A THEORY OF DUALITY FOR ALGEBRAIC CURVES

TRISTRAM DE PIRO

Abstract. We formulate a refined theory of g_n^r, using the methods of [9], and use the theory to give a geometric interpretation of the genus of an algebraic curve. Using principles of duality, we prove generalisations of Plucker's formulae for algebraic curves. The results hold for arbitrary characteristic of the base field L, with some occasional exceptions when $char(L) = 2$, which we observe in the course of the paper.

1. A refined theory of g_n^r

The purpose of this section is to refine the general theory of g_n^r, in order to take into account the notion of a branch for a projective algebraic curve. We will rely heavily on results proved in [9]. We also refer the reader there for the relevant notation. Unless otherwise stated, we will assume that the characteristic of the field L is zero, making the modifications for non-zero characteristic in the final section. By an algebraic curve, we always mean a projective irreducible variety of dimension 1.

Definition 1.1. *Let $C \subset P^w$ be a projective algebraic curve of degree d and let Σ be a linear system of dimension R, contained in the space of algebraic forms of degree e on P^w. Let ϕ_λ belong to Σ, having finite intersection with C. Then, if $p \in C \cap \phi_\lambda$ and γ_p is a branch centred at p, we define;*

$$I_p(C, \phi_\lambda) = I_{italian}(p, C, \phi_\lambda)$$

$$I_p^\Sigma(C, \phi_\lambda) = I_{italian}^\Sigma(p, C, \phi_\lambda)$$

$$I_p^{\Sigma,mobile}(C, \phi_\lambda) = I_{italian}^{\Sigma,mobile}(p, C, \phi_\lambda)$$

$$I_{\gamma_p}(C, \phi_\lambda) = I_{italian}(p, \gamma_p, C, \phi_\lambda)$$

$$I_{\gamma_p}^\Sigma(C, \phi_\lambda) = I_{italian}^\Sigma(p, \gamma_p, C, \phi_\lambda)$$

$I^{\Sigma,mobile}_{\gamma_p}(C,\phi_\lambda)=I^{\Sigma,mobile}_{italian}(p,\gamma_p,C,\phi_\lambda)$

where $I_{italian}$ was defined in [9].

It follows that, as λ varies in Par_Σ, we obtain a series of weighted sets;

$W_\lambda=\{n_{\gamma^1_{p_1}},\ldots,n_{\gamma^{n_1}_{p_1}},\ldots,n_{\gamma^1_{p_m}},\ldots,n_{\gamma^{nm}_{p_m}}\}$

where;

$\{p_1,\ldots,p_i,\ldots,p_m\}=C\cap\phi_\lambda$, for $1\leq i\leq m$,

$\{\gamma^1_{p_i},\ldots,\gamma^{j(i)}_{p_i},\ldots,\gamma^{n_i}_{p_i}\}$, for $1\leq j(i)\leq n_i$, consists of the branches of C centred at p_i

and

$I_{\gamma^{j(i)}_{p_i}}(C,\phi_\lambda)=n_{\gamma^{j(i)}_{p_i}}$

By the branched version of the Hyperspatial Bezout Theorem, see [9], the total weight of any of these sets, which we will occasionally abbreviate by $C\sqcap\phi_\lambda$, is always equal to de. Let r be the least integer such that every weighted set W_λ is defined by a linear subsystem $\Sigma'\subset\Sigma$ of dimension r.

Definition 1.2. *We define;*

$Series(\Sigma)=\{W_\lambda:\lambda\in Par_\Sigma\}$

$dimension(Series(\Sigma))=r$

$order(Series(\Sigma))=de$

We then claim the following;

Theorem 1.3. .

(i). $r\leq R$, with equality iff every weighted set W_λ of the series is cut out by a single *form of Σ.*

(ii). $r \lneqq R$ iff there exists a form ϕ_λ in Σ, containing all of C.

Proof. We first show the equivalence of (i) and (ii). Suppose that (i) holds and $r \lneqq R$. Then, we can find a weighted set W and distinct elements $\{\lambda_1, \lambda_2\}$ of Par_Σ such that $W = W_{\lambda_1} = W_{\lambda_2}$. Let $\{\phi_{\lambda_1}, \phi_{\lambda_2}\}$ be the corresponding algebraic forms of Σ and consider the pencil $\Sigma_1 \subset \Sigma$ defined by these forms. We claim that;

$W = C \sqcap (\mu_1\phi_{\lambda_1} + \mu_2\phi_{\lambda_2})$, for $[\mu_1 : \mu_2] \in P^1$ $(*)$

This follows immediately from the results in [9] that the condition of multiplicity at a branch is *linear* and the branched version of the Hyperspatial Bezout Theorem. Now choose a point $p \in C$ not contained in W. Then, the condition that an algebraic form ϕ_λ passes through p defines a hyperplane condition on Par_k, hence, intersects Par_{Σ_1} in a point. Let ϕ_{λ_0} be the algebraic form in Σ_1 defined by this parameter. Then, by $(*)$, we have that;

$W \cup \{p\} \subseteq C \sqcap \phi_{\lambda_0}$

Hence, the total multiplicity of intersection of ϕ_{λ_0} with C is at least equal to $de + 1$. By the branched version of the Hyperspatial Bezout Theorem, C must be contained in ϕ_{λ_0}. Conversely, suppose that (i) holds and there exists a form ϕ_{λ_0} in Σ containing all of C. Let W be cut out by ϕ_{λ_1} and consider the pencil $\Sigma_1 \subset \Sigma$ generated by $\{\phi_{\lambda_0}, \phi_{\lambda_1}\}$. By the same argument as above, we can find ϕ_{λ_2} in Σ_1, distinct from ϕ_{λ_1}, which also cuts out W. Hence, by (i), we must have that $r \lneqq R$. Therefore, (ii) holds.

