphism to $A^1(L)$ and letting x be the canonical coordinate, we have that $f^*(x) = t$, hence $ord_a(f^*(x)) = 1$. This shows that f is etale in an open neighborhood of a by Theorem 9.2 and Remarks 9.3. (†)

As etale morphisms have multiplicity coprime to p, it is sufficient to check the result for a branched cover $\pi : F \rightarrow P^1(L)$. If $a \in D$ is not wildly ramified for this cover, then we can follow through arguments (2) *and* (3) *of Theorem 10.6. The argument from* (4) *to* (8) *is the same and we obtain the result of* (9) *again using the fact that m there is coprime to p. This proves the result.*

Theorem 10.6 also holds with the modification that $char(L) = p \neq 0$ and the cover $pr : F \rightarrow D$ is seperable. *However, the proof requires more sophisticated methods, which we consider in* [21]. *We can, however, handle a special case by an elementary counting argument. First observe that we can replace the argument* (1) *by observing that there exists a seperable morphism f from D to $P^1(L)$. This either follows from the argument* (†) *above or using the classical result that the function field $L(C)$ admits a seperating transcendence basis over L, (see p27 of* [9]*). Hence, it is sufficient to check the result for a finite seperable cover $\pi : F \rightarrow P^1(L)$. By a classical result, (see Proposition 2.2, p300, of* [9]*), there exist finitely many ramification points, in particularly finitely many wild ramification points $\{a_1, \ldots, a_n\}$, for the cover π. By the previous proof, we need only check the result of Theorem 10.6 for these finitely many points.*

Special Case. a is a wild ramification point for the cover with the property that that there exist no other wild ramification points in the fibre $\pi^{-1}(\pi(a))$.

As both F and $P^1(L)$ are non-singular, the finite morphism π is flat, by Lemma 9.11. By a result in [16]*, (Corollary of Proposition 2, p218), we have that;*

$\sum_{y\in\pi^{-1}(x)} mult_y^{alg}(F/P^1)$ is independent of $x \in P^1(L)$, and equals the cardinality of a generic fibre.

By Lemma 8.3, a corresponding result also holds for Zariski multiplicities. Hence, by the result of the previous proof in this remark, the claim follows.

Unfortunately, one can have;

a is a wild ramification point for the cover with the property that there exist other wild ramification points $\{a_1, \ldots, a_r\}$, distinct from a, in the fibre $\pi^{-1}(\pi(a))$.

It seems difficult to find any way of reducing this scenario to the special case. However, one can still use a local method, which is done in [21].

Theorem 10.9. *Let hypotheses be as in Theorem 10.6, with the modification that $char(L) = p \neq 0$. If e denotes the Zariski multiplicity and d the algebraic multiplicity at $a \in F$, then $d = ep^n$ and π factors as $F \rightarrow_h F' \rightarrow_g D$ with $h = Frob^n$ and g having algebraic multiplicity e at $h(a)$.*

Proof. By Theorem 10.3, we can factor π into a purely inseperable morphism $h : F \rightarrow F'$ and a seperable morphism $g : F' \rightarrow D$ with F' a smooth projective curve. We then have a corresponding sequence of field extensions $L(D) \subset L(F') \subset L(F)$, with $L(F)$ a purely inseperable extension of $L(F')$. As $L(F)$ is a purely inseperable field extension of $L(F')$, it has degree p^n for some $n \geq 1$. Hence, $L(F) = L(F')^{1/p^n}$ and we may, without loss of generality, assume that $h = Frob^n$, see also Proposition 2.5 (p302) of [9]. By Remarks 10.8, the notions of Zariski multiplicity and algebraic multiplicity coincide for the morphism g. By Remarks 6.5, the Frobenius morphism $Frob$ may be identified with Fr, without effecting Zariski or algebraic multiplicities. Clearly, Fr is a bijection on points, hence it is Zariski unramified. Fr has algebraic multiplicity p everywhere, as, for any point $x \in F'$, we can choose a local uniformiser t at x such that $Fr^*(t) = t^p$. It follows that h has algebraic multiplicity p^n everywhere and is Zariski unramified. The result now follows immediately from Lemma 8.5 and Remarks 9.7.

□

We now give local versions of Theorem 10.1 in the case of algebraic curves, where we allow F to be a singular algebraic curve, over a field L with $char(L) = 0$ and find an analogous version of Theorem 10.3, in the case when $char(L) = p$.

Theorem 10.10. *Let hypotheses be as in Theorem 7.3, with the additional assumption that $char(L) = 0$ and D is a curve. Let pr be the projection map of F onto D. Then, if $(ab) \in F$ is non-singular;*

$$Mult_{ab}(F/D) = mult_{ab}^{alg}(F/D)$$

that is Zariski multiplicity and algebraic multiplicity coincide. In particular, the cover (F/D) is Zariski unramified at (ab) iff there exists an open $U \subset F$, containing (ab), such that $pr : U \to D$ is etale.

Proof. For the first part of the theorem, we follow the proof of Theorem 10.6, the difference between the hypotheses there is that we do *not* assume that F is non-singular. Using the fact that D is smooth and the result of Theorem 6.6, we may, without loss of generality, assume that $D = P^1(L)$. Now, one can follow through the proof of Theorem 10.6, using the fact that (ab) is non-singular, in order to obtain the result. One should make the modification that Zariski multiplicity is well defined for any finite cover $F' \to F$ at (abc) lying over (ab). This follows from an easy extension of Theorem 7.3, to show that a nonsingular open subvariety of an irreducible projective variety of dimension 1 is presmooth (see [29]). For the second part of the theorem, suppose that there exists an open $U \subset F$, containing (ab), such that $pr : U \to D$ is etale. As (ab) is non-singular, we may assume that U defines a non-singular open subvariety of F. Following the argument of Theorem 10.6, from the end of (4) to the end of (7), we obtain that the cover (F/D) is Zariski unramified at (ab). For the converse, assume that the cover is Zariski unramified at (ab). By Theorem 9.2, Remarks 9.3 and the fact that (ab) is non-singular, it is sufficient to prove that $d(pr) : (m_{(ab)}/m_{(ab)}^2)^* \to (m_a/m_a^2)^*$ is an isomorphism. Equivalently, we need to show that the algebraic multiplicity $mult_{(ab)}^{alg}(F/D)$ of pr at $(ab) \in F$ equals 1. This follows from the first part of the theorem.

□

Theorem 10.11. *Let hypotheses be as in Theorem 7.3, with the additional assumption that $char(L) = p$, D is a curve and the projection map pr of F onto D is seperable. Then, if $(ab) \in F$ is non-singular;*

$Mult_{ab}(F/D) = mult_{ab}^{alg}(F/D)$

that is Zariski multiplicity and algebraic multiplicity coincide. In particular, the cover (F/D) is Zariski unramified at (ab) iff there exists an open $U \subset F$, containing (ab), such that $pr : U \rightarrow D$ is etale.

Proof. The proof is similar to the previous theorem. By Remarks 10.8, we can assume that $D = P^1(L)$. Using the fact that (ab) is non-singular, one can either follow through the proof of Theorem 10.6, if (ab) is not wildly ramified for the cover, or one can use the method in [21], if (ab) is wildly ramified for the cover. For the second part of the theorem, one can use the same reasoning as in Theorem 10.10. □

Part 3

In this part of the paper ,we show that the Zariski structure analysis can be useful in algebraic geometry, by providing a more geometric framework for understanding intersections of algebraic curves in the plane. We assume some familiarity with certain notions from algebraic and analytic geometry, as well as the material from Part 2.

11. Etale Morphisms Preserve Zariski and Algebraic Multiplicity

We recall material from the previous part of the paper, in particular Definition 9.1, Lemma 9.4 and the subsequent proof and Theorem 9.5.

The following theorem requires some knowledge of Zariski structures, see sections 1-4 of [19], or sections 5 and 7 of this paper.

Theorem 11.1. *Zariski multiplcity is preserved by etale morphisms*
Let $\pi : X \rightarrow Y$ be an etale morphism with Y smooth, then any $(ab) \in graph(\pi) \subset X \times Y$ is unramified in the sense of Zariski structures.

The proof is contained in Theorem 10.6; using Fact 10.7, and the argument up to (7).

We also review some facts about algebraic multiplicity and show that algebraic multiplicity is preserved by etale morphisms. We recall Definition 9.8 and the subsequent Remarks 9.9, observing the definition holds even without the the assumption of smoothness.

We now have the following;

Theorem 11.2. *Algebraic multiplicity is preserved by etale morphisms;*

Given finite morphisms $f : X_3 \rightarrow X_2$ *and* $g : X_2 \rightarrow X_1$ *with* f *etale. If* $a \in X_3$*, then* $mult^{alg}_{a,gf(a)}(X_3/X_1) = mult^{alg}_{f(a),gf(a)}(X_2/X_1)$.

Proof. This result is essentially given in [16]. Let $O^{\wedge}_{f(a),X_2}$ be the Henselisation of the local ring at $f(a)$. By base change, we have an etale morphism $f' : X' = X_3 \times_{X_2} Spec(O^{\wedge}_{f(a)}) \rightarrow Spec(O^{\wedge}_{f(a)})$. By the definition of an etale morphism given above, we may write this cover locally in the form $Spec(O^{\wedge}_{f(a)}\frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n})$, with $det(\frac{\partial f_i}{\partial x_j}) \neq 0$ at each closed point in the fibre over $f(a)$. At the closed point a, let a_i be the residues of the x_i in L, then we have that $(a_1, \ldots a_n)$ is a common root for $\{\bar{f}_1, \ldots, \bar{f}_n\}$ where $\bar{f}_i$ is obtained by reducing f_i with respect to the maximal ideal $m_{f(a),X_2}$ of $O^{\wedge}_{f(a)}$. As $O^{\wedge}_{f(a)}$ is Henselian, by the above, and the determinant condition, we can lift the roots a_i to roots α_i of the f_i in $O^{\wedge}_{f(a)}$. We therefore obtain a subscheme $Z = Spec(O^{\wedge}_{f(a)}\frac{[x_1,\ldots,x_n]}{<x_1-\alpha_1,\ldots,x_n-\alpha_n} >)$ of X' which is isomorphic to $Spec(O^{\wedge}_{f(a)})$ under the restriction of f. Let Q be the $O_{X'}$ ideal defining Z, we then have that $m_{a,X'} = f^* m_{f(a),X_2} \oplus Q_a$. As f is etale, by $(**)$ after definition 1.1 above, $m_{a,X'} = f^* m_{f(a),X_2}$, therefore $Q_a = 0$ and by Nakayama's lemma $Q = 0$ in an open neighborhood of a in X'. This gives that $Z = X'$ in an open neighborhood of a. Hence we obtain the sequence $O_{f(a),X_2} \rightarrow_{f^*} O_{a,X_3} \rightarrow_{i^*} O_{a,X'}$ (***) where the map $i^* f^*$ is the inclusion of $O_{f(a),X_2}$ inside $O^{\wedge}_{f(a),X_2}$. Now if $n \subset m_{f(a),X_2}$ is the pullback $g^* m_{gf(a),X_1}$, we have that $length(O_{f(a),X_2}/n) = length(O^{\wedge}_{f(a),X_2}/n)$, hence the result follows by $(***)$ as required.

□

12. More Results on Zariski Multiplicity

We work in the context of Theorem 7.3. Namely, W (we used the notation V before) will denote a smooth projective variety defined over an algebraically closed field L, considered as a Zariski structure with closed sets given by algebraic subvarieties defined over L. All notions connected to the definition of Zariski multiplicity will come from the fixed specialisation map $\pi : W(K_\omega) \rightarrow W(L)$ where K_ω denotes a "universal" algebraically closed field containing $L = K_0$. We consider D a smooth subvariety of some cartesian power W^m and a finite cover, with respect to projection onto the first coordinate, $F \subset D \times W^k$, all defined over L (*). This allows us to make sense of Zariski multiplicity. In general, we can move freely between Zariski structure notation and algebraic geometry notation. Clearly $(*)$ makes sense algebraically. Conversely, if X and Y denote fixed projective varieties defined over L with Y smooth and a finite morphism $f : X \rightarrow Y$ over L is given , then we can reduce to the situation of $(*)$ by taking F to be $graph(f) \subset X \times Y$ with the projection map onto the second factor and W to be the corresponding projective space $P^n(L)$ where $X, Y \subset P^n(L)$. We can even take W to be the 1-dimensional Zariski structure $P^1(L)$ by using the embedding of $P^n(L)$ into the N'th Cartesian power of $P^1(L)$ for sufficiently large N.

We use Definition 8.1 of Zariski multiplicity for irreducible finite covers. We will also require the following generalisation;

Definition 12.1. *Let $F \subset D \times W^k$ be an equidimensional, finite cover of smooth D, with irreducible components $C_1, \ldots, C_n$. Then for $(ab) \in F$, we define $Mult_{ab}(F/D) = \sum_{(ab) \in C_i} Mult_{ab}(C_i/D)$.*

Clearly this is well defined using the definition of Zariski multiplicity for irreducible covers. However, until Lemma 12.9, the assumption that F is irreducible will be in force.

Lemma 12.2. *Zariski multiplicity is multiplicative over composition*

Suppose that F_1, F_2 and F_3 are smooth, irreducible, with $F_2 \subset F_1 \times W^k$ and $F_3 \subset F_2 \times W^l$ finite covers. Let $(abc) \in F_3 \subset F_1 \times W^k \times W^l$. Then $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1) mult_{abc}(F_3/F_2)$.

Proof. To see this, let $m = mult_{ab}(F_2/F_1)$ and $n = mult_{abc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1(K_\omega)$ generic over L. By definition, we can find distinct $b_1 \ldots b_m$ in $W^k(K_\omega) \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1, we have that $dim(a'b_i/L) = dim(a'/L) = dim(F_1) = dim(F_2)$, so each $(a'b_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over L. Again by definition, we can find distinct $c_{i1} \ldots c_{in}$ in $W^l(K_\omega) \cap \mathcal{V}_c$ such that $F_3(a'b_ic_{ij})$ holds. Then the mn distinct elements $(a'b_ic_{ij})$ are in $\mathcal{V}_{abc}$, so by definition of multiplicity $mult_{abc}(F_3/F_1) = mn$ as required.

□

Lemma 12.3. *Let hypotheses be as in the above lemma with the extra condition that the cover F_3/F_2 is etale. Then for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$*

Proof. This is an immediate consequence of Lemma 12.2 and Theorem 11.1.