The argument that (ii) implies (i) is similar.

We now prove that (ii) holds. Using the Hyperspatial Bezout Theorem, the condition on Par_Σ that a form ϕ_λ contains C is linear. Let H be the linear subsystem of Σ, consisting of forms containing C and let $h = dim(H)$. Let $K \subset \Sigma$ be a maximal linear subsystem, having finite intersection with C. Then K has no form in common with H and $dim(K) = R - h - 1$. We claim that every weighted set in $Series(\Sigma)$ is cut out by a unique form from K. For suppose that $W = C \sqcap \phi_\lambda$ is such a weighted set and consider the linear system defined by $< H, \phi_\lambda >$. If ϕ_μ belongs to this system and has finite intersection with C, then clearly $(C \cap \phi_\lambda) = (C \cap \phi_\mu)$. Using linearity of multiplicity at a branch and the Hyperspatial Bezout Theorem again (by convention, a form containing C has infinite multiplicity at a branch), we must have that $(C \sqcap \phi_\lambda) = (C \sqcap \phi_\mu)$. Now consider $K \cap < H, \phi_\lambda >$. We have that;

$$\begin{aligned} codim(K \cap < H, \phi_\lambda >) &\leq codim(K) + codim(< H, \phi_\lambda >) \\ . \quad &= (h+1) + (R - (h+1)) = R. \end{aligned}$$

Hence, $dim(K \cap < H, \phi_\lambda >) \geq 0$. We can, therefore, find a form ϕ_μ belonging to K such that $W = (C \sqcap \phi_\mu)$. We need to show that ϕ_μ is the unique form in K defining W. This follows by the argument given above. It follows immediately that $r = dim(K) = R - h - 1$. Hence, $r \lneq R$ iff $h \geq 0$. Therefore, (ii) is shown.

□

Using this theorem, we give a more refined definition of a g_n^r.

Definition 1.4. *Let $C \subset P^w$ be a projective algebraic curve. By a g_n^r on C, we mean the collection of weighted sets, without repetitions, defined by $Series(\Sigma)$ for* some *linear system Σ, such that $r = dimension(Series(\Sigma))$ and $n = order(Series(\Sigma))$. If a branch γ_p^j appears with multiplicity at least s in every weighted set of a g_n^r, as just defined, then we allow the possibility of removing some multiplicity contribution $s' \leq s$ from each weighted set and adjusting n to $n' = n - s'$.*

Remarks 1.5. *The reader should observe carefully that a g_n^r is defined independently of a particular linear system. However, by the previous theorem, for any g_n^r, there exists a $g_{n'}^r$ with $n \leq n'$ such that the following property holds. The $g_{n'}^r$ is defined by a linear system of dimension r, having finite intersection with C, such that each there is a bijection between the weighted sets W in the $g_{n'}^r$ and the W_λ in $Series(\Sigma)$. The original g_n^r is obtained from the $g_{n'}^r$ by removing some fixed point contribution.*

We now reformulate the results of Section 2 and Section 5 in [9] for this new definition of a g_n^r. In order to do this, we require the following definition;

Definition 1.6. *Suppose that $C \subset P^w(L)$ is a projective algebraic curve and $C^{ext} \subset P^w(K)$ is its non-standard model. Let a g_n^r be given on C, defined by a linear system Σ after removing some fixed point contribution. We define the extension $g_n^{r,ext}$ of the g_n^r to the nonstandard model C^{ext} to be the collection of weighted sets, without repetitions, defined by $Series(\Sigma)$ on C^{ext}, after removing the same fixed point contribution. Note that, by definability of multiplicity at a branch, see*

Theorem 6.5 of [9]*, if* γ_p^j *is a branch of* C *and;*

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) \geq k, \ (\lambda \in Par_{\Sigma(L)})$$

then;

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) \geq k, \ (\lambda \in Par_{\Sigma(K)})$$

Hence, it is *possible to remove the same fixed point contribution of* $Series(\Sigma)$ *on* C^{ext}*. See also the proof of Lemma 1.7.*

It is a remarkable fact that, after introducing the notion of a branch, the definition is independent of the particular linear system Σ. This is the content of the following lemma;

Lemma 1.7. *The previous definition is independent of the particular choice of linear system* Σ *defining the* g_n^r*.*

Proof. We divide the proof into the following cases;

Case 1. $\Sigma \subset \Sigma'$;

By the proof of Theorem 1.3, we can find a linear system $\Sigma_0 \subset \Sigma \subset \Sigma'$ of dimension r, having finite intersection with C, such that the g_n^r is defined by removing some fixed contribution from Σ_0. Here, we have also used the fact that the base point contributions (at a branch) of $\{\Sigma_0, \Sigma, \Sigma'\}$ are the same. Again, by Theorem 1.3, if $W_{\lambda'}$ is a weighted set defined by Σ' on C^{ext}, then it appears as a weighted set $V_{\lambda''}$ defined by Σ_0 on C^{ext}. Hence, it appears as a weighted set $V_{\lambda''}$ defined by Σ on C^{ext}. By the converse argument and the remark on base point contributions, the proof is shown.

Case 2. Σ are Σ' are both linear systems of dimension r, having finite intersection with C, such that $degree(\Sigma) = degree(\Sigma') = n$;

By Theorem 1.3, every weighted set W in the g_n^r is defined uniquely by weighted sets W_{λ_1} and V_{λ_2} in $Series(\Sigma_1)$ and $Series(\Sigma_2)$ respectively. Let (C^{ns}, Φ^{ns}) be a non-singular model of C. Using the method of Section 5 in [9] to avoid the technical problem of presentations of Φ^{ns} and base point contributions, we may, without loss of generality, assume that there exist finite covers $W_1 \subset Par_\Sigma \times C^{ns}$ and $W_2 \subset Par_{\Sigma'} \times C^{ns}$

such that;

$$j_{k,\Sigma}(\lambda,p_j)\equiv Mult_{(W_1/Par_\Sigma)}(\lambda,p_j)\geq k \text{ iff } I_{italian}(p,\gamma_p^j,C,\phi_\lambda)\geq k$$

$$j_{k,\Sigma'}(\lambda',p_j)\equiv Mult_{(W_2/Par_{\Sigma'})}(\lambda',p_j)\geq k \text{ iff } I_{italian}(p,\gamma_p^j,C,\psi_{\lambda'})\geq k$$

Then consider the sentences;

$$(\forall\lambda\in Par_\Sigma)(\exists!\lambda'\in Par_{\Sigma'})\forall x\in C^{ns}[\bigwedge_{k=1}^n(j_k(\lambda,x)\leftrightarrow j_k(\lambda',x))]$$

$$(\forall\lambda'\in Par_\Sigma)(\exists!\lambda\in Par_\Sigma)\forall x\in C^{ns}[\bigwedge_{k=1}^n(j_k(\lambda',x)\leftrightarrow j_k(\lambda,x))]\ (*)$$

in the language of $< P^1(L), C_i >$, considered as a Zariski structure with predicates $\{C_i\}$ for Zariski closed subsets defined over L, (see [16]). We have, again by results of [16] or [7], that $< P^1(L), C_i >\prec< P^1(K), C_i >$, for the nonstandard model $P(K)$ of $P(L)$. It follows immediately from the algebraic definition of j_k in [16], that, for any weighted set $W_{\lambda'_1}$ defined by $Series(\Sigma)$ on C^{ext}, there exists a unique weighted set $V_{\lambda'_2}$ defined by $Series(\Sigma')$ on C^{ext} such that $W_{\lambda'_1}=V_{\lambda'_2}$, and conversely. Hence, the proof is shown.

Case 3. Σ are Σ' are both linear systems of dimension r, having finite intersection with C;

Let $n_1=degree(\Sigma)$ and $n_2=degree(\Sigma')$. Then the original g_n^r is obtained from $Series(\Sigma)$, by removing a fixed point contribution of multiplicity n_1-n, and, is obtained from $Series(\Sigma')$, by removing a fixed point contribution of multiplicity n_2-n. We now imitate the proof of Case 2, with the slight modification that, in the construction of the sentences given by $(*)$, we make an adjustment of the multiplicity statement at the finite number of branches where a fixed point contribution has been removed. The details are left to the reader. □

Now, using Definition 1.6, we construct a specialisation operator $sp: g_n^{r,ext}\rightarrow g_n^r$. We first require the following simple lemma;

Lemma 1.8. *Let $C\subset P^w(L)$ be a projective algebraic curve and let $C^{ext}\subset P^w(K)$ be its nonstandard model. Let $p'\in C^{ext}$ be a non-singular point, with specialisation $p\in C$. Then there exists a unique branch γ_p^j such that $p'\in\gamma_p^j$.*

Proof. We may assume that $p' \neq p$, otherwise p would be non-singular and, by Lemma 5.4 of [9], would be the origin of a single branch γ_p. Let (C^{ns}, Φ) be a non-singular model of C, then p' must belong to the canonical set $V_{[\Phi]}$, hence there exists a unique $p'' \in C^{ns}$ such that $\Phi(p'') = p'$. By properties of specialisations, $p'' \in C^{ns} \cap \mathcal{V}_{p_j}$ for some $p_j \in \Gamma_{[\Phi]}(x,p)$. Hence, by definition of a branch given in Definition 5.15 of [9], we must have that $p' \in \gamma_p^j$. The uniqueness statement follows as well. □

We now make the following definition;

Definition 1.9. *Let $C \subset P^w(L)$ be a projective algebraic curve and let $C^{ext} \subset P^w(K)$ be its non-standard model. Given a g_n^r on C with extension $g_n^{r,ext}$ on C^{ext}, we define the specialisation operator;*

$sp : g_n^{r,ext} \rightarrow g_n^r$

by;

$sp(\gamma_{p'}) = \gamma_p^j$, *for* $p' \in NonSing(C^{ext})$ *and* γ_p^j *as in Lemma 1.8.*