□

Lemma 12.4. *Zariski multiplicity is summable over specialisation*

Suppose that $F\subset D\times W^k$ is a finite irreducible cover with D smooth. Suppose $(ab)\in F$, $a'\in\mathcal{V}_a\cap D$ and $a''\in\mathcal{V}_{a'}\cap D$ with a'' generic over L. Then

$$Mult_{ab}(F/D)=\Sigma_{b'\in\mathcal{V}_b\cap F(a'y)}Mult_{a'b'}(F/D)$$

Proof. Suppose $F(a''b_1),\ldots F(a''b_n)$ hold with $b_i\in\mathcal{V}_b$, so $\{b_1,\ldots,b_n\}$ witness the fact that $Mult_{ab}(F/D)=n$. Write $\{b_1,\ldots b_n\}$ as $\{b_{11},\ldots,b_{1m_1},b_{21},\ldots,b_{2m_2},\ldots,b_{i1},\ldots b_{ij},\ldots,b_{im_i},\ldots,b_{nm_n}\}$ (*), where b_{ij} maps to a_i in the specialisation taking a'' to a'. To prove the lemma, it is sufficient to show that $F(a'y)\cap\mathcal{V}_b=\{a_1,\ldots,a_n\}$ and $Mult_{(a'a_i)}(F/D)=m_i$. The second statement just follows from the fact that a'' is generic in D over L in $\mathcal{V}_{a'}$. To prove the first statement, suppose we can find a_{n+1} with $F(a'a_{n+1})$ and $a_{n+1}\in\mathcal{V}_b$ but $a_{n+1}\notin\{a_1,\ldots a_n\}$. By Theorem 7.3, we can find c with $F(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$. As $a_{n+1}\in\mathcal{V}_b$, $(a'a_{n+1})$ specialises to (ab), hence so does $(a''c)$. Therefore, c must witness the fact that $Mult_{ab}(F/D)=n$ and appear in the set $\{b_1,\ldots,b_n\}$. This clearly contradicts the arrangement of $\{b_1,\ldots,b_n\}$ given in $(*)$.

□

Definition 12.5. *Let $F\subset U\times V\times W^k$ be an irreducible finite cover of $U\times V$ with U and V smooth.*

Given $(u,v,x)\in F$ we define;

$LeftMult_{u,v,x}(F/D)=Card(\mathcal{V}_x\cap F(u',v))$ for $u'\in\mathcal{V}_u\cap U$ generic over L.

$RightMult_{u,v,x}(F/D)=Card(\mathcal{V}_x\cap F(u,v'))$ for $v'\in\mathcal{V}_v\cap V$ generic over L.

We first show that both left and right multiplicity are well defined. In order to see this, observe that the fibres $F(u,V)$ and $F(U,v)$ are finite covers of V and U respectively with U and V smooth. Moreover, the fibres $F(u,V)$ and $F(U,v)$ are equidimensional covers of V and

U respectively. In order to see this, as U is smooth, it satisfies the presmoothness axiom with the smooth projective variety W^k given in Definition 5.1. The fibre $F(u,V)=F\cap(W^k\times\{u\}\times V)$. By presmoothness, each irreducible component of the intersection has dimension at least $dim(F)+dim(W^k\times V)-dim(U\times V\times W^k)=dim(F)-dim(U)=dim(V)$. As $F(u,V)$ is a finite cover of V, it has exactly this dimension. Now we can use the definition of Zariski multiplicity given in Definition 12.1.

We then claim the following;

Lemma 12.6. *Factoring Multiplicity*

In the situation of the above definition, we have that;

$Mult_{u,v,x}(F/U\times V)=\Sigma_{x'\in(\mathcal{V}_x\cap F(y,u',v))}RightMult_{x',u',v}(F/U\times V)$ *for* u' *generic in* U *over* L.

$Mult_{u,v,x}(F/U\times V)=\Sigma_{x'\in(\mathcal{V}_x\cap F(y,u,v'))}LeftMult_{x',u,v'}(F/U\times V)$ *for* v' *generic in* V *over* L.

Proof. We just prove the first statement, the proof of the second is apart from notation identical. By the construction in Section 3 and Lemma 7.2, we can choose algebraically closed fields $L=K_0\subset K_{n_1}\subset K_{n_2}\subset K_\omega$, and tuples $u'\in K_{n_1}$, $v'\in K_{n_2}$ such that u' is generic in U over L, v' is generic in V over K_{n_1} with specialisations $\pi_1:P^n(K_{n_1})\rightarrow P^n(L)$ and $\pi_2:P^n(K_{n_2})\rightarrow P^n(K_1)$ such that $\pi_2(u'v')=(u'v)$ and $\pi_1(u'v)=(uv)$. Now $dim(u'v'/L)=dim(v'/L(u'))+dim(u'/L)=dim(V)+dim(U)$, hence $u'v'$ is generic in $U\times V$ over L. Therefore $Mult_{u,v,x}=Card(\mathcal{V}_x\cap F(u'v'))$. Let $S=\{y_{11},\ldots,y_{1m_1},\ldots,y_{ij_i},\ldots,y_{n1},\ldots,y_{nm_n}\}$ be distinct elements in $\mathcal{V}_x\cap W^k$ witnessing this multiplicity such that for $1\leq j_i\leq m_i$, $\pi_2(y_{ij_i})=z_i\in\mathcal{V}_x\cap W^k$. It is sufficient to show that $RightMult_{u'v,z_i}(F/U\times V)=m_i$ and $\{z_1,\ldots z_n\}$ enumerates $\mathcal{V}_x\cap F(y,u',v)$. The first statement follows as $v'\in\mathcal{V}_v\cap V$ is generic in V over $L(u')$. For the second statement, suppose that we can find $z_{n+1}\in\mathcal{V}_x\cap F(y,u',v)$ with $z_{n+1}\notin\{z_1,\ldots z_n\}$. Consider $F(u',V)$ as a finite cover of V, defined over $L(u')$, so by the above $F(u',V)$ is an equidimensional finite cover of V. Then, as v' was chosen to be generic in V over $L(u')$, choosing an irreducible component of $F(u',V)$ passing through $(z_{n+1},u'v)$, by the lifting result of Theorem 7.3, we can find

$y_{n+1} \in \mathcal{V}_{z_{n+1}} \cap W^k$ such that $F(y_{n+1}, u', v')$. Clearly, $y_{n+1} \in S$ which contradicts the definition of S.

□

Theorem 7.3 of does not hold in the case when D fails to be smooth. However, in the case of etale covers, we still have the following result;

Lemma 12.7. *Lifting Lemma for Etale Covers*

Let $F \subset D \times W^k$ be an etale cover of D defined over L, with the projection map denoted by f. Then given $a \in D$, $(ab) \in F$ and $a' \in \mathcal{V}_a \cap D$ generic over L, we can find $b' \in \mathcal{V}_b$ such that $F(a', b')$ holds. Moreover b' is unique, hence $Mult_{ab}(F/D) = 1$. Moreover, in the situation of Lemma 12.3, without requiring that F_2 is smooth, we have that for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$.

Proof. Using the definition of etale we recalled in Section 11 above, we can assume that the cover is given algebraically in the form $f^* : L[D] \rightarrow L[D]\frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n}$ with $det(\frac{\partial f_i}{\partial x_j})_{ij}(x) \neq 0$ for all $x \in F$. So we can present the cover in the form $f_1(x, y) = 0, f_2(x, y) = 0, \ldots, f_n(x, y) = 0$, with y in D and x in $A^n(L)$. Let L_m be the algebraic closure of the field generated by L and $\bar{g}(a)$ where $\bar{g}$ is a tuple of functions defining D locally. Consider the system of equations $f_1(x, a) = f_2(x, a) = \ldots = f_n(x, a) = 0$ defined over L_m. Then this system is solved by b in L_m with the property that $det(\frac{\partial f_i}{\partial x_j})_{ij}(b) \neq 0$ (*). Now suppose that $a' \in \mathcal{V}_a \cap D$ is chosen to be generic over L. By the construction given in Section 3, we may assume that a' lies in an algebraic extension R of $L_s\{t\}$, the completion of $L_s(t)$, with respect to a valuation on $L_s(t)$, for some algebraically closed field L_s extending L_m. R is a henselian ring, hence if we consider the system of equations $f_1(x, a') = f_2(x, a') = \ldots = f_n(x, a') = 0$ with coefficients in R, by the fact that the system specialises to a solution in L_s with the condition (*) we can find a solution b' in R. Then $(a'b')$ lies in F and by construction $b' \in \mathcal{V}_b$. The uniqueness result follows from the proof of Theorem 11.1. For the last part, suppose that $mult_{ab}(F_2/F_1) = n$, then we can find $a' \in \mathcal{V}_a \cap F_1$ generic over L and $\{b_1, \ldots b_n\} \in \mathcal{V}_b \cap W^k$ distinct such that $F(a', b_i)$ holds. Each $(a'b_i)$ is generic in F_2 over L, hence by the previous part of the lemma, we can find a unique $c_i \in \mathcal{V}_c \cap W^l$ such that $F_3(a'b_ic_i)$ holds. This show that $mult_{abc}(F_3/F_1) = n$ as required.

□

Lemma 12.8. *Lifting Lemma for Etale Covers with Right(Left) Multiplicity*

Let hypotheses be as in Lemma 12.2, with the additional assumption that $F_1 = U \times V$, F_2 is a smooth irreducible cover of F_1 and F_3 is an irreducible etale cover of F_2. Then, with notion as in Definition 12.5, given $(uvbc) \in F_3$, $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$. Similarly for left multiplicity.

Proof. Suppose that $RightMult_{uvb}(F_2/F_1) = n$, then for $v' \in \mathcal{V}_b$ generic in V over L, we can find $\{b_1, \ldots, b_i, \ldots b_n\} \in \mathcal{V}_b$ with $F_2(uv'b_i)$ holding. For each b_i we claim that there exists a unique $c_i \in \mathcal{V}_c$ such that $F_3(uv'b_ic_i)$ holds. For the existence, we can use Lemma 12.7, with the simple modification that, with the notation there, if L_m is the algebraic closure of the field generated by $\bar{g}(uv)$, then provided $dim(V) \geq 1$, we can find $v' \in \mathcal{V}_v \cap V$ generic over L with $uv' \in L_s[[t^{1/r}]]$ for some algebraically closed field L_s containing L_m. For the uniqueness, we can use the fact that Zariski multiplicity is summable over specialisation (Lemma 12.4) and the fact that for generic $(u'v'b_i') \in \mathcal{V}_{uvb} \cap F_2$, we can find a unique $c_i' \in \mathcal{V}_c$ such that $F_3(u'v'b_i'c_i')$ holds. Finally, we claim that $\{b_1c_1, \ldots, b_nc_n\}$ enumerate $F_3(uv'xy) \cap \mathcal{V}_{bc}$. This is clear by the above proof and the fact that $\{b_1, \ldots, b_n\}$ enumerates $F_2(uv'x) \cap \mathcal{V}_b$. □

Lemma 12.9. *The following versions of the above properties hold when we consider finite equidimensional covers, possibly with components, with the definition of Zariski multiplicity given in Definition 12.1.*

Proof. For Lemma 12.3, we replace the hypotheses with F_1 is smooth irreducible, F_2 is an equidimensional finite cover of F_1 and F_3 is an etale cover of F_2. We then claim, using notation as in Lemma 12.2, that $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$. By definition $mult_{abc}(F_3/F_1) = \sum_{(abc)\in C_i}(mult_{abc}(C_i/F_1))$, where C_i are the irreducible components of F_3 passing through (abc). As F_3 is an etale cover of F_2, the images of the C_i are precisely the irreducible components D_i of F_2 passing through (ab), each C_i is an etale cover of D_i and $mult_{ab}(F_2/F_1) = \sum_{(ab)\in D_i}(mult_{ab}(D_i/F_1))$. Hence, it is sufficient to prove the result in the case when F_2 and F_3 are irreducible. This is just Lemma 12.3

For Lemma 12.4, we replace the hypothesis with F is an equidimensional finite cover of D. The proof then goes through exactly as in the lemma with the observation that if we find $a_{n+1} \in \mathcal{V}_b$ and $F(a'a_{n+1})$ then we can find an irreducible component C passing through $(a'a_{n+1})$ which allows us to apply Theorem 7.3 to obtain c with $C(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$.

For Definition 2.5, we alter the hypothesis to F is an equidimensional finite cover of $U \times V$. Again, we can use an identical proof to show that left multiplicity and right multiplicity are well defined. The proof of Lemma 12.6 with the new hypothesis on F is identical

We don't require a modified version of Lemma 12.7, the result we need is contained in the modified proof of Lemma 12.3

For Lemma 12.8, we alter the hypotheses to F_2 is an equidimensional cover of F_1 and F_3 is an etale cover of F_2. We then claim that for (uvb) a non-singular point of F_2 and $(uvbc) \in F_3$, necessarily non-singular as well, that $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$ and similarly for left multiplicity. To prove this, note that as (uvb) and $(uvbc)$ are non-singular points, there exist unique components C and D passing through (uvb) and $(uvbc)$ respectively. Now replacing C and D by the open subsets C' and D' of smooth points, we can apply the definition of Right Multiplicity and the proof of Lemma 12.8.

□

13. Analytic Methods

In order to use the method of etale morphisms, which preserve Zariski multiplicity, we need to work inside the Henselisation of local rings $L[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$. In the next section, we will only need the result for the local ring in 2 variables $L[x, y]_{(x,y)}$.

We let $L[[x_1, \ldots, x_n]]$ denote the ring of formal power series in n variables, which is the formal completion of $L[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ with respect to the canonical order valuation. The following is a classical result, requiring the fact that etale morphisms are flat, used in the proof of the Artin approximation theorem. This relates the Henselisation of the ring $L\{x_1, \ldots, x_n\}$ of strictly convergent power series in several variables with its formal completion $L[[x_1, \ldots, x_n]]$, see [3] or [25];

$$\text{Henselisation}(L[x_1, \ldots x_n]_{(x_1,\ldots x_n)}) = L[[x_1, \ldots x_n]] \cap L(x_1, \ldots x_n)^{alg}$$

This implies that

$$O^{\wedge}_{\bar{0},A^n} \cong L[[x_1, \ldots x_n]] \cap L(x_1, \ldots x_n)^{alg}$$

The following result, which can be found in [5], is essential for the next section

Lemma 13.1. *Weierstrass Preparation*

Let $F(x_1,\ldots x_n)$ be a polynomial in $L[x_1,\ldots,x_n]$ which is regular in the variable x_n. Then we have $F(x_1,\ldots,x_n)=U(x_1,\ldots,x_n)G(x_1,\ldots,x_n)$ where $U(x_1,\ldots,x_n)$ is a unit in the local ring $L[[x_1,\ldots,x_n]]$ and $G(x_1,\ldots,\ldots x_n)$ is a Weierstrass polynomial in x_n with coefficients in $L[[x_1,\ldots,x_{n-1}]]$

We will require the Weierstrass decomposition to hold inside Henselisation($L[x_1,\ldots,x_n]$), therefore we need to show that the Weierstass data can be found inside $L(x_1,\ldots,x_n)^{alg}$. This is acheived by the following lemma;

Lemma 13.2. *Definability of Weierstrass data*

Let $F(x_1,\ldots,x_n)$ be a polynomial with coefficients in L such that F is regular in x_n, then if $F(x_1,\ldots,x_n)=U(x_1,\ldots,x_n)G(x_1,\ldots,x_n)$ is the Weierstrass decomposition of F with $G(x_1,\ldots,x_n)=x_n^m+a_1(x_1,\ldots,x_{n-1})x_n^{m-1}+\ldots+a_m(x_1,\ldots,x_{n-1})$, and $a_i\in L[[x_1,\ldots,x_{n-1}]]$, $U(x_1,\ldots,x_n)\in L[[x_1,\ldots,x_n]]$, then $a_i(x_1,\ldots,x_{n-1})\in L(x_1,\ldots,x_{n-1})^{alg}$ and $U(x_1,\ldots,x_n)\in L(x_1,\ldots,x_n)^{alg}$.