$sp(\gamma_p^j) = \gamma_p^j$, *for* $p \in Sing(C^{ext}) = Sing(C)$ *and* $\{\gamma_p^1, \ldots, \gamma_p^j, \ldots, \gamma_p^s\}$ *enumerating the branches at p.*

$sp(n_1\gamma_{p_1}^{j_1} + \ldots + n_r\gamma_{p_r}^{j_r}) = n_1 sp(\gamma_{p_1}^{j_1}) + \ldots + n_r sp(\gamma_{p_r}^{j_r})$,

for a linear combination of branches with $n_1 + \ldots + n_r = n$

It is also a remarkable fact that, after introducing the notion of a branch, the specialisation operator *sp* is well defined. This is the content of the following lemma;

Lemma 1.10. *Let hypotheses be as in the previous definition, then, if W is a weighted set belonging to $g_n^{r,ext}$, its specialisation $sp(W)$ belongs to g_n^r.*

Proof. We may assume that there exists a linear system Σ, having finite intersection with C, such that $dimension(\Sigma) = r$ and $degree(\Sigma) = n_1$, with the g_n^r and $g_n^{r,ext}$ both defined by $Series(\Sigma)$, after removing some fixed point contribution W_0 of multiplicity $n_1 - n$. Let W be a weighted set of the $g_n^{r,ext}$, then $W \cup W_0 = (C \sqcap \phi_{\lambda'})$, for some unique $\lambda' \in Par_\Sigma$.

We claim that $sp(W\cup W_0)=C\sqcap\phi_\lambda$, for the specialisation $\lambda\in Par_\Sigma$ of λ' $(*)$. As $sp(W_0)=W_0$, it then follows immediately from linearity of sp, that $sp(W)$ belongs to the g_n^r as required. We now show $(*)$. Let $p\in C$ and let γ_p be a branch centred at p. By γ_p^{ext}, we mean the branch at p, where p is considered as an element of C^{ext}. We now claim that;

$$I_{\gamma_p}(C,\phi_\lambda)=I_{\gamma_p^{ext}}(C,\phi_{\lambda'})+\sum_{p'\in(\gamma_p\setminus p)}I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})\ (**)$$

Let $(C^{ns},\Phi)\subset P^{w'}(L)$ be a non-singular model of C, such that γ_p corresponds to $C^{ns,ext}\cap\mathcal{V}_q$, where $q\in\Gamma_{[\Phi]}(x,p)$ and $\mathcal{V}_q$ is defined relative to the specialisation from $P(K)$ to $P(L)$. Let $C^{ns,ext,ext}\subset P^{w'}(K')$ be a non-standard model of $C^{ns,ext}$, such that γ_q^{ext} corresponds to $C^{ns,ext,ext}\cap\mathcal{V}_q$, where $\mathcal{V}_q$ is defined relative to the specialisation from $P(K')$ to $P(K)$. Then, for $p'\in(\gamma_p\setminus p)$, we can find $q'\in\mathcal{V}_q\cap C^{ns,ext}$ such that $\gamma_{p'}$ corresponds to $\mathcal{V}_{q'}\cap C^{ns,ext,ext}$. We may choose a suitable presentation Φ_{Σ_1} of Φ, such that $Base(\Sigma_1)$ is disjoint from $\Gamma_{[\Phi]}(x,p)$, and, therefore, disjoint from $\Gamma_{[\Phi]}(x,p')$, for $p'\in(\gamma_p\setminus p)$. Let $\{\overline{\phi_\lambda}\}$ denote the lifted family of on C^{ns} from the presentation $\Phi_{\Sigma'}$. In this case, we have, by results of [9], that;

$$I_{\gamma_p}(C,\phi_\lambda)=I_q(C^{ns},\overline{\phi_\lambda})$$

$$I_{\gamma_p^{ext}}(C,\phi_{\lambda'})=I_q(C^{ns},\overline{\phi_{\lambda'}})$$

$$I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})=I_{q'}(C^{ns},\overline{\phi_{\lambda'}})\ (1)$$

By summability of specialisation, see [7];

$$I_q(C^{ns},\overline{\phi_\lambda})=I_q(C^{ns},\overline{\phi_{\lambda'}})+\sum_{q'\in C^{ns}\cap(\mathcal{V}_q\setminus q)}I_{q'}(C^{ns},\overline{\phi_{\lambda'}})\ (2)$$

Combining (1) and (2), the result $(**)$ follows, as required. Now, suppose that a branch γ_p occurs with non-trivial multiplicity in $sp(C\sqcap\phi_{\lambda'})$. By Definition 1.9, the contribution must come from either $I_{\gamma_p^{ext}}(C,\phi_{\lambda'})$ or $I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})$, for some $p'\in(\gamma_p\setminus p)$. Applying sp to $(**)$, one sees that the branch γ_p occurs with multiplicity $I_{\gamma_p}(C,\phi_\lambda)$. It follows that $sp(C\sqcap\phi_{\lambda'})=C\sqcap\phi_\lambda$, hence $(*)$ is shown. The lemma then follows.

□

We can now reformulate the results of Section 2 and Section 5 of [9] in the language of this refined theory of g_n^r. We first make the following definition;

Definition 1.11. *Let $C \subset P^w$ be a projective algebraic curve and let a g_n^r be given on C. Let W be a weighted set in this g_n^r or its extension $g_n^{r,ext}$ and let γ_p be a branch centred at p. Then we say that;*

γ_p is s-fold (s-plo) for W if it appears with multiplicity at least s.

γ_p is multiple for W if it appears with multiplicity at least 2.

γ_p is simple for W if it is not multiple.

γ_p is counted (contato) s-times in W if it appears with multiplicity exactly s.

γ_p is a base branch of the g_n^r if it appears in every *weighted set.*

γ_p is s-fold for the g_n^r if it is s-fold in W for every *weighted set W of the g_n^r.*

γ_p is counted s-times for the g_n^r if it is s-fold for the g_n^r and is counted s-times in some weighted set W of the g_n^r.

We then have the following;

Theorem 1.12. *Local Behaviour of a g_n^r*

Let C be a projective algebraic curve and let a g_n^r be given on C. Let γ_p be a branch centred at p, such that γ_p is counted s-times for the g_n^r. If γ_p is counted t times in a given weighted set W, then there exists a weighted set W' in $g_n^{r,ext}$ such that $sp(W') = W$ and $sp^{-1}(t\gamma_p)$ consists of the branch γ_p counted s-times and $t-s$ other distinct branches $\{\gamma_{p_1}, \ldots, \gamma_{p_{t-s}}\}$, each counted once in W'.

Proof. Without loss of generality, we may assume that the g_n^r is defined by a linear system Σ of dimension r, having finite intersection with C. Let W be the weighted set defined by ϕ_λ in Σ. Suppose that $s = 0$, then γ_p is not a base branch for Σ. Hence, by Lemma 5.25 of [9], we can find $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, and distinct $\{p_1, \ldots, p_t\} = C^{ext} \cap \phi_{\lambda'} \cap (\gamma_p \setminus p)$ such that the intersections at these points are transverse. Let W' be the weighted set defined by $\phi_{\lambda'}$ in $g_n^{r,ext}$. By the proof of $(*)$ in Lemma 1.10, we have that $sp(W') = W$. By the construction of sp in Definition 1.9, we have that $sp^{-1}(t\gamma_p)$ consists of the distinct branches $\{\gamma_{p_1},\ldots,\gamma_{p_t}\}$, each counted once in W'. If $s\geq 1$, then γ_p is a base branch for Σ. By Lemma 5.27 of [9], we have that $I_{italian}^{\Sigma,mobile}(p,\gamma_p,C,\phi_\lambda)=t-s$. The result then follows by application of Lemma 5.28 in [9] and the argument given above. □

We now note the following;

Lemma 1.13. *Let a g_n^r be given on a projective algebraic curve C. Let W_0 be* any *weighted set on C with total multiplicity n'. Then the collection of weighted sets given by $\{W\cup W_0\}$ for the weighted sets W in the g_n^r defines a $g_{n+n'}^r$.*

Proof. Let the original g_n^r be obtained from a linear system Σ of dimension r and degree n'', having finite intersection with C, after removing some fixed point contribution J of total multiplicity $n''-n$. Let $\{\phi_0,\ldots,\phi_r\}$ be a basis for Σ and let $\{n_1\gamma_{p_1}^{j_1},\ldots,n_m\gamma_{p_m}^{j_m}\}$ be the branches appearing in W_0 with total multiplicity $n_1+\ldots+n_m=n'$ (†). Let $\{H_1,\ldots,H_m\}$ be hyperplanes passing through the points $\{p_1,\ldots,p_m\}$ and let G be the algebraic form of degree n' defined by $H_1^{n_1}\centerdot\ldots\centerdot H_m^{n_m}$. Let Σ' be the linear system of dimension r defined by the basis $\{G\centerdot\phi_0,\ldots,G\centerdot\phi_r\}$. As we may assume that C is not contained in any hyperplane section, Σ' has finite intersection with C. We claim that $g_{n''}^r(\Sigma)\subset g_{n''+n'deg(C)}^r(\Sigma')$, in the sense that every weighted set W_λ defined by $g_{n''+n'deg(C)}^r(\Sigma')$ is obtained from the corresponding V_λ in $g_{n''}^r(\Sigma)$ by adding a *fixed* weighted set $W_1\supset W_0$ of total multiplicity $n'deg(C)$ $(*)$. The proof then follows as we can recover the original g_n^r by removing the fixed point contribution $J\cup(W_1\setminus W_0)$ from $g_{n''+n'deg(C)}^r(\Sigma')$. In order to show $(*)$, let W_1 be the weighted set defined by $C\sqcap G$. By the branched version of the Hyperspatial Bezout Theorem, see Theorem 5.13 of [9], this has total multiplicity $n'deg(C)$. We claim that $W_0\subset W_1$ $(**)$. Let γ_p^j be a branch appearing in (†) with multiplicity s. By construction, we can factor G as $H^s\centerdot R$, where H is a hyperplane passing through s. We need to show that;

$$I_{\gamma_p^j}(C,H^s\centerdot R)\geq s$$

or equivalently,

$$I_{p_j}(C^{ns},\overline{H^s\centerdot R})=I_{p_j}(C^{ns},\overline{H}^s\centerdot\overline{R})\geq s$$

for a suitable presentation C^{ns} of a non-singular model of C, see Lemma 5.12 of [9], where we have used the "lifted" form notation there. Using the method of conic projections, see section 4 of [9], we can find a plane projective curve C' birational to C^{ns}, such that the point p_j corresponds to a non-singular point q of C' and;

$$I_{p_j}(C^{ns},\overline{H}^s\,\centerdot\,\overline{R}) = I_q(C',\overline{\overline{H}^s\,\centerdot\,\overline{R}}) = I_q(C',\overline{\overline{H}}^s\,\centerdot\,\overline{\overline{R}})$$