Proof. This can be proved by rigid analytic methods. Equip L with a complete non-trivial non-archimedean valuation v and corresponding norm $||.||_v$, this can be done for example by assuming that L is the completion of an algebraically closed field with any non-archimidean valuation, see [5]. Let $T_{n-1}(L)$ be the free Tate algebra in the indeterminate variables $x_1,\ldots,x_{n-1}$ over L, that is the subalgebra of strictly convergent power series in $L[[x_1,\ldots,x_{n-1}]]$. By the proof of Weierstrass preparation in [5], as $F\in T_{n-1}(L)[x_n]$, the coefficients a_i lie in $T_{n-1}(L)$ and $U(x_1,\ldots,x_n)\in T_{n-1}(L)[x_n]$. Now choose $(u_1,\ldots u_{n-1})\subset L$ transcendental over the coefficients of F with $max(\{||u_i||\})\leq 1$. Then if $s_1(\bar u),\ldots,s_m(\bar u)$ denote the roots of $F(\bar u,x_n)$ with $||s_i(\bar u)||\leq 1$, then both $U(\bar u,s_i(\bar u))$ and $G(\bar u,s_i(\bar u))$ define elements of L and moreover, by a theorem in [23], we have that the coefficients $a_i(\bar u)$ are symmetric functions of the $s_i(\bar u)$. Hence the $a_i(\bar u)$ belong to $L(\bar u)^{alg}$. As $\bar u$ was transcendental, we have that each $a_i\in L[x_1,\ldots,x_{n-1}]^{alg}$. As $U(x_1,\ldots x_n)=F/G(x_1,\ldots,x_n)$, we clearly have that $U(x_1,\ldots,x_n)\in L[x_1,\ldots,x_n]^{alg}$ as well.

□

14. Families of Curves in $P^2(L)$

We consider the family Q_d of projective curves in $P^2(L)$ with degree d. An element of Q_d may be written;

$$\sum_{0\leq i+j\leq d} a_{ij}(X/Z)^i(Y/Z)^j = 0$$

which, rewriting in homogenous form, becomes;

$$\sum_{0\leq i+j\leq d} a_{ij}X^iY^jZ^{d-(i+j)} = 0$$

For ease of notation, we will use affine coordinates $x = X/Z$ and $y = Y/Z$. More generally, if we give an affine cover, we implicitly assume that it can be projectivized by taking $\bar{y} = (y_1, \ldots, y_n) = (Y_1/Z, \ldots, Y_n/Z)$. As the notion of Zariski multiplicity is local, this will not effect our calculations.

Now consider two such families Q_d and Q_e. Then we have the cover obtained by intersecting degree d and degree e curves

$$Spec(L[x, y, u_{ij}, v_{ij}]/ < s(u_{ij}, x, y), t(v_{ij}, x, y) >) \rightarrow Spec(L[u_{ij}, v_{ij}]).$$
(*)

where

$$s(u_{ij}, x, y) = \sum_{0\leq i+j\leq d} u_{ij}x^iy^j$$

$$t(v_{ij}, x, y) = \sum_{0\leq i+j\leq e} v_{ij}x^iy^j$$

We denote the parameter space for degree d curves by U and the parameter space for degree e curves by V. These are affine spaces of dimension $(d+1)(d+2)/2$ and $(e+1)(e+2)/2$ respectively. The cover (*) is generically finite, that is there exists an open subset $U' \subset Sp(L[u_{ij}, v_{ij}])$ for which the restricted cover has finite fibres. Throughout this section, we will denote the base space of the cover by $U \times V$, bearing in mind that we implicitly mean by this $(U \times V) \cap U'$. Now, given 2 fixed parameters sets $\bar{u}$ and $\bar{v}$, with $(\bar{u}, \bar{v}) \in U'$, corresponding to curves $C_{\bar{u}}$ and $C_{\bar{v}}$, the algebraic multiplicity of the cover $(*)$ at $(00, \bar{u}, \bar{v})$ is exactly the intersection multiplicity $I(C_{\bar{u}}, C_{\bar{v}}, 00)$ of the curves at (00). The cover (*) is equidimensional as $U \times V$ satisfies the presmoothness axiom with the smooth projective variety $P^2(L)$. Restricting to a finite cover over U', by Definition 12.1 we can also define the Zariski multiplicity of the cover at the point $(00,\bar{u},\bar{v})$. The main result that we shall prove in this part of the paper is the following, which generalises an observation given in [13];

Theorem 14.1. *In all characteristics, the algebraic multiplicity and Zariski multiplicity of the cover* $(*)$ *coincide at* $(00,\bar{u},\bar{v})$.

Definition 14.2. *We say that a monic polynomial* $p(x,\bar{y})$ *is Weierstrass in* x *if* $p(x,\bar{y})=x^n+\ldots+q_j(\bar{y})x^{n-j}+\ldots+q_n(\bar{y})$ *with* $q_j(\bar{0})=0$.

Definition 14.3. *Let* $F(x,\bar{y})$ *be a polynomial in* x *with coefficients in* $L[\bar{y}]$. *We say the cover*

$$Spec(L[x\bar{y}]/<F(x,\bar{y})>)\rightarrow Spec(L[\bar{y}])$$

is generically reduced if for generic $\bar{u}\in Spec(L[\bar{y}])$, $F(x,\bar{u})$ *has no repeated roots.*

Definition 14.4. *Let* $F\rightarrow U\times V$ *be a finite cover with* U *and* V *smooth, such that for* $(\bar{u},\bar{v})\in U\times V$ *the fibre* $F(\bar{u},\bar{v})$ *consists of the intersection of algebraic curves* $F_{\bar{u}},F_{\bar{v}}$. *We call the family sufficiently deformable at* $(\bar{u}_0,\bar{v}_0)$ *if there exists* $\bar{u}'\in U$ *generic over* L *such that* $F_{\bar{u}'}$ *intersects* $F_{\bar{v}_0}$ *transversely at simple points.*

We now require a series of lemmas;

Lemma 14.5. *Let* $F(x,\bar{y})$ *be a Weierstrass polynomial in* x *with* $F(0,\bar{0})=0$ *then algebraic multiplicity and Zariski multiplicity coincide at* $(0,\bar{0})$ *if the cover*

$$Spec(L[x\bar{y}]/<F(x,\bar{y})>)\rightarrow Spec(L[\bar{y}])$$

is generically reduced.

Proof. We have that $F(x,\bar{y})=x^n+q_1(\bar{y})x^{n-1}+\ldots+q_n(\bar{y})$ where $q_i(\bar{0})=0$. The algebraic multiplicity is given by $length(L[x]/F(x,\bar{0}))=ord(F(x,\bar{0})=n$ in the ring $L[x]$ with the canonical valuation. We first claim that the Zariski multiplicity is the number of solutions to $x^n+q_1(\bar{\epsilon})x^{n-1}+\ldots+q_n(\bar{\epsilon})=0$ (†), where $\bar{\epsilon}$ is generic in $\mathcal{V}_{\bar{0}}$. For suppose that $(a,\bar{\epsilon})$ is such a solution, then $F(a,\bar{\epsilon})=0$ and by specialisation $F(\pi(a),\bar{0})=0$. As F is a Weierstrass polynomial in x, $\pi(a)=0$, hence $a\in\mathcal{V}_0$, giving the claim. We have that $Disc(F(x,\bar{y}))=Res_{\bar{y}}(F,\frac{\partial F}{\partial x})$ is a regular polynomial in $\bar{y}$ defined over L. By the fact that the cover is generically reduced, this defines a proper closed subset of $Spec(L[\bar{y}])$. Therefore, $Disc(F(x,\bar{y}))|\bar{\epsilon} \neq 0$, hence (†) has no repeated roots. This gives the lemma. □

Lemma 14.6. *Let $F(x,\bar{y})$ be any polynomial with $F(x,\bar{0}) \neq 0$ and $F(0,\bar{0}) = 0$. Then if the cover $Spec(L[x,\bar{y}]/ < F(x,\bar{y}) >) \rightarrow Spec(L[\bar{y}])$ is generically reduced, the Zariski multiplicity at $(0,\bar{0})$ equals $ord(F(x,\bar{0}))$ in $L[x]$.*

Proof. By the Weierstrass Preparation Theorem, Lemma 13.1, we can write $F(x,\bar{y}) = U(x,\bar{y})G(x,\bar{y})$ with $U(x,\bar{y}), G(x,\bar{y}) \in L[[x,\bar{y}]]$, $G(x,\bar{y})$ a Weierstrass polynomial in x and $deg(G) = ord(F(x,\bar{0}))$, see also the more closely related statement given in [2]. By Lemma 13.2, we may take the new coefficients to lie inside the Henselized ring $L[x,\bar{y}]^{\wedge}_{\bar{0}}$, hence inside some finite etale extension $L[x,\bar{y}]^{ext}$ of $L[x,\bar{y}]$ (possibly after localising $L[x,\bar{y}]$ corresponding to an open subset of $Spec(L[x,\bar{y}])$ containing $(0,\bar{0})$). Now we have the sequence of morphisms;

$$Sp(L[x,\bar{y}]^{ext}/UG) \rightarrow Spec(L[x,\bar{y}]/F) \rightarrow Spec(L[\bar{y}])$$

The left hand morphism is etale at $\bar{0}$, hence by Lemma 12.3 or Lemma 12.7, to compute the Zariski multiplicity of the right hand morphism, we need to compute the Zariski multiplicity of the cover

$$Spec(L[x,\bar{y}]^{ext}/UG) \rightarrow Spec(L[\bar{y}])$$

at $(0,\bar{0})^{lift}$, the marked point in the cover above $(0,\bar{0})$. Choose $\bar{\epsilon} \in \mathcal{V}_{\bar{0}}$, the fibre of the cover is given formally analytically by $L[[x,\bar{y}]]/ < UG > \otimes_{L[\bar{y}],\bar{y}\mapsto\bar{\epsilon}} L$, hence by solutions to $U(x,\bar{\epsilon})G(x,\epsilon)$. By definition of Zariski multiplicity, we consider only solutions $(x\bar{\epsilon})$ in $\mathcal{V}_{(0,\bar{0})^{lift}}$. As $U(x,\bar{y})$ is a unit in the local ring $L[x,\bar{y}]^{ext}_{(0,\bar{0})^{lift}}$, we must have $U(x,\bar{\epsilon}) \neq 0$ for such solutions, otherwise by specialisation $U((0,\bar{0})^{lift}) = 0$. Hence, the solutions are given by $G(x,\bar{\epsilon}) = 0$. Now, we use the previous lemma to give that the Zariski multiplicity is exactly $deg(G)$ as required.

□

Now return to the cover

$$Sp(L[x,y,u_{ij},v_{ij}]/ < s(u_{ij},x,y), t(v_{ij},x,y) >) \rightarrow Sp(L[u_{ij},v_{ij}]) \ (*)$$

We will show below, Lemma 14.12, that this is a sufficiently deformable family at $(\bar{u}_0, \bar{v}_0)$ when $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define reduced curves. We claim the following;

Lemma 14.7. *Suppose parameters $\bar{u}^0$ and $\bar{v}^0$ are chosen such that $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are reduced Weierstrass polynomials in x. Then the Zariski multiplicity of the cover $(*)$ at $(0,0,\bar{u}^0,\bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof. Introduce new parameters $\bar{u}'$ and $\bar{v}'$. Let $C_{\bar{u}^0}^{\bar{u}'}$ and $C_{\bar{v}^0}^{\bar{v}'}$ denote the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ deformed by the parameters $\bar{u}'$ and $\bar{v}'$ respectively. That is $C_{\bar{u}^0}^{\bar{u}'}$ is given by the new equation $\Sigma_{1\leq i+j\leq d}(u_{ij}^0 + u'_{ij})x^i y^j$. Let $F(y,\bar{u}',\bar{v}') = Res(C_{\bar{u}^0}^{\bar{u}'}, C_{\bar{v}^0}^{\bar{v}'})$. Then,

$$F(0,\bar{0},\bar{0}) = Res(s(u_{ij}^0, x, 0), t(v_{ij}^0, x, 0)) = 0$$

as $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass in x and share a common solution at $(0,0)$. By a result due to Abhyankar, see for example [1], $ord_y(F(y,\bar{0},\bar{0})) = \Sigma_x I(C_{\bar{u}^0}, C_{\bar{v}^0}, (x0))$ at common solutions $(x,0)$ to $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ over $y=0$. As $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass polynomials in x, this is just $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$. By the previous lemma and the fact that $F(y,\bar{u},\bar{v})$ is generically reduced (see argument (†) below), it is therefore sufficient to prove that the Zariski multiplicity of the cover $(*)$ at $(00,\bar{u}^0,\bar{v}^0)$ equals the Zariski multiplicity of the cover $Spec(L[y,\bar{u}',\bar{v}']/ < F(y,\bar{u}',\bar{v}') >) \rightarrow Spec(L[\bar{u}',\bar{v}'])$ $(**)$ at $(0,\bar{0},\bar{0})$. Suppose the Zariski multiplity of $(**)$ equals n. Then there exist distinct $y_1,\ldots,y_n \in \mathcal{V}_0$ and $(\bar{\delta},\bar{\epsilon})$ generic in $\mathcal{V}_{(\bar{0},\bar{0})} \cap U \times V$ such that $F(y_i,\bar{\delta},\bar{\epsilon})$ holds. Consider $Q(\bar{u}',\bar{v}') = Res(F(y,\bar{u}',\bar{v}'), \partial F/\partial y(y,\bar{u}',\bar{v}'))$. This defines a closed subset of $U \times V$ defined over L, we claim that this in fact proper closed (†). By the fact that the family is sufficently deformable at $(\bar{u}_0,\bar{v}_0)$, we can find $(\bar{u},\bar{v}_0)$ such that $C_{\bar{u}}$ intersects $C_{\bar{v}_0}$ transversely at simple points. Without loss of generality, making a linear change of coordinates, we may suppose that for there do not exists points of intersection of the form $(x_1 y)$ and $(x_2 y)$ for $x_1 \neq x_2$. By Abhyankar's result, this implies that $F(y,\bar{u}',\bar{v}_0)$ has no repeated roots. Then, by genericity of $(\bar{\delta},\bar{\epsilon})$, we have that $Q(\bar{\delta},\bar{\epsilon}) \neq 0$. Hence $F(y_i,\bar{\delta},\bar{\epsilon})$ is a non-repeated root. By Abhyankar's result, we can find a unique x_i with $(x_i y_i)$ a common solution to the deformed curves $C_{\bar{u}^0}^{\bar{\delta}}$ and $C_{\bar{v}^0}^{\bar{\epsilon}}$. We claim that each $(x_i y_i) \in \mathcal{V}_{00}$. As $C_{\bar{u}^0}^{\bar{\delta}}(x_i y_i) = 0$, by the fact $(\bar{u}^0,\bar{\delta},y_i)$ specialises to $(\bar{u}^0,\bar{0},0)$ and $C_{\bar{u}^0}$ is a Weierstrass polynomial in x, we have that $\pi(x_i) = 0$ as well. This shows that the Zariski multiplicity of the cover $(*)$ is at least n. A virtually identical argument shows that the Zariski multiplicity of the cover $(*)$ is at most n as well.