The result then follows by results of the paper [8] for the intersections of plane projective curves. This shows $(**)$. We now need to prove that, for an algebraic form ϕ_λ in Σ and a branch γ_p^j of C;

$$I_{\gamma_p^j}(C,\phi_\lambda\,\centerdot\,G) = I_{\gamma_p^j}(C,\phi_\lambda) + I_{\gamma_p^j}(C,G)$$

This follows by exactly the same argument, reducing to the case of intersections between plane projective curves and using the results of [8]. The result is then shown.

□

Theorem 1.14. *Birational Invariance of a g_n^r*

Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map between projective algebraic curves. Then, given a g_n^r on C_2, there exists a canonically defined g_n^r on C_1, depending only on the class $[\Phi]$ of the birational map. Conversely, given a g_n^r on C_1, there exists a canonically defined g_n^r on C_2, depending only on the class $[\Phi^{-1}]$ of the birational map. Moreover, these correspondences are inverse.

Proof. By Lemma 5.7 of [9], $[\Phi]$ induces a bijection;

$$[\Phi]^* : \bigcup_{O\in C_2}\gamma_O \rightarrow \bigcup_{O\in C_1}\gamma_O$$

of branches, with inverse given by $[\Phi^{-1}]^*$.

Then $[\Phi]^*$ extends naturally to a map on weighted sets of degree n by the formula;

$$[\Phi]^*(n_1\gamma_{p_1}^{j_1} + \ldots + n_r\gamma_{p_r}^{j_r}) = n_1[\Phi]^*(\gamma_{p_1}^{j_1}) + \ldots + n_r[\Phi]^*(\gamma_{p_r}^{j_r})$$

for a linear combination of branches $\{\gamma_{p_1}^{j_1},\ldots,\gamma_{p_r}^{j_r}\}$ with $n = n_1 + \ldots + n_r$. Therefore, given a g_n^r on C_2, we obtain a canonically defined collection $[\Phi]^*(g_n^r)$ of weighted sets on C_1 of degree n $(*)$. It is trivial to see that $[\Phi^{-1}]^* \circ [\Phi]^*(g_n^r)$ recovers the original g_n^r on C_2, by the fact the map $[\Phi]^*$ on branches is invertible, with inverse given by $[\Phi^{-1}]^*$. Let C^{ns} be a non-singular model of C_1 and C_2 with morphisms $\Phi_1 : C^{ns} \rightarrow C_1$ and $\Phi_2 : C^{ns} \rightarrow C_2$ such that $\Phi \circ \Phi_1 = \Phi_2$ and $\Phi^{-1} \circ \Phi_2 = \Phi_1$ as birational maps (see the proof of Lemma 5.7 in [9]). We then have that $[\Phi]^*(g_n^r) = [\Phi_1^{-1}]^* \circ [\Phi_2]^*(g_n^r)$. It remains to prove that this collection given by $(*)$ defines a g_n^r on C_1. We will prove first that $[\Phi_2]^*(g_n^r)$ defines a g_n^r on C^{ns} $(\dagger)$. Let the original g_n^r on C_2 be defined by a linear system Σ, having finite intersection with C_2, such that $dimension(\Sigma) = r$ and $degree(\Sigma) = n'$, after removing some fixed branch contribution of multiplicity $n' - n$. We may assume that $n' = n$, as if the fixed branch contribution in question is given by W_0 and $g_n^r \cup W_0 = g_{n'}^r$, then $[\Phi_2]^*(g_n^r) \cup [\Phi_2]^*(W_0) = [\Phi_2]^*(g_{n'}^r)$, hence it is sufficient to prove that $[\Phi_2]^*(g_{n'}^r)$ defines a $g_{n'}^r$. Let W_1 be the fixed branch contribution of the g_n^r on C_2 and let $g_{n''}^r \subset g_n^r$ be obtained by removing this fixed branch contribution. It will be sufficient to prove that $[\Phi_2]^*(g_{n''}^r)$ defines a $g_{n''}^r$ on C^{ns} as $[\Phi_2]^*(g_n^r) = [\Phi_2]^*(g_{n''}^r) \cup [\Phi_2]^*(W_1)$ and we may then use Lemma 1.13. Let Φ_{Σ_1} and Φ_{Σ_2} be presentations of the morphisms Φ_1 and Φ_2. We may assume that $Base(\Sigma_1)$ and $Base(\Sigma_2)$ are disjoint. Let $\{\overline{\phi_\lambda}\}$ denote the lifted family of forms on C^{ns}, defined by the linear system Σ and the presentation Φ_{Σ_2}. We claim that $[\Phi_2]^*(g_{n''}^r)$ is defined by this system after removing its fixed branch contribution. In order to see this, we first show that for any branch γ_p^j of C;

$$I_{\gamma_p^j}^{\Sigma,mobile}(C, \phi_\lambda) = I_{p_j}^{\Sigma,mobile}(C^{ns}, \overline{\phi_\lambda}) \ (*) \ (1)$$

where p_j corresponds to γ_p^j in the fibre $\Gamma_{[\Phi_2]}(x, p)$, see Section 5 of [9]. By Definition 2.20 and Lemma 5.23 of [9], we have that;

$$I_{p_j}^{\Sigma,mobile}(C^{ns}, \overline{\phi_\lambda}) = Card(C^{ns} \cap (\mathcal{V}_{p_j} \setminus p_j) \cap \overline{\phi_{\lambda'}}) \text{ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } Par_\Sigma$$

$$I_{\gamma_p^j}^{\Sigma,mobile}(C, \phi_\lambda) = Card(C \cap (\gamma_p^j \setminus p) \cap \phi_{\lambda'}) \text{ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } Par_\Sigma$$

As $(\gamma_p^j \setminus p)$ is in biunivocal correspondence with $(\mathcal{V}_{p_j} \setminus p_j)$ under the morphism Φ_2, we obtain immediately the result $(*)$. Now, using Lemma 5.27 of [9], we have that, if γ_p^j appears in a weighted set W_λ of the $g_{n''}^r$ with multiplicity s, then the corresponding branch γ_{p_j} appears in the weighted set $[\Phi_2]^*(W_\lambda)$ with multiplicity equal to $s = I_{p_j}^{mobile}(C^{ns}, \overline{\phi_\lambda})$. Again, using Lemma 5.27 of [9], we obtain that $[\Phi_2]^*(W_\lambda)$ is given by $C^{ns} \sqcap \overline{\phi_\lambda}$, after removing all fixed point contributions of the linear system Σ. We, therefore, obtain that $[\Phi_2]^*(g_{n''}^r)$ is defined by Σ, after removing all fixed point contributions, as required. This proves $(\dagger)$.

We now claim that, for the given g_n^r on C^{ns}, $[\Phi^{-1}]^*(g_n^r)$ defines a g_n^r on C_1, (††). Let Φ_{Σ_3} be a presentation of the morphism Φ^{-1}. If ϕ_λ is a form belonging to the linear system Σ defined on C^{ns}, using the presentations Φ_{Σ_1} and Φ_{Σ_3} of Φ and Φ^{-1}, we obtain a lifted form $\overline{\phi_\lambda}$ on C_1 and a lifted form $\overline{\overline{\phi_\lambda}}$ on C^{ns} again. We now claim that, for $p \in C^{ns}$;

$$I_p^{\Sigma,mobile}(C^{ns}, \phi_\lambda) = I_p^{\Sigma,mobile}(C^{ns}, \overline{\overline{\phi_\lambda}}) \; (2)$$

In order to see this, first observe that we can obtain the lifted system of forms $\{\overline{\overline{\phi_\lambda}}\}$ directly from the linear system Σ_4, obtained by composing bases of the linear systems Σ_1 and Σ_3. The corresponding morphism Φ_{Σ_4} defines a birational map of C^{ns} to itself, which is equivalent to the identity map Id. Now the result follows immediately from Definition 2.20 and Lemma 2.16 of [9], both multiplicities are witnessed inside the canonical set W of Φ_{Σ_4}, which, in this case, is just the domain of definition of Φ_{Σ_4} on C^{ns}, see Definition 1.30 of [9]. Now, returning to the proof of (††), we may suppose that the given g_n^r on C^{ns} is defined by the linear system Σ, after removing all fixed point contributions. Combining (1) and (2), we have that, for a branch γ_p^j of C_1;

$$I_{\gamma_p^j}^{\Sigma,mobile}(C_1, \overline{\phi_\lambda}) = I_{p_j}^{\Sigma,mobile}(C^{ns}, \overline{\overline{\phi_\lambda}}) = I_{p_j}^{\Sigma,mobile}(C^{ns}, \phi_\lambda)$$

The result now follows from the same argument as above, using Lemma 5.27 of [9]. This completes the theorem.

Remarks 1.15. *Using the quoted Theorem 1.33 of* [9], *one can use the Theorem to reduce calculations involving g_n^r on projective algebraic curves to calculations on plane projective curves. We will use this property extensively in the following sections.*

□

We finally note the following;

Lemma 1.16. *For a given g_n^r, we always have that $r \leq n$.*

Proof. The proof is almost identical to Lemma 2.24 of [9]. We leave the details to the reader.

□

2. A Theory of Complete Linear Series on an Algebraic Curve

We now develop further the theory of g_n^r on an algebraic curve C, analogously to classical results for divisors on non-singular algebraic curves. We will first assume that C is a plane projective algebraic curve, defined by some homogeneous polynomial $F(X,Y,Z)$. Without loss of generality, we will use the coordinates $x=X/Z$ and $y=Y/Z$ for local calculations on the curve C, defined in this system by $f(x,y)=0$. Using Theorem 1.14, we will later derive general results for g_n^r on an algebraic curve from the corresponding calculations for the plane case.