□

We now have the following result;

Lemma 14.8. *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be reduced curves, having finite intersection, then the Zariski multiplicity of the cover $(*)$ at $((0,0),\bar{u}^0,\bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof. We have $C_{\bar{u}^0} = s(u_{ij}^0,x,y)$ and $C_{\bar{v}^0} = t(v_{ij}^0,x,y)$. By making the substitutions $\bar{U} = \bar{u}^0 + \bar{u}$ and $\bar{V} = \bar{v}^0 + \bar{v}$, we may assume that $\bar{u}^0 = \bar{v}^0 = \bar{0}$. Moreover, we can suppose that;

$s(\bar{0}_{ij},x,0) \neq 0$ and $t(\bar{0}_{ij},x,0) \neq 0$. (**)

This can be achieved by making the invertible linear change of variables $(x' = x, y' = \lambda x + \mu y)$ with $(\lambda,\mu) \in L^2$ and $\mu \neq 0$, noting that as $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ are curves, for some choice of (λ,μ), the corresponding polynomials $s(u_{ij}^0,x,y)$ and $t(v_{ij}^0,x,y)$ do not vanish identically on the line $\lambda x + \mu y = 0$. It is trivial to check that the transformation preserves both Zariski multiplicity and intersection multiplicity, so our calculations are not effected.

We may then apply the Weierstrass preparation theorem, Lemma 13.1, in the ring $L[[u_{ij},v_{ij},x,y]]$, obtaining factorisations $s(u_{ij},x,y) = U_1(u_{ij},x,y)S(u_{ij},x,y)$ and $t(v_{ij},x,y) = U_2(v_{ij},x,y)T(v_{ij},x,y)$ where U_1 and U_2 are units in the local rings $L[[u_{ij},x,y]]$ and $L[[v_{ij},x,y]]$, S,T are Weierstrass polynomials in x with coefficients in $L[[u_{ij},y]]$ and $L[[v_{ij},y]]$ respectively. A close inspection of the Weierstrass preparation theorem, see [2], shows that we can obtain the following uniformity in the parameters $\bar{u}$ and $\bar{v}$;

Namely, if $U = \{u_{ij} : s(u_{ij},x,0) \neq 0\}$ and $V = \{v_{ij} : t(v_{ij},x,0) \neq 0\}$, are the constructible sets for which $(*)$ holds, then if we let R_U and R_V denote the coordinate rings of U and V, we may assume U_1,U_2 lie in $R_U[[x,y]]$ and the coefficients of S,T lie in $R_U[[y]]$ and $R_V[[y]]$ respectively. By Lemma 13.2, we may assume that U_1,U_2,S and T lie in a finite etale extension $R_{U\times V}[x,y]^{ext}$ of the algebra $A = R_{U\times V}[x,y]$ (again, possibly after localisation corresponding to an open subvariety of $Spec(A)$. Now we have the sequence of morphisms;

$$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S,U_2T>})\rightarrow Spec(\frac{R_{U\times V}[x,y]}{<s,t>})\rightarrow Spec(R_{U\times V}).$$

We claim that the left hand morphism is etale at the point $(\bar{0},\bar{0},(00)^{lift})$. This follows from the fact that $R_{U\times V}[x,y]^{ext}$ is an etale extension of $R_{U\times V}[x,y]$ and the maximal ideal given by $(\bar{0},\bar{0},(00)^{lift})$ contains $< U_1S, U_2T >$. Now consider the cover;

$$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S,U_2T>})\rightarrow Spec(R_{U\times V})\ (***)$$

For $\bar{u},\bar{v}$ in $U\times V$, the fibre of this cover over $\bar{u},\bar{v}$ corresponds exactly to the intersection of the reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ which lift the original curves $C_{\bar{u}}$ and $C_{\bar{v}}$ to an etale cover of $Spec(L[xy])$. By Theorem 11.2 and Lemma 12.3 (in the case when $C_{\bar{u}_0},C_{\bar{v}_0}$ intersect at simple points) or Lemma 12.7 (for singular points of intersection) and the corresponding Lemma 12.9 for reducible covers, it is sufficent to show that the Zariski multiplicity of the cover (***) at $(\bar{0},\bar{0},(00)^{lift})$ corresponds to the intersection multiplicity of the curves $C'_{\bar{u}_0},C'_{\bar{v}_0}$ at $(00)^{lift}$. The idea now is to apply Lemma 14.7 to the Weierstrass factors of $C'_{\bar{u}}$ and $C'_{\bar{v}}$. This will be achieved by the "unit removal" lemma below (Lemma 14.15).

□

In order to prove the "unit removal lemma", we first require some more definitions and a moving lemma for curves;

Definition 14.9. *Let $X\rightarrow Spec(L[x,y])$ be an etale cover in a neighboorhood of $(0,0)$, with distiguished point $(0,0)^{lift}$. We call a curve C on X passing through $(0,0)^{lift}$ Weierstrass if, in the power series ring $L[[x,y]]$, the defining equation of C may be written as a Weierstrass polynomial in x with coefficients in $L[[y]]$.*

Definition 14.10. *Let $F\rightarrow U\times V$ be a finite equidimensional cover of a smooth base of parameters $U\times V$ with a section $s:U\times V\rightarrow F$. We call the cover Weierstrass with units if the fibres $F(\bar{u},\bar{v})$ can be written as the intersection of reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ in an etale cover $A_{\bar{u},\bar{v}}$ of $U_{\bar{u},\bar{v}}\subset Spec(L[x,y])$ with the distinguished point $s(\bar{u},\bar{v})$ lying above $(0,0)$ and $C'_{\bar{u}},C'_{\bar{v}}$ factoring as $U_{\bar{u}}F_{\bar{u}}$ and $U_{\bar{v}}F_{\bar{v}}$ with $U_{\bar{u}},U_{\bar{v}}$ units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$ and $F_{\bar{u}},F_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.*

Let hypotheses on F, U and V be as above. We call the cover Weierstrass if the fibres $F(\bar{u},\bar{v})$ can be written as above but with $C'_{\bar{u}}, C'_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.

We say that a Weierstrass cover (with units) factors through the family of projective degree d and degree e curves if the cover $F \rightarrow U \times V$ factors as $F \rightarrow F' \rightarrow U \times V$ where $F' \rightarrow U \times V$ is the finite equidimensional cover obtained by intersecting the families Q_d and Q_e restricted to U and V.

Lemma 14.11. *The cover (***) in Lemma 14.8 is a Weierstrass cover with units factoring through the family of projective degree d and degree e curves.*

Proof. Clear by the above definitions.

□

Lemma 14.12. *Moving Lemma for Reduced Curves*

Let Q_d and Q_e be the families of all *projective degree d and degree e curves. That is, with the usual coordinate convention $x = X/Z, y = Y/Z$, Q_d consists of all curves of the form $s(\bar{u}, x, y) = \sum_{0\leq i+j\leq d} u_{ij}x^iy^j$. Then, if $\bar{u}, \bar{v}$ are chosen in L, so that the reduced curves $C_{\bar{u}}$ and $C_{\bar{v}}$ are defined over L, if the tuple $\bar{u}'$ is chosen to be generic in U over L, the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}$ transversely at simple points.*

Proof. We can give an explicit calculation;

Let $C_{\bar{u}}^{\bar{u}'}$ be defined by the equation $s(\bar{u}', x, y) = \sum_{0\leq i+j\leq d} u'_{ij}x^iy^j$ and $C_{\bar{v}}$ by $t(\bar{v}, x, y) = \sum_{0\leq i+j\leq e} v_{ij}x^iy^j$ with $\{v_{ij} : 0 \leq i + j \leq e\} \subset L$ and $\{u'_{ij} : 0 \leq i + j \leq d\}$ algebraically independent over L. Let (x_0y_0) be a point of intersection, then $dim(x_0y_0/L) = 1$, otherwise $dim(x_0y_0/L) = 0$ and, as L is algebraically closed, we must have that $x_0, y_0 \in L$. Substituting (x_0y_0) into the equation $s(\bar{u}', x, y) = 0$, we get a non trivial linear dependence over L between u'_{00} and u'_{ij} for $1 \leq i + j \leq d$ which is impossible. Now, the locus of singular points for $C_{\bar{v}}$ is defined over L and hence (x_0y_0) is a simple point of $C_{\bar{v}}$. Now we further claim that $s(\bar{u}', x, y) = 0$ defines a non-singular curve in $P^2(K_\omega)$ with transverse intersection to $C_{\bar{v}}$ Consider the conditions Sing$(\bar{u})$ given by $\exists x_0\exists y_0((\frac{\partial s}{\partial x}(x_0y_0), \frac{\partial s}{\partial y}(x_0y_0)) = (0, 0))$ and Non-Transverse$(\bar{u})$ by $\exists x_0\exists y_0(\frac{\partial s}{\partial x}(x_0y_0)\frac{\partial t}{\partial y}(x_0y_0) - \frac{\partial s}{\partial y}(x_0y_0)\frac{\partial t}{\partial x}(x_0y_0) = 0)$ By the properness of $P^2(K_\omega)$, these conditions define closed subsets of the parameter space U defined over L. We claim that this in fact a proper closed subset. This can be proved in a number of ways. In the case where we restrict ourselves to affine curves, the result follows from a classical result of Kleiman, see [9], as affine space $A^2(K_\omega)$ is homogenous for the action of the additive group $(A^2(K_\omega),+)$. More generally, we can use the moving lemma, given in [7], by observing that the class of all degree d projective curves is closed under rational equivalence. We can also give an explicit proof using Bertini's theorem;

Observe that the curve $C_{\bar{u}}$ defines a complete linear system $|C_{\bar{u}}|$ corresponding exactly to the zero loci of sections σ of the bundle $\mathcal{O}_{P^2}(d)$. We claim the following;

(i). The system $|C_{\bar{u}}|$ is base point free.

(ii). The system $|C_{\bar{u}}|$ separates points.

(i) is obvious. Just observe that if $F(x,y)$ defines a polynomial of degree d, then we can form a pencil of projective degree d curves $G(X,Y,Z)+\lambda Z^d$ such that $F(X/Z,Y/Z)=G(X,Y,Z)/Z^d$ where $x=X/Z, y=Y/Z$. Clearly, for $\lambda_1\neq\lambda_2$, the curves $F(x,y)=\lambda_1$ and $F(x,y)=\lambda_2$ are disjoint.

(ii) is also clear. Given distinct points p and q in $P^2(K)$, we can find a line l passing through p but not q. Then the d-fold line l^d belongs to the system $|C_{\bar{u}}|$.

Now we can define a morphism $\Phi_d : P^2(K)\rightarrow P^{d(d+3)/2}(K)$, by sending $x\in P^2$ to the hyperplane $H_x\subset U$ of curves of degree d, passing through x. By (i) and (ii), the restriction of Φ_d to $C_{\bar{v}}$ is injective. By arguments on Frobenius for curves, given in part 2, we can assume that Φ_d is an immersion. Using Bertini's Theorem, a generic hyperplane $\mathcal{H}_{\bar{u}'}$ of $P^{d(d+3)/2}(K)$ will intersect $Im(C_{\bar{v}})$ transversely in simple points. By definition of the morphism Φ_d, and the fact that it is an immersion, the corresponding curve $C_{\bar{u}}^{\bar{u}'}$ also intersects $C_{\bar{v}}$ transversely in simple points.

One can also give an enumerative calculation, which was done in an older version of this part of the paper, see [20], but it seems unnecessary.

□

Remarks 14.13. *If we restrict the family of curves, the result in general fails. A simple example is given by the family of all projective degree 3 curves* $Q_3^{0,0}$ *passing through* $(0,0)$ *with* $x = X/Z$ *and* $y = Y/Z$*. If we take* $C_{\bar{v}}$ *to be the cusp* $x^2 - y^3$*, then any curve in* $Q_3^{0,0}$ *will have a non-transverse intersection with* $C_{\bar{v}}$ *at the origin.*

Lemma 14.14. *Moving Lemma for Curves with Finitely Many Marked Points*

Let hypotheses be as in the previous lemma with $C_{\bar{u}}$ *and* $C_{\bar{v}}$ *defining reduced curves. Suppose also that there exists finitely many marked points* $\{p_1, \ldots, p_n\}$ *on* $C_{\bar{v}}$ *defined over* L*. Then for* $\bar{u}' \in U$ *generic over* L *the deformed curve* $C_{\bar{u}}^{\bar{u}'}$ *intersects* $C_{\bar{v}}$ *transversely at finitely many simple points excluding the set* $\{p_1, \ldots, p_n\}$*.*

Proof. As before, the condition that $\bar{u}'$ defines a curve $C_{\bar{u}}^{\bar{u}'}$ either with non-transverse intersection to $C_{\bar{v}}$ or passing through at least one of the points $\{p_1, \ldots, p_n\}$ is a closed subset of U defined over L. Using the above proof and the obvious fact that we can find a curve $C_{\bar{u}}^{\bar{u}'}$ not passing through any of the points $\{p_1, \ldots, p_n\}$, we see that it is proper closed.

□

Lemma 14.15. *Unit Removal for Reduced Curves*

Let $(\pi, s) : F \rightarrow U \times V$ *be a Weierstrass cover with units factoring through projective degree d and degree e curves. Let* $(\bar{u}, \bar{v}) \in U \times V$*, then there exists a Weierstrass cover* $(\pi', s') : F^{-} \rightarrow U' \times V'$ *with* $U' \subset U$ *and* $V' \subset V$ *open subsets,* $(\bar{u}, \bar{v}) \in U' \times V'$*, such that* $Mult_{(\bar{u},\bar{v},s(\bar{u},\bar{v}))}(F/U \times V) = Mult_{\bar{u},\bar{v},s'(\bar{u},\bar{v}))}(F^{-}/U' \times V')$*.*

Proof. Let $C'_{\bar{u}}$ and $C'_{\bar{v}}$ be the Weierstrass curves with units in $A_{\bar{u},\bar{v}}$ lifting the curves $C_{\bar{u}}$ and $C_{\bar{v}}$. Now suppose that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F/U \times V) = n$. Then we can find $(\bar{u}', \bar{v}') \in \mathcal{V}_{\bar{u}\bar{v}} \cap U \times V$ generic over L such that the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}^{\bar{v}'}$ at the n distinct points $x_1, \ldots, x_n$ in $\mathcal{V}_{s(\bar{u},\bar{v})}$. Now using the Weierstrass factorisations of $C_{\bar{u}}^{\bar{u}'}$ and $C_{\bar{v}}^{\bar{v}'}$, we claim that $U_{\bar{u}}^{\bar{u}'}(x_i) \neq 0$ and $U_{\bar{v}}^{\bar{v}'}(x_i) \neq 0$. Suppose not, then $U_{\bar{u}}^{\bar{u}'}(x_i) = U_{\bar{v}}^{\bar{v}'}(x_i) = 0$ and as $(\bar{u}', \bar{v}', x_i)$ specialises to $(\bar{u}, \bar{v}, s(\bar{u}, \bar{v}))$, then $U_{\bar{u}}(s(\bar{u}, \bar{v})) = U_{\bar{v}}(s(\bar{u}, \bar{v})) = 0$. This contradicts the fact that $U_{\bar{u}}$ and $U_{\bar{v}}$ are units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$. Therefore, we must have that $F_{\bar{u}}^{\bar{u}'}(x_i) = F_{\bar{v}}^{\bar{v}'}(x_i) = 0$. This shows that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^{-}/U \times V) \geq n$ where $F^{-} \rightarrow U \times V$ is the cover of $U \times V$ obtain by taking as fibres $F^{-}(\bar{u}, \bar{v})$ the intersection of the Weierstrass factors $F_{\bar{u}}$ and $F_{\bar{v}}$.

Formally, if F is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S,U_2T>})$ then F^{-} is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<S,T>})$. Clearly as $F^{-}\subset F$ is a union of components of F, we have that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^{-}/U\times V)\leq n$ as well. This proves the lemma.

□

We now complete the proof of Lemma 14.8. By unit removal, it is sufficient to compute the Zariski multiplicity of the cover

$$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<S,T>})\rightarrow Spec(R_{U\times V})$$

The fibre over $(\bar{u},\bar{v})$ of this cover corresponds exactly to the intersection of the Weierstrass curves $F_{\bar{u}}$ and $F_{\bar{v}}$ lifting $C_{\bar{u}}$ and $C_{\bar{v}}$. We then use Lemma 12.7, noting that the Weierstrass factors are still reduced, see [2], to finish the result, with the straightforward modification that we work in a uniform family of etale covers.

We now turn to the problem of non-reduced curves. We will show the following stronger version of Lemma 14.8

Lemma 14.16. *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be non-reduced curves having finite intersection, then the Zariski multiplicity of the cover (*) at $((0,0),\bar{u}^0,\bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

First, we will require some more lemmas.

Lemma 14.17. *Let $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ be reduced curves intersecting transversely at $(0,0)$. Then the Zariski multiplicity, left multiplicity and right multiplicity of the cover (*) at $((0,0),\bar{u}^0,\bar{v}^0)$ equals 1.*

Proof. First note that by Lemma 12.6 (and corresponding Lemma 12.9), and the fact that a generic deformation $C_{\bar{v}_0}^{\bar{v}'}$ will still intersect $C_{\bar{u}_0}$ transversely by Lemma 14.12, it is sufficient to prove the result for right multiplicity.

In order to show this we require the following result, given for analytic curves in [2], we will only need the result for polynomials;

Implicit Function Theorem:

If $G(X,Y)$ is a power series with $G(0,0)=0$ then $G_Y(0,0)\neq 0$ implies there exists a power series $\eta(X)$ with $\eta(0)=0$ such that

$G(X,\eta(X))=0$.

In order to show that $RightMult_{(0,0),\bar{u}^0,\bar{v}^0}(F'/U\times V)=1$, where F' is the family obtained by intersecting degree d and degree e curves, we apply the implicit function theorem to the curve $C_{\bar{u}^0}$ at the point $(0,0)$ of intersection with $C_{\bar{v}^0}$. Let $G(X,Y)$ and $H(X,Y)$ denote the polynomials defining the curves. We have that $G(0,0)=H(0,0)=0$. Moreover, as the first curve is non-singular at $(0,0)$, we may also assume that $G_Y(0,0)\neq 0$. Now let $\eta(X)$ be given by the theorem. As the intersection of the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ is transverse, $ord_X H(X,\eta(X))=1$.Now we have the sequence of maps;

$$L[\bar{v}]\rightarrow \frac{L[X,Y][\bar{v}]}{< G(u^0,X,Y),H(\bar{v},X,Y) >}\rightarrow \frac{L[X]^{ext}[Y][\bar{v}]}{< Y-\eta(X),H(\bar{v},X,Y) >}.$$

where $L[X]^{ext}$ is an etale extension of $L[X]$ containing $\eta(X)$. (Note that $\eta(X)$ is trivially algebraic over $L(X)$). This corresponds to a sequence of finite covers $F_1\rightarrow F'(u_0,V)\rightarrow Spec(L[\bar{v}])$. The left hand morphism is trivially etale at $(\bar{v}^0,(00)^{lift})$, hence it is sufficient to compute the Zariski multiplicity of $F'\rightarrow Spec(L[\bar{v}])$ at $(\bar{v}^0,(00)^{lift})$ by Lemma 12.3 (or corresponding Lemma 12.9). This is a straightforward calculation, the fibre over $\bar{v}^0$ consists of the scheme $Spec(\frac{L[X,\eta(X)]}{G(X,\eta(X))})=Spec(L)$ as $ord_X(H(X,\eta(X)))=1$, hence is etale at the point $(\bar{v}^0,(00)^{lift})$. By Theorem 11.1, the Zariski multiplicity is 1.

□

Lemma 14.18. *Let hypotheses be as in Lemma 14.17, then for any* $(\bar{u}',\bar{v}')\in\mathcal{V}_{(\bar{u}^0,\bar{v}^0)}$*, we have that* $Card(F'(\bar{u}',\bar{v}')\cap\mathcal{V}_{0,0})=1$

Proof. This follows immediately from Lemma 14.17 and Lemma 12.4. □

Definition 14.19. *For ease of notation, given curves* $C_{\bar{u}}$ *and* $C_{\bar{v}}$ *of degree* d *and degree* e *intersecting at* $x\in P^2(K_\omega)$*, we define* $Mult_x(C_{\bar{u}},C_{\bar{v}})$ *to be the corresponding Zariski multiplicity of the cover* $F'\rightarrow U\times V$ *at the point* $(x,\bar{u},\bar{v})$*. Similarly for left/right multiplicity.*

We can now give the proof of Lemma 14.16;

Proof. Case 1. $C_{\bar{v}_0}$ is a reduced curve (possibly having components). Write $C_{\bar{u}^0}$ as $G_1^{n_1}(X,Y)\ldots G_m^{n_m}(X,Y)=0$ with G_i the reduced irreducible components of $C_{\bar{u}_0}$ with degree d_i passing through $(0,0)$. Choose $\bar{\epsilon}_1^1,\ldots\bar{\epsilon}_1^{n_1},\ldots\bar{\epsilon}_i^j,\ldots\bar{\epsilon}_m^1,\ldots,\bar{\epsilon}_m^{n_m}$ independent generic in U_i, the parameter space for degree d_i projective curves with $\bar{\epsilon}_i^j \in \mathcal{V}_{\bar{u}_i^0}$, where $\bar{u}_i^0$ defines G_i. By repeated application of Lemma 14.14, the deformed curves $G_i^{\bar{\epsilon}_i^j}=0$ intersect $C_{\bar{v}_0}$ transversely at disjoint sets of points We denote by $Z_{\bar{\epsilon}_i^j}$ those points lying in $\mathcal{V}_{00}$. Now the curve defined by $\prod_{ij} G_i^{\bar{\epsilon}_i^j}=0$ is a deformation $C_{\bar{u}^0}^{\bar{\epsilon}}$ of $C_{\bar{u}^0}$. We let $Z_{\bar{\epsilon}}$ denote the points of intersection of $C_{\bar{u}^0}^{\bar{\epsilon}}$ with $C_{\bar{v}_0}$ in $\mathcal{V}_{00}$. Then we have;

$$Z_{\bar{\epsilon}} = \bigcup_{ij} Z_{\bar{\epsilon}_i^j}$$

$$Card(Z_{\bar{\epsilon}}) = \sum_{ij} Card(Z_{\bar{\epsilon}_i^j})$$

By Lemma 12.4, we have that

$$\begin{aligned} LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) &= \sum_{x\in Z_{\bar{\epsilon}}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0}) \\ &= \sum_{i,j}\sum_{x\in Z_{\bar{\epsilon}_i^j}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0}) \ (*) \end{aligned}$$

We now claim that for a point $x \in Z_{\bar{\epsilon}_i^j}$,

$$LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0}) = LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}_0}) \ (**)$$

This follows as both the reduced curves $C_{\bar{u}_0}^{\bar{\epsilon}}$ and $G_i^{\bar{\epsilon}_i^j}$ intersect $C_{\bar{v}_0}$ transversely at x. Hence, in both cases the left multiplicity is 1, by Lemma 14.17.

Combining $(*)$ and $(**)$, we obtain;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i,j}\sum_{x\in Z_{\bar{\epsilon}_i^j}} LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}^0})$$

Now using Lemma 12.4 again gives that;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^m n_i LeftMult_{(00)}(G_i, C_{\bar{v}^0}) \ (***)$$

If we go through exactly the same calculation with Mult replacing Left Mult, we see as well that

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^m n_i Mult_{(00)}(G_i, C_{\bar{v}^0})$$

By Lemma 14.8, this gives

$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i I(G_i, C_{\bar{v}^0}, (00))$

By a straightforward algebraic calculation, see the references below at the end of the proof for the required more general result, this gives

$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$

as required.

Case 2. Both $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define non-reduced curves. Write $C_{\bar{u}_0}$ as above and $C_{\bar{v}_0}$ as $H_1^{e_1} \ldots H_n^{e_n}$ with H_i the reduced compoments with degree c_i of $C_{\bar{v}_0}$ passing through (00). Then $H_1 \ldots H_n = 0$ defines a reduced curve passing through (00). Now repeat the argument in Case 1 for the curves $C_{\bar{u}_0}$ and $H_1 \ldots H_n = 0$. Again let $Z_{\bar{\epsilon}}$ be the intersection points of the deformed curve $C^{\epsilon}_{\bar{u}_0}$ with $H_1 \ldots H_n = 0$ in $\mathcal{V}_{(00)}$. By (***) of Case 1, Lemma 12.4 and Lemma 14.18 with the fact that the intersection of $C^{\bar{\epsilon}}_{\bar{u}^0}$ with $H_1 \ldots H_n$ is transverse, we have;

$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i Mult_{(00)}(G_i, H_1 \ldots H_n)$

Now using the argument in Case 1 applied to the reduced curves G_i and $H_1 \ldots H_n$, we have;

$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i \sum_{j=1}^{n} I(G_i, H_j, (00))$ (*)

We claim that for any component H_j

$Card(H_j \cap Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i I(G_i, H_j, (00))$

This follows as the deformed curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ a fortiori intersects H_j transversely at simple points. Therefore, again by Case 1, gives the expected multiplicity. Now, using this together with (*), we write $Z_{\bar{\epsilon}}$ as $\cup_j Z^j_{\bar{\epsilon}}$ where $Z^j_{\bar{\epsilon}}$ are the disjoint sets consisting of the intersection of $C^{\bar{\epsilon}}_{\bar{u}_0}$ with H_j. Then by Lemma 12.6, we have that

$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_j \sum_{x \in Z^j_{\bar{\epsilon}}} RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0})$

We can now calculate the Right Mult term by applying Case 1 to the intersection of $C_{\bar{v}_0}$ with the reduced curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ at the points of intersection $x \in Z^j_{\bar{\epsilon}}$. At a point $x \in Z^j_{\bar{\epsilon}}$, we have that

$RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0}) = e_j I(C^{\bar{\epsilon}}_{\bar{u}^0}, H_j, x) = e_j$

as the intersection is transverse. Finally this gives;

$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} \sum_{j=1}^{n} n_i e_j I(G_i, H_j, (00))$

By an algebraic result, see [11] for the case of complex algebraic curves, or [6] for its generalisation to algebraic curves in arbitrary characteristics, we have

$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$

as required.

□

The following version of Bezout's theorem in all characteristics is now an easy generalisation from the above lemma. For curves C_1 and C_2 in $P^2(L)$, we let $M(C_1, C_2, x)$ denote the intersection multiplicity or the Zariski multiplicity, we know from the above that the two are equivalent. One can still prove a general Bezout's theorem using general Zariski structure methods, as is done in, for example, [28]; this is required for other reasons, namely to prove the purity of the field structure for arbitrary Zariski geometries. However, should the intersection multiplicities fail to coincide in the case of algebraically closed fields, this would effect geometric calculations performed using a non-standard approach, see the next part of the paper. One might think of the Lefschetz trace formula as an example where the "correct" calculation of intersection multiplicity is required. Of course, historically, Bezout's theorem was originally proved using Poncelet's principle of continuity, relying on an infinitesimal definition of multiplicity; this served as a foundation for the work of Severi and others.

Theorem 14.20. *Zariski Structure Formulation of Bezout's Theorem*

Let C_1 and C_2 be projective curves of degree d and degree e in $P^2(L)$, possibly with non-reduced components, intersecting at finitely many points $\{x_1, \ldots, x_i, \ldots x_n\}$, then we have;

$\sum_{i=1}^{n} M(C_1, C_2, x_i) = de.$

Of course, we could just quote the algebraic result given in [9] (though this in fact only holds for reduced curves). Instead we can give a more geometric proof, which in many ways is conceptually simpler and doesn't involve any algebra;

Proof. Let Q_d and Q_e be the families of all projective degree d and degree e curves. Then we have the cover $F \rightarrow U \times V$ with $F \subset U \times V \times P^2(L)$ obtained by intersecting the families Q_d and Q_e. We have that

$\sum_{i=1}^{n} M(C_1, C_2, x_i) = \sum_{i=1}^{n} Mult_{x_i \in F(\bar{u}_0, \bar{v}_0)}(F/U \times V)$

where $(\bar{u}_0, \bar{v}_0)$ define C_1 and C_2. By Lemma 8.3, this equals

$\sum_{x \in F(\bar{u}, \bar{v})} Mult_{x, \bar{u}, \bar{v}}(F/U \times V)$

where $(\bar{u}, \bar{v})$ is generic in $U \times V$. Using, for example, the proof of Lemma 14.12, generically independent curves $C_{\bar{u}}$ and $C_{\bar{v}}$ intersect transversely at a finite number of simple points. Hence, by Lemma 14.17, the Zariski multiplicity calculated at these points is 1. As the cover F has degree de, there is a total number de of these points as required.

□

15. A First Order Definition of Intersection Multiplicity and Bezout's Theorem

We now formulate a definition of intersection multiplicity in the language $\mathcal{L}_{spec}$, which we introduced in Part 1, Section 2 We will do this only in the case of projective curves inside $P^2(L)$, the reader may wish to try formulating a corresponding definition in higher dimensions.

Let C_1 and C_2 be projective curves of degree d and degree e in $P^2(K)$ defined over L. The parameter spaces for such curves are affine spaces of dimension $(d+1)(d+2)/2$ and $(e+1)(e+2)/2$ respectively. We can give them a projective realisation by noting that if $(\bar{l})$ is a non-zero vector defining a curve of degree d, then multiplying it by a constant μ defines the same curve. Let $P^{d(d+3)/2}(K)$ and $P^{e(e+3)/2}(K)$ define these spaces which we will denote by P_d and P_e for ease of notation. Let $Curve_d$ and $Curve_e$ be the closed projective subvarieties of $P_d \times P^2(K)$ and $P_e \times P^2(K)$, defined over the prime subfield of L, such that, for

$l \in P_d$, the fibre $Curve_d(l)$ defines the corresponding projective curve of degree d in $P^2(K)$. For l in $P^n(L)$, we denote its infinitesimal neighborhood $\mathcal{V}_l$ to be the inverse image under the specialisation π_n, see the introduction to Section 2.

Now suppose that C_1 and C_2 (which may not be reduced or irreducible), of degrees d and e respectively, are defined by parameters l_1 and l_2 and intersect at an isolated point l in $P^2(L)$. Then we define;

$Mult(C_1, C_2, l) \geq n$

iff

$\exists x_1, x_2 \in \mathcal{V}_{l_1}, \mathcal{V}_{l_2}, \exists_{y_1 \neq \ldots \neq y_n} \in \mathcal{V}_l(\{y_1, \ldots, y_n\} \subset Curve_d(x_1) \cap Curve_e(x_2))$

Then define $Mult(C_1, C_2, l) = n$

iff

$Mult(C_1, C_2, l) \geq n$ and $\neg Mult(C_1, C_2, l) \geq n + 1$.

Clearly, the statement that $Mult(C_1, C_2, l) = n$ naturally defines a sentence in the language $\mathcal{L}_{spec}$. One consequence of the completeness result given in Theorem 2.1 is that the statement "The curves C_1 and C_2 intersect with multiplicity n at l" depends only on the theory T_{axioms} and is independent of the particular structure $(P(K), P(L), \pi)$. We have shown that the non-standard definition of multiplicity is equivalent to the algebraic definition of multiplicity when computed in the structure $(P(K_{univ}), P(L), \pi_{univ})$, recall from Section 3. It therefore follows that the non-standard definition of multiplicity is equivalent to the algebraic definition even when computed in a prime model of T_{axioms} which I will denote by $(P(K_{prime}), P(L), \pi_{prime})$.

We now turn to the statement of Bezout's theorem. In algebraic language, this says that if projective algebraic curves C_1 and C_2 of degree d and degree e in $P^2(L)$ intersect at finitely many points $\{l_1, \ldots, l_n\}$, then;

$\sum_{i=1}^{n} I(C_1, C_2, l_i) = de$

where $I(C_1, C_2, l_i)$ is the algebraic intersection multiplicity. This result can be formulated in the language $\mathcal{L}_{spec}$ by the sentence;

$$Bezout(C_1, C_2) \equiv \exists_{m_1,\ldots,m_n;m_1+\ldots+m_n=de}(\bigwedge_{i=1}^{n} Mult(C_1, C_2, l_i) = m_i)$$

We proved the *algebraic* version of Bezout's theorem by Zariski structure methods in the structure $(P(K_{univ}), P(L), \pi_{univ})$. It follows that the sentences $Bezout(C_1, C_2)$ are all proved by the theory T_{axioms} and therefore hold in the structure $(P(K_{prime}), P(L), \pi_{prime})$ as well. This demonstrates the fact that we can prove an algebraic statement of Bezout's theorem using only infinitesimals from a straightforward extension of L, namely $L(\epsilon)^{alg}$, in particular in a structure such that the infinitesimal neighborhoods $\mathcal{V}_l$ are all recursively enumerable. This seems to provide some answer to a general objection concerning the use of infinitesimals, originating in [4]. It may also provide an effective alternative method to compute intersection multiplicities generally in algebraic geometry.

Part 4

In the final part of this paper, we consider Robinson's theory of non-standard analysis, as developed in [24]. We successfully reinterpret the results, obtained above for algebraic curves using a Zariski geometry approach, in terms of Robinson's theory of enlargements and monads. We are able to derive standard analogues of classical results for algebraic curves, by transfer.

16. Robinson's Non-Standard Analysis

The idea of non-standard analysis is to introduce an enlargement of a given structure of mathematical interest, to include infinitesimal elements. The simplest example is that of the ordered real field $< \mathcal{R}, +, ., <, 0, 1 >$, considered as a first order structure. We consider the theory;

$$T = Th(\mathcal{R}, R) \cup \{0 < \epsilon < r : r \in R_{>0}\}$$

where we have added a set of constants R for the individuals of the field, and a new constant ϵ. Any finite set of sentences from this theory is consistent, realised by the structure $< \mathcal{R}, +, ., 0, 1 >$ itself. Namely, given any finite set of constants $\{r_1, \ldots, r_n\}$, we can choose ϵ to be a real number between 0 and $min\{r_1, \ldots, r_n\}$. Such a real number exists by the completeness property of the reals. However, observe that by the same property the structure $<\mathcal{R},+,.,<,0,1>$ does not realise the full set of sentences T. However, by compactness, the theory T is consistent and has a model $<{}^*\mathcal{R},+,.,<,0,1>$, which is an elementary extension of $<\mathcal{R},+,.,<,0,1>$, that is $\mathcal{R}\prec{}^*\mathcal{R}$. In particular ${}^*R$ is also an ordered field and contains an infinitely small element ϵ, with respect to the elements of $\mathcal{R}$. In fact, it contains infinitely many infinitely small elements, for example consider the sequence $\{\epsilon,\epsilon^2,\epsilon^3,\ldots\}$. We refer to the set of infinitesimals in ${}^*\mathcal{R}$ as the monad of 0. Every element $r\in\mathcal{R}$ has a monad, consisting of the elements in ${}^*\mathcal{R}$, infinitely close to r. This simple construction is the basis for Robinson's work in [24], in which he constructs an enlargement for *any* structure, and, for structures with a topology, the notion of a monad.

We have, of course, encountered the notion of an infinitesimal, in our considerations of algebraically closed fields, see, for example the last section. However, we have to proceed more carefully in how we define such objects, as there is no total ordering available to us, as in the case of the reals.

I wish to consider briefly Robinson's theory of higher order structures. This theory is not strictly necessary for the construction of enlargements in cases of interest, such as the reals, and, certainly, in the context of this paper, we can work entirely within first order logic, in order to obtain the results we need. However, the construction *is* necessary in the context of other fields of mathematics, for example in probability and stochastic processes. It is the author's hope that, at some point in the future, a non-standard approach will be able to foster links between subjects as diverse as algebraic geometry and probability theory, so it seems a good idea to at introduce this increased level of generality. Robinson begins by defining the notion of a type;

Definition 16.1. *A type is constructed inductively by requiring that;*

(i). 0 *is a type.*

(ii). If n *is a positive integer,* $\{\tau_1,\ldots,\tau_n\}$ *are types, then* $(\tau_1,\ldots,\tau_n)$ *is a type.*

We denote the set of types by T*.*

He then defines higher order relations on a set of individuals A;

Definition 16.2. *Higher Order Relations*

Given a set of individuals A*, and a type* $\tau = (\tau_1, \ldots, \tau_n)$*, we define the set of higher relations of type* τ*,* A_τ*, inductively, by requiring that;*

$A_0 = A$

A_τ *consists of subsets of* $A_{\tau_1} \times \ldots \times A_{\tau_n}$.

The definition of a higher order structure is as follows;

Definition 16.3. *A higher order structure* M*, based on* A*, is a set of higher order relations* $\{B_\tau\}$*, indexed by types* τ*, such that;*

(i). For each type τ*,* $B_\tau \subseteq A_\tau$*, (possibly with repetitions)*

(ii). $B_0 = A_0 = A$.

(iii). If $R \in B_\tau$ *and* $R(R_1, \ldots, R_n)$*, then* $R_i \in B_{\tau_i}$*, for* $1 \leq i \leq n$.

Remarks 16.4. *The notion of repetitions is mainly a technical convenience to prove the existence of enlargements. Robinson defines a higher order structure to be normal, if there are no repetitions in* (i)*, and proves that an enlargement can eventually be chosen to be normal. He also defines a structure to be full if each* B_τ *contains all the relations of* A_τ*. It is very important to note that, in general, an enlargement will* not *be full.*

To proceed further, we need the notions of a higher order language and constants;

Definition 16.5. *A higher order language* Λ *consists of an* $(n+1)$*-ary relation symbol* ϕ_τ*, for each type* $\tau = (\tau_1, \ldots, \tau_n) \neq 0$*, and a set of (typed) constants denoting individuals or higher order relations.*

Definition 16.6. *We define an interpretation of* Λ *by* M *to be a map from the constants of* Λ onto *the higher order relations and individuals of* M.

We define a sentence of the language Λ in the usual way. Robinson gives the following inductive definition on the length of a sentence;

Definition 16.7. *Satisfaction of a sentence from Λ in a Structure M, given an interpretation*

Suppose that we are given an interpretation of the constants appearing in Λ.

The sentence σ from Λ is true in M, if;

(i). Base Case. σ is atomic, of the form $\phi_\tau(a, b_1, \ldots, b_n)$, with a mapping to a relation R, $(b_1, \ldots, b_n)$ mapping to relations $(R_{\tau_1}, \ldots, R_{\tau_n})$ such that $R(R_{\tau_1}, \ldots, R_{\tau_n})$ holds in M.

(ii). σ is of the form, $\sigma_1 \wedge \sigma_2$, then σ is true in M iff σ_1 and σ_2 are true in M. The cases involving $\vee$ and $\neg$ are handled similarly.

(iii). σ is of the form $\exists y Z$; if y doesn't occur in Z, σ is true iff Z is true, if y does occur in Z, σ is true iff there exists a constant a mapping to a relation R of M, such that $Z(a)$ holds in M.

(iv). σ is of the form $\forall y Z$; if y doesn't occur in Z, σ is true iff Z is true, if y does occur in Z, σ is true iff for all constants a mapping to a relation R of M, $Z(a)$ holds in M

Remarks 16.8. *The truth value of σ in M depends only on the interpretation of the constants appearing in σ. It is easy to check that if σ is a sentence of two languages Λ_1 and Λ_2, and we have 2 interpretations of these languages in M,* extending *the partial interpretation of the constants appearing in σ, then, σ is true in M, given the interpretation of Λ_1 iff σ is true in M, given the interpretation of Λ_2. The same is true for an infinite set of sentences.*

Definition 16.9. *A set of sentences K in a language Λ, is consistent for higher order logic, if, for some interpretation of Λ, and some higher order structure M, all sentences in K hold in M.*

We assume the reader is familiar with first order logic. We make the following definitions;

Definition 16.10. *We define the first order fragment* M_1 *of a higher order structure* M *to be the restriction of its higher order relations to types of the form* $\tau = (0, \ldots, 0)$ *or* $\tau = 0$. *We define the first order fragment* Λ_1 *of a higher order language* Λ *to be the restriction of the relation symbols* ϕ_τ *to types of the form* $\tau = (0, \ldots, 0)$ *and the restriction of constants to individuals and relations of type* $(0, \ldots, 0)$. *We call a sentence of* Λ *first order, if it is a sentence of* Λ_1*, in which the quantifiers are restricted to being of type* $\tau = 0$. *Given an interpretation by* M *of the constants in* Λ*, we call the restriction to an interpretation by* M_1 *of the constants in* Λ_1*, a first order interpretation.*

We then have that;

Lemma 16.11. *Let* σ *be a first order sentence of* Λ*, and let* M_1 *be the first order fragment of a higher order structure* M. *Suppose that we are given an interpretation in* M *of the constants in* Λ*, with the corresponding first order interpretation in* M_1. *Then;*

$M \models \sigma$ *iff* $M_1 \models \sigma$

Proof. The proof is an easy induction on the length of the sentence. The important point to note is that the set of individuals of M constitute the full domain of the first order structure M_1.

□

Remarks 16.12. *In general, the set of higher order relations is a proper subset of all possible relations on the set of individuals. Such relations are called* internal.

Robinson proves the following property of sentences from a language;

Theorem 16.13. *Compactness*

Let K *be a a set of sentences in a higher order language* Λ*, then* K *is consistent iff every finite subset of* K *is consistent.*

Proof. The idea is to interpret sentences from Λ as sentences in a first order language and to invoke the compactness theorem from first order logic. The reader is referred to [24] □

In order to define enlargements, we recall the following construction of Robinson;

Let K be a set of sentences, and let Γ be the constants occurring in K. Let $b \in \Gamma$ of type $\tau = (\tau_1, \tau_2)$. Let;

$$\Delta_b = \{g \in \Gamma : K \vdash \exists y \phi_\tau(b, g, y)\}$$

be the domain of b. We say that b is concurrent if for every finite subset $\{g_1, \ldots, g_n\}$ of Δ_b;

$$K \vdash \exists y(\phi_\tau(b, g_1, y) \wedge \ldots \wedge \phi_\tau(b, g_n, y)) \ (*)$$

We let Γ_0 denote the set of concurrent elements of Γ. For $b \in \Gamma_0$, choose a new constant a_b and, for $b \in \Gamma_0$, let;

$$K_b = \{\phi_\tau(b, g, a_b : g \in \Delta_b\}$$

Let $K_0 = \bigcup_{b \in \Gamma_0} K_b$ and $H = K \cup K_0$. Then;

Definition 16.14. *H is an enlargement of K.*

Moreover, we have the following;

Lemma 16.15. *If K is consistent, then its enlargement H is consistent.*

Proof. The proof uses compactness, Theorem 16.13. It is easy to give a direct proof when K is a set of first order sentences, which will be our primary concern, later in the paper. In this case, let H' be a finite subset of H. Then we can find finitely many elements $\{b_1, \ldots, b_n\}$, denoting binary relations, individuals $\{g_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$, and sentences $\{\sigma_1, \ldots \sigma_m\}$ from K such that H' consists of;

$$\bigcup_{1 \leq i \leq n, 1 \leq j \leq k} R_{b_i}(g_{ij}, a_{b_i}) \cup \{\sigma_1, \ldots, \sigma_m\} \ (**)$$

Now, by concurrency of each b_i, we can take a model M of K and interpret the new constants a_{b_i} by individuals c_i of K, witnessing $(*)$. Clearly, then, $(**)$ holds in M. By compactness, H is consistent. □

Now in the case that M is a higher order structure, which we usually assume to be full and normal, we let K be the set of sentences in a language Λ naming all the relations and individuals of M. By the previous theorem, the enlargement of K is consistent.

Definition 16.16. *We denote a model of the enlargement of K by ${}^*M$ and call it an enlargement of M.*

Robinson proves the following;

Theorem 16.17. *Transfer*

There exists an embedding f of M into ${}^*M$, such that for all sentences σ in $\mathcal{L}_M$, the language obtained by adding constants for all the elements and relations of M, we have;

$M \models \sigma$ *iff* ${}^*M \models \sigma$ $(***)$

One can choose an enlargement ${}^*M$ to be normal.

Proof. We define the embedding f by sending an individual m of M to the element of ${}^*M$ interpreting the constant representing m in Λ, similarly for the higher order relations. If $\{r_1, r_2\}$ are higher order relations with $r_1 \neq r_2$, then by normality of M, $\exists x(r_1(x) \wedge \neg r_2(x))$ holds. As this is a sentence from K, it holds in ${}^*M$, with the interpretations $f(r_1)$ and $f(r_2)$, clearly this implies that $f(r_1)$ and $f(r_2)$ are distinct. The claim $(***)$ follows obviously from the fact that ${}^*M$ is a model of K, consisting of all sentences true in M, with constants naming individuals and relations of M. We refer the reader to [24] for the final claim.

□

Remarks 16.18. *One should be careful in interpreting the transfer theorem. For example, it is true in M, that "there exists a set consisting of the non-empty subsets of the individuals of M";*

$$\exists r \forall s(\phi_{((0))}(r; s) \leftrightarrow \exists x \phi_{(0)}(s; x))$$

By transfer, this is true in ${}^*M$, hence, one might be tempted to conclude that every non-empty subset of ${}^*M$ is internal. This is false, in general, a counterexample being a monad in an enlargement ${}^*\mathcal{R}$ of $\mathcal{R}$. We resolve the contradiction, by observing that the universal quantifier refers to internal subsets of ${}^*M$, not to the collection of all subsets.

Once we have defined enlargements, it is a short step to develop a general theory of monads for topological spaces. We let $T = (A, \Omega)$ be a topological space, that is a set A, with a collection of open sets Ω satisfying the following axioms;

(i). $\{\emptyset, A\} \subset \Omega$.

(ii). If $\{a,b\} \subset \Omega$, then the intersection $a \cap b \in \Omega$

(iii). If I is an index set and $\{a_i : i \in I\} \subset \Omega$, then $\bigcup_{i\in I} a_i \in \Omega$.

We consider T as a full,normal structure in which the open sets Ω are represented by a higher order relation of type $\tau = ((0))$. We let ${}^*T$ be an enlargement of T, and ${}^*\Omega$ the collection of open sets in ${}^*T$. Observe that ${}^*\Omega$ does not necessarily form a topology, but that, by transfer, if $\Phi \subset {}^*\Omega$ is internal, then $\bigcup \Phi \in {}^*\Omega$.

Definition 16.19. *Let $p \in T$, and let $\Omega_p \subset \Omega$ consist of the open sets containing p, then we define the monad;*

$\mu(p) = \bigcap_{U_\nu \in \Omega_p} {}^*U_\nu$

We observe;

Lemma 16.20. *Let $p \in T$, and suppose that $\{p\} \notin \Omega_p$, then $(\mu(p) \setminus \{p\})$ is non-empty.*

Proof. Consider the relation $R \subset \Omega_p \times A$ given by;

$R(U_\nu, q)$ iff $p \neq q$ and $q \in U_\nu$

Then R is concurrent, by property (ii) of the definition of a topology, and the fact that $\{p\} \notin \Omega_p$. Hence, by the construction of an enlargement, there exists a ${}^*p \neq p$, in ${}^*T$ belonging to all open sets $U_\nu \in \Omega_p$. □

An element of a monad $\mu(p) \setminus \{p\}$ is infinitesimally close to p, in the sense of being indistinguishable from p in the topology of T. Provided that p is not discrete for the topology, such elements exist. In particular, we can see the construction of monads as generalising the construction given at the beginning of this section, by taking the topology on $\mathcal{R}$ induced by the Euclidean metric.

We require the following further properties of monads;

Lemma 16.21. *If T is Haussdorf, then for $p \neq q$, $(\mu(p) \cap \mu(q)) = \emptyset$*

Proof. Given $p \neq q$ in T, we can find open sets $U_1 \in \Omega_p$ and $U_2 \in \Omega_q$ with $U_1 \cap U_2 = \emptyset$. As $\mu(p) \subset U_1$ and $\mu(q) \subset U_2$, we have $(\mu(p) \cap \mu(q)) = \emptyset$ as well. □

Definition 16.22. *We call a point of* ${}^*T$ *near standard if it lies in the monad of some* $p \in T$.

Lemma 16.23. *If* T *is compact, then every point of* ${}^*T$ *is near standard.*

Proof. Suppose there exists $q \in^* T$ which is not near standard. Then for all $p \in T$, there exists an open set U_p containing p, with $q \notin^* U_p$. Let $\{U_p : p \in T\}$ be the collection of all such sets, then $\bigcup_{p \in T} U_p = A$. Hence, by compactness, there exist $\{p_1, \ldots, p_n\}$ with $A = U_{p_1} \cup \ldots \cup U_{p_n}$. As this is expressible by a sentence in $\mathcal{L}_T$, we have, by transfer, that ${}^*A =^* U_{p_1} \cup \ldots \cup^* U_{p_1}$. This implies that $q \in^* U_{p_j}$ for some j, a contradiction.

□

We consider the Cartesian products T^n as equipped with the product topology. We call a subset of T^n closed if its complement is open.

Lemma 16.24. *If* T *is compact and Haussdorf, then we can define a standard part map* $\pi : {}^*T \rightarrow T$, *with the property that, for any closed subset* C *of* T^n, *and tuple* $\bar{a} \in T^n$, *if* ${}^*C(\bar{a})$ *then* $C(\pi(\bar{a}))$.

Proof. As T is compact, every point $q \in^* T$ is near standard, and, as T is Haussdorf, q belongs to a unique monad $\mu(p)$ for some $p \in T$. We define the map π by sending q to p. The products $\{U_{p_1} \times \ldots \times U_{p_n} : U_{p_i} \in \Omega_{p_i}\}$ form a basis for $\Omega_{(p_1, \ldots, p_n)}$, hence, if $\pi(\bar{q}) = \bar{p}$, then $\bar{q} \in \mu(\bar{p})$, the monad defined for the topology on T^n. Now suppose that C is a closed subset of T^n with ${}^*C(\bar{q})$, and that $\bar{p} \notin C$, then $\bar{p} \in U$, the complement of C, which is an open subset of T^n. By definition of a monad, $\bar{q} \in^* U$, but ${}^*U$ is the complement of ${}^*C$, using transfer. This is a contradiction.

□

17. Universal Specialisations and Enlargements

We refer the reader to the results of Part 1 of this paper. The aim of this section is to construct an enlargement which is simultaneously a universal specialisation. In such a structure, monads coincide with infinitesimal neighborhoods, hence we can rephrase our results from

Parts 2 and 3, in a non-standard context.

It is clear from the above that an enlargement of the reals ${}^*\mathcal{R}$ contains infinitesimal elements. It is also clear, working within first order logic, that any $\aleph_2$-saturated model of $Th(\mathcal{R}, R)$ will also have this property, see the construction at the beginning of this section. For our purposes, we will work in a first order structure;

$$< \mathcal{C}, +_{\mathcal{C}}, \cdot_{\mathcal{C}}, d, 0, 1, \mathcal{R}, +_{\mathcal{R}}, \cdot_{\mathcal{R}}, <, 0, 1 >$$

where we have 2 sorts for the complex numbers and for the reals. As before, we consider $\mathcal{R}$ as an ordered field. We equip $\mathcal{C}$ with the Euclidean metric d. This define the complex topology on $\mathcal{C}$. We abbreviate this structure by $< \mathcal{C}, \mathcal{R}, d >$, and the corresponding complete first order theory by $T_{\mathcal{C},\mathcal{R},d}$, where we include constants to name the individuals of $\mathcal{C}$ and $\mathcal{R}$. It should be clear from the above discussion, that an $\aleph_2$-saturated model of this theory, or a model of the enlargement of $T_{\mathcal{C},\mathcal{R},d}$, contains infinitesimal elements $\epsilon \neq 0$, in the sense that for any $z \in \mathcal{C}$ and $r \in \mathcal{R} > 0$, $d(z, z + \epsilon) < r$. However, there is no standard part map completely defined on such structures. In order to find such a map, we consider the following structure;

$$< \mathcal{C}, \mathcal{R}, d, P(\mathcal{C}), \mathcal{C}_{top}, pr >$$

where we have individual sorts for $P(\mathcal{C}) = \bigcup_{n\geq 1} P^n(\mathcal{C})$, a sort for the underlying field $\mathcal{C}$, predicate symbols to denote open sets of cartesian powers of $P^n(\mathcal{C})$, for $n \geq 1$, in the complex topology, and symbols for the projection maps;

$$pr_n : (\mathcal{C}^{n+1} \setminus \{0\}) \rightarrow P^n(\mathcal{C}) \text{ for } n \geq 1.$$

We denote the corresponding theory by $T_{\mathcal{C},\mathcal{R},d,P(\mathcal{C}),pr}$, again including constants for the individuals. We make the following observations;

(i). $T_{\mathcal{C},\mathcal{R},d} \subset T_{\mathcal{C},\mathcal{R},d,P(\mathcal{C}),pr}$.

(ii). The fragment $< \mathcal{C}, P(\mathcal{C}), pr >$ is interpretable in the fragment $< \mathcal{C}, +, ., 0, 1 >$. This follows from the fact that the equivalence relation on $(\mathcal{C}^{n+1} \setminus \{0\}) \times (\mathcal{C}^{n+1} \setminus \{0\})$;

$E(z_0, \ldots, z_n, w_0, \ldots, w_n)$ iff $\exists\lambda_{\neq 0}[\lambda(z_0, \ldots, z_n) = (w_0, \ldots, w_n)]$

is definable.

It follows that any $\aleph_2$-saturated model of $T_{\mathcal{C},\mathcal{R},d,P(\mathcal{C}),pr}$ is of the form;

$< {}^*\mathcal{C}, {}^*\mathcal{R}, d, P({}^*\mathcal{C}), \mathcal{C}_{top}, pr >$

where $<^* \mathcal{C},^* \mathcal{R}, d >$ is an enlargement of $< \mathcal{C}, \mathcal{R}, d >$ in the sense discussed above. We now fix such a structure, and define a standard part map $\pi_{st} : P({}^*\mathcal{C}) \rightarrow P({}^*\mathcal{C})$ by sending the monad $\mu(p)$ of $p \in \mathcal{C}$ to p. Such a map exists by Lemmas 16.23 and 16.24, by Lemma 16.20, it is non-trivial, and its image $P(\mathcal{C})$ is fixed (†). We add function symbols π_n to denote the standard part mappings on each $P^n({}^*\mathcal{C})$ and consider the corresponding structure;

$< {}^*\mathcal{C}, {}^*\mathcal{R}, d, P({}^*\mathcal{C}), \mathcal{C}_{top}, pr, \pi > (*)$

Observe that the fragment $< P({}^*\mathcal{C}), P(\mathcal{C}), \pi >$ is definable in this structure, and satisfies the theory T_{axioms} given in Theorem 2.1, this follows from (†) and Lemma 16.24. We denote the theory of the structure defined in $(*)$, with constants for the individuals of $(\mathcal{C}, \mathcal{R}, P(\mathcal{C}))$ by $T_{\mathcal{C},\mathcal{R},d,P(\mathcal{C}),pr,\pi}$. Now fix an $\aleph_2$-saturated model of $T_{\mathcal{C},\mathcal{R},d,P(\mathcal{C}),pr,\pi}$, and consider the corresponding structure;

$< \mathcal{C}_{univ}, \mathcal{R}_{univ}, d, P(\mathcal{C}_{univ}), \mathcal{C}_{top}, pr, \pi_{res} > (**)$

The fragment $< P(\mathcal{C}_{univ}), P(\mathcal{C}), \pi_{res})$ satisfies the theory T_{axioms}. By Theorem 1.10, we can find an algebraically closed field $\mathcal{C}_{univ,res}$, with $\mathcal{C} \subset \mathcal{C}_{univ,res} \subset \mathcal{C}_{univ}$ such that $P(\mathcal{C}_{univ,res})$ is the fixed image of the specialisation π_{res}. Now observe that we can define $\mathcal{C}_{univ,res}$ in the home sort $\mathcal{C}_{univ}$ by;

$\{x \in \mathcal{C}_{univ} : \exists y \in P^1(\mathcal{C}_{univ})[\pi_{res}(y) = pr_1(x, 1)]\}$

and we can define the valuation ring $\mathcal{O}_{univ}$ for the specialisation π_{res} by;

$\{x \in \mathcal{C}_{univ} : (\pi_{res} \circ pr_1)(x, 1) \in U\}$

where U is the open subset of $P^1(\mathcal{C})$ obtained by removing the point $[1:0]$. The specialisation π_{res} naturally induces a definable map

$\pi_{1,res}:\mathcal{O}_{univ}\rightarrow\mathcal{C}_{univ,res}$ in the home sort.

We claim that;

For $r\in\mathcal{R}_{>0}$, and $x\in\mathcal{O}_{univ}$, $d(x,\pi_{1,res}(x))<r$. $(\dagger\dagger)$

This follows from the fact that the standard part map π_{st} satisfies this property in the structure defined by $(*)$. We now observe that the structure $<\mathcal{C}_{univ},\mathcal{R}_{univ},d>$ is an enlargement of $<\mathcal{C},\mathcal{R},d>$ and, hence, we can define a standard part map $\pi_{st,univ}:P(C_{univ})\rightarrow P(\mathcal{C})$, which fixes $P(\mathcal{C})$. We consider the restriction of this map to $P(C_{univ,res})$ and let $\mathcal{O}_{univ,res}$ denote the valuation ring. By definition of the standard part map, we have that;

For $r\in\mathcal{R}_{>0}$, and $x\in\mathcal{O}_{univ,res}$, $d(x,\pi_{1,st,univ}(x))<r$. $(\dagger\dagger\dagger)$

We now compose the specialisations $<P(\mathcal{C}_{univ}),P(\mathcal{C}_{univ,res}),\pi_{res}>$ and $<P(\mathcal{C}_{univ,res}),P(\mathcal{C}),\pi_{st,univ}>$ to obtain a specialisation;

$<P(\mathcal{C}_{univ}),P(\mathcal{C}),\pi_{univ}>$

We claim first that $<P(\mathcal{C}_{univ}),P(\mathcal{C}),\pi_{univ}>$ is a universal specialisation in the sense of Section 3. This follows from the fact that the structure $<P(\mathcal{C}_{univ}),P(\mathcal{C}_{univ,res}),\pi_{res}>$ is a saturated model of T_{axioms} and, following through the proof in Section 3, we have replaced the specialisation of Lemma 2.1, with a standard part mapping $<P(\mathcal{C}_{univ,res}),P(\mathcal{C}),\pi_{st,univ}>$.

Let $\mathcal{O}$ denote the valuation ring of the specialisation π_{univ}. We claim second that;

For $r\in\mathcal{R}_{>0}$, and $x\in\mathcal{O}$, $d(x,\pi_{1,univ}(x))<r$. $(\dagger\dagger\dagger\dagger)$

By construction, $\pi_{1,res}(x)\in\mathcal{O}_{univ,res}$, and for $r\in\mathcal{R}_{>0}$, using $(\dagger\dagger)$, $(\dagger\dagger\dagger)$ and the triangle inequality for d;

$d(x,\pi_{1,univ}(x))=d(x,(\pi_{1,st,univ}\circ\pi_{1,res})(x))$

$\leq d(x,\pi_{1,res}(x))+d(\pi_{1,res}(x),(\pi_{1,st,univ}\circ\pi_{1,res})(x))<\frac{r}{2}+\frac{r}{2}=r$

We claim third that the maps $\pi_{st,univ}$ and π_{univ} coincide, $(***)$. Let $\mathcal{V}_p$ be an infinitesimal neighborhood of $p\in\mathcal{C}$, that is $\pi_{univ}^{-1}(p)$, and let $\mu(p)$ be the monad of p, that is $\pi_{st,univ}^{-1}(p)$. We show that $\mathcal{V}_p\subset\mu(p)$. Write p in coordinates $[p_0:\ldots:p_n]$, let $\pi_{n,univ}(q)=p$, and write $q=[q_0,\ldots q_n]$ with $q_i\in\mathcal{O}$, for $0\leq i\leq n$, and $\pi_{1,univ}(q_0,\ldots,q_n)=(p_0,\ldots,p_n)$. Using $(\dagger\dagger\dagger\dagger)$, we know that $d(\bar{q},\bar{p})<r$ for any $r\in\mathcal{R}_{>0}$, where d is the induced Euclidean distance on $\mathcal{C}_{univ}^{n+1}$. The sets $pr_n(U_r(\bar{p}))$, where $U_r(\bar{p})=\{\bar{x}\in(\mathcal{C}^{n+1}\setminus 0):d(\bar{x},\bar{p})<r\}$ form a basis at p for the complex topology in $P^n(\mathcal{C})$. It follows that q lies in every open set $U_{\nu,p}$ about p, hence $q\in\mu(p)$, as required. The third claim is now clear. Suppose there exists $x\in(\mu(p)\setminus\mathcal{V}_p)$, then $\pi_{univ}(x)=q\neq p$, and $x\in\mathcal{V}_q$. Therefore, $x\in\mu(q)$. This implies that $\mu(q)\cap\mu(p)\neq\emptyset$, contradicting Lemma 16.21. It follows that $\mathcal{V}_p=\mu(p)$ and the maps $\pi_{st,univ}$ and π_{univ} coincide, showing $(***)$.

We summarise what we have proved in the following theorem;

Theorem 17.1. *For $n\geq 1$, Let $<P^n(\mathcal{C}),\mathcal{C}_{top,open},\mathcal{C}_{top,closed}>$ be the first order structure structure with domain complex projective space $P^n(\mathcal{C})$, and predicates denoting open sets and closed sets for the complex topology on Cartesian powers. Let $T_{\mathcal{C}}$ be the theory of this structure. Then there exists a model of this theory $<P^n(\mathcal{C}_{univ}),\mathcal{C}_{top,open},\mathcal{C}_{top,closed}>$, with the following properties;*

(i). The fragment $<P^n(\mathcal{C}_{univ}),\mathcal{C}_{top,open}>$ is an enlargement of the fragment $<P^n(\mathcal{C}),\mathcal{C}_{top,open}>$ considered as a topological space.

Let π_{univ} denote the standard part map for this extension.

(ii). The fragment $<P^n(\mathcal{C}_{univ}),\mathcal{C}_{closed,alg}>$ is an elementary extension of the fragment $<P^n(\mathcal{C}),\mathcal{C}_{closed,alg}>$ considered as a Zariski structure, with predicates for the closed algebraic subvarieties of Cartesian powers. Moreover, the standard part map π_{univ} is a universal specialisation, in the sense of Section 3.

18. Non-Standard and Standard Results for Algebraic Curves

In this final section, we realise the power of Theorem 17.1, by reformulating classical results for algebraic curves in characteristic 0, in the complex topology. We first observe the following;

Lemma 18.1. *Let $f : C_1 \rightarrow C_2$ be a dominant* algebraic *morphism between smooth projective algebraic curves. Let $p_1 \in C_1$ and $p_2 \in C_2$ with $f(p_1) = p_2$, then there exist complex connected open neighborhoods U, V around p_1, p_2 such that $f : (U \setminus p_1) \rightarrow (V \setminus p_2)$ is k to 1.*

Proof. As C_1 and C_2 are complex manifolds, we can find charts $\phi_1 : U \rightarrow D(0,1)$ and $\phi_2 : V \rightarrow D(0,1)$, centred at p_1 and p_2. As f is continuous, we can assume that $f(U) \subset V$. As f is algebraic, the composition $(\phi_2 \circ f \circ \phi_1^{-1}) : D(0,1) \rightarrow D(0,1)$ is holomorphic. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series representation of f, given by Taylor's theorem. We can write this series as $z^k h(z)$ with $h(0) \neq 0$ and h holomorphic. Let $g(z) = h(z)^{1/k} = exp(log(h(z))/k)$ be a choice of k'th root for h. Then g is holomorphic with $g(0) \neq 0$, on a disc $D' \subset D$ containing 0. It follows that $f(z) = (zg(z))^k$. We have that $(zg(z))' = g(z) + zg'(z)$, in particular, $(zg(z))'|0 \neq 0$. By the inverse function theorem, $\theta(z) = zg(z)$ is locally a homeomorphism. Clearly, $f(z) = (\theta(z))^k$ is $k - 1$ on D'. This proves the result.

□

Remarks 18.2. *Clearly, the value k is independent of the choice of U, V. We leave this as an exercise for the interested reader. We can, therefore, define the complex multiplicity of f at p_1 to be k.*

We recall the definition of algebraic multiplicity, Definition 9.6;

Theorem 18.3. *Let hypotheses be as in Theorem 8.1, then the notions of complex multiplicity and algebraic multiplicity coincide.*

Proof. By Theorem 10.6, algebraic multiplicity k coincides with Zariski multiplicity. Working in the structure $< P^n(C_{univ}), \mathcal{C}_{alg} >$ provided by Theorem 17.1, and using Definition 8.1, the maximum cardinality of the fibre $f^{-1}(p) \cap \mathcal{V}_{p_2}$ for $p \in \mathcal{V}_{p_1}$ is k. By Theorem 17.1, working in the structure $< P^n(C_{univ}), \mathcal{C}_{top,open} >$ the maximum cardinality of the fibre $f^{-1}(p) \cap \mu_{p_2}$ for $p \in \mu_{p_1}$ is k, $(*)$. By definition of a monad, for any connected open neighborhoods U, V of p_1, p_2, we can find $p_2' \in V$, with $Card(f^{-1}(p_2') \cap U) \geq k$. As this can be written as a first order statement in $\mathcal{L}_{\mathcal{C}_{top,open}}$, it follows by transfer, that this holds in the structure $< P^n(\mathcal{C}), \mathcal{C}_{top,open} >$. This clearly implies that the complex multiplicity is at least k. Suppose the complex multiplicity is l, strictly greater than k. Then, the following set of statements is consistent;

$\{\bigwedge_{i=1}^{l}(U(c_i)\wedge V(f(c_i)))\wedge(c_1\neq\ldots\neq c_l)\wedge(f(c_1)=\ldots=f(c_l)\}$

where U,V range over all the complex open neighborhoods of p_1 and p_2. This follows from compactness and the fact that any finite subset is realised in the structure $<P^n(\mathcal{C}),\mathcal{C}_{top,open}>$. As $<P^n(C_{univ}),\mathcal{C}_{top,open}>$ is an enlargement, we can realise these statements to find $\{c_1,\ldots,c_l\}$ distinct in $\mu(p_1)\cap C_1$, mapping to an element $d\in(\mu(p_2)\cap C_2)$. This contradicts $(*)$. Hence, $l=k$, and algebraic multiplicity coincides with complex multiplicity.

□

We now turn to Bezout's Theorem. We formulate a complex version of intersection multiplicity for plane projective algebraic curves, and prove that this coincides with the algebraic definition, see Section 14. We claim the following;

Lemma 18.4. *Let C_1 and C_2 be plane, possibly non-reduced and non-irreducible, projective curves of degree d and e, centred at $(0,0)$ in an affine coordinate system. Let C_1 and C_2 be defined by the equations;*

$f_1(x,y,\bar{c})=\sum_{i+j\leq d}c_{ij}x^iy^j=0$

$f_2(x,y,\bar{d})=\sum_{i+j\leq e}d_{ij}x^iy^j=0$

Let $D(\bar{0},r)$ be an open disc of radius r about the origin in $\mathcal{C}^2$, and let $D(\bar{c},\bar{d},s)$ be an open disc of radius s centred at the parameter $(\bar{c},\bar{d})$ in $\mathcal{C}^N$, where $N=[(d+1)(d+2)+(e+1)(e+2)]/2$. Let;

$m_{r,s}=max\{Card(f_1(x,y,\bar{c}')\cap f_2(x,y,\bar{d}')\cap D(0,r)):(\bar{c}',\bar{d}')\in D(\bar{c},\bar{d},s)\}$

Then $lim_{r,s\rightarrow 0}m_{r,s}$ exists. We define this to be the complex intersection multiplicity $I_{\mathcal{C}}(f_1,f_2,\bar{0})$.

Proof. For fixed r, the function $g_r(s)=m_{r,s}$ is decreasing in s. Similarly, for fixed s, the function $h_s(r)=m_{r,s}$ is decreasing in r, $(*)$. Suppose the limit does not exist, then as $m_{r,s}$ takes only finitely many integer values, we can find sequences (r_i,s_j) and (r_i',s_j') converging to $(0,0)$ in $\mathcal{R}^2$, such that m is constant on each sequence, and $m_{(r_i,s_j)}<m_{(r_i',s_j')}$, $(**)$. Let (r_1,s_1) be the first element in the sequence, then we can find (r_{i_1}',s_{j_1}') with $r_{i_1}'\leq r_1$ and $s_{j_1}'\leq s_1$. Then, by $(*)$, $m_{(r_{i_1}',s_{j_1}')}\leq m_{r_1,s_1}$. This contradicts $(**)$. □

Remarks 18.5. *As the limit exists and $m_{r,s}$ takes integer values, m must be constant on an open neighborhood of $(0,0)$ in $\mathcal{R}^2$. That is, the complex intersection multiplicity depends only on the value of $m_{r,s}$ for sufficiently small r and s.*

We can now show;

Theorem 18.6. *Let hypotheses be as in Lemma 18.4, then the notions of complex intersection multiplicity and algebraic intersection multiplicity coincide. In particular the definition of complex intersection multiplicity is independent of the choice of coordinates, and a complex version of Bezout's theorem holds, that is $\sum_{x\in f_1\cap f_2} I_{\mathcal{C}}(f,g,x) = de$.*

Proof. We refer the reader to Section 14, By Theorem 14.1, algebraic intersection multiplicity k coincides with Zariski multiplicity for the cover;

$$\theta : Spec(C_{univ}[x,y,\bar{u}\bar{v}]/ < f_1(x,y,\bar{u}), f_2(x,y,\bar{v}) >) \rightarrow Spec(C_{univ}[\bar{u}\bar{v}])$$

at the point $(0,0,\bar{c},\bar{d})$. By Definition 8.1, the maximum cardinality of the fibre $\theta^{-1}(\bar{c}',\bar{d}')\cap\mathcal{V}_{(0,0)}$ for $(\bar{c}',\bar{d}')\in\mathcal{V}_{(\bar{c},\bar{d})}$ is k as well. By Theorem 17.1, the maximum of the cardinality $\theta^{-1}(\bar{c}',\bar{d}')\cap\mu_{(0,0)}$ for $(\bar{c}',\bar{d}')\in\mu_{(\bar{c},\bar{d})}$ is also k, $(*)$. By definition of a monad, for any discs $D(\bar{0},r)$ and $D(\bar{c},\bar{d},s)$, we can find $(\bar{c}',\bar{d}')\in D(\bar{c},\bar{d},s)$ with $Card(\theta^{-1}(\bar{c}',\bar{d}')\cap D(\bar{0},r))\geq k$. As this can be written as a first order statement in $\mathcal{L}_{\mathcal{C}_{top,open}}$, it follows by transfer, that this holds in the structure $< P^n(\mathcal{C}),\mathcal{C}_{top,open} >$. By the definition in Lemma 18.4, this implies the complex intersection multiplicity is at least k. Suppose the complex intersection multiplicity is l, strictly greater than k. Then the following set of statements is consistent;

$$\{\bigwedge_{i=1}^{l}[\bar{t}_i\in D(\bar{0},r)\wedge\theta(\bar{t}_i,\bar{c}',\bar{d}')\in D(\bar{c},\bar{d},s)]\wedge(t_1\neq\ldots\neq t_l)$$

where r,s range over the positive real numbers. This follows from compactness and the fact that any finite subset is realised in the structure $< P^n(\mathcal{C}),\mathcal{C}_{top,open} >$. As $< P^n(C_{univ}),\mathcal{C}_{top,open} >$ is an enlargement, we can realise these statements to find $\{t_1,\ldots,t_l\}$ distinct in $\mu_{\bar{0}}$, mapping to an element $(\bar{c}',\bar{d}')\in\mu_{(\bar{c},\bar{d})}$. This contradicts $(*)$. Hence, $l=k$, and algebraic intersection multiplicity coincides with complex intersection multiplicity. The final statement is a direct consequence of the invariance of algebraic intersection multiplicity and the classical Bezout's Theorem.

□

Implicit in the above proofs is the following theorem;

Theorem 18.7. *Let hypotheses be as in Lemma 18.1, we define the non-standard multiplicity as;*

$$max\{Card(f^{-1}(p_1')) : p_1' \in \mu(p)\}$$

Then the non-standard multiplicity coincides with the complex, Zariski and algebraic multiplicities.

Let hypotheses be as in Lemma 18.4, we define the non standard multiplicity as;

$$max\{Card(f_1(x, y, \bar{c}') \cap f_2(x, y, \bar{d}') \cap \mu(0, 0)) : (\bar{c}', \bar{d}') \in \mu(\bar{c}, \bar{d})\}$$

Then the non-standard multiplicity coincides with the complex, Zariski and algebraic multiplicities.

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