We consider first the case when $r=1$. By results of the previous section, a g_n^1 is defined by a pencil Σ of algebraic curves $\{\phi(x,y)+\lambda\phi'(x,y)=0\}_{\lambda\in P^1}$ (in affine coordinates), after removing some fixed point contribution, where, by convention, we interpret the algebraic curve $\phi(x,y)+\infty\phi'(x,y)=0$ to be $\phi'(x,y)=0$. We assume that the g_n^1 is, in fact, cut out by this pencil. Now suppose that γ_p is a branch of C. We may assume that p corresponds to the origin O of the affine coordinate system (x,y), (use a linear transformation and the result of Lemma 2.1) By Theorem 6.1 of [9], we can find algebraic power series $\{x(t),y(t)\}$, with $x(t)=y(t)=0$, parametrising γ_p. We can now substitute the power series in order to obtain a formal expression of the form;

$\frac{\phi(x(t),y(t))}{\phi'(x(t),y(t))}=\frac{t^iu(t)}{t^jv(t)}=t^{i-j}u(t)v(t)^{-1}$, where $\{u(t),v(t),u(t)v(t)^{-1}\}$ are units in $L[[t]]$.

We then define;

$ord_{\gamma_p}(\frac{\phi}{\phi'})=i-j$,

$val_{\gamma_p}(\frac{\phi}{\phi'})=0$, if $i>j$, ($\frac{\phi}{\phi'}$ has a zero of order $i-j$)

$ord_{\gamma_p}(\frac{\phi}{\phi'})=j-i$,

$val_{\gamma_p}(\frac{\phi}{\phi'})=\infty$, if $i<j$, ($\frac{\phi}{\phi'}$ has a pole of order $j-i$)

$ord_{\gamma_p}(\frac{\phi}{\phi'})=ord_t(h(t)-h(0))$,

$val_{\gamma_p}(\frac{\phi}{\phi'})=h(0)$, if $i=j$ and $h(t)=u(t)v(t)^{-1}$

Observe that in all cases, ord_{γ_p} gives a *positive* integer, while val_{γ_p} determines an element of P^1. In order to see that this construction does not depend on the particular power series representation of the branch, we require the following lemma;

Lemma 2.1. *Let $\{C, \gamma_p, \phi, \phi', g_n^1, \Sigma\}$ be as defined above, then;*

$ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}(C, \phi - \lambda\phi')$, *if γ_p is not a base branch for the g_n^1 and $\frac{\phi}{\phi'}(p) = val_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda$.*

$ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}^{\Sigma,mobile}(C, \phi - \lambda\phi')$, *if γ_p is a base branch for the g_n^1 and $\lambda = val_{\gamma_p}(\frac{\phi}{\phi'})$ is unique such that, for $\mu \neq \lambda$;*
$I_{\gamma_p}(C, \phi - \lambda\phi') > I_{\gamma_p}(C, \phi - \mu\phi')$.

Proof. Suppose that γ_p is not a base branch for the g_n^1, then $\frac{\phi}{\phi'}(p) = \lambda$ is well defined, if we interpret $(c/0) = \infty$ for $c \neq 0$, and $\phi - \lambda\phi'$ is the unique curve in the pencil passing through p. It is trivial to check, using the facts that $\phi(p) = \phi(x(0), y(0))$ and $\phi'(p) = \phi'(x(0), y(0))$, that, in all cases, $val_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda$ as well. By Theorem 6.1 of [9], we have that;

$$I_{\gamma_p}(C, \phi - \lambda\phi') = ord_t[(\phi - \lambda\phi')(x(t), y(t))]$$

If $\lambda = 0$, then $\phi(p) = 0$ and $\phi'(p) \neq 0$, hence, by a straightforward algebraic calculation, $\phi(x(t), y(t)) = t^i u(t)$, for some $i \geq 1$, and $\phi'(x(t), y(t)) = v(t)$ for $\{u(t), v(t)\}$ units in $L[[t]]$. Therefore, $ord_{\gamma_p}(\frac{\phi}{\phi'}) = ord_t\phi(x(t), y(t))$ and the result follows.

If $\lambda = \infty$, then $\phi(p) \neq 0$ and $\phi(p) = 0$, hence, $\phi(x(t), y(t)) = u(t)$ and $\phi'(x(t), y(t)) = t^j v(t)$, for some $j \geq 1$, and $\{u(t), v(t)\}$ units in $L[[t]]$. Therefore, $ord_{\gamma_p}(\frac{\phi}{\phi'}) = ord_t\phi'(x(t), y(t))$ and the result follows.

If $\lambda \neq \{0, \infty\}$, then $\phi(x(t), y(t)) = u(t)$ and $\phi'(x(t), y(t)) = v(t)$ with $\{u(t), v(t)\}$ units in $L[[t]]$. As $v(t)$ is a unit in $L[[t]]$, we have that;

$$ord_t(\frac{u(t)}{v(t)} - \frac{u(0)}{v(0)}) = ord_t(v(t)(\frac{u(t)}{v(t)} - \frac{u(0)}{v(0)})) = ord_t(u(t) - \frac{u(0)}{v(0)}v(t))$$

Hence, by definition of ord_{γ_p};

$ord_{\gamma_p}(\frac{\phi}{\phi'}) = ord_t[(\phi - \lambda\phi')(x(t), y(t))]$

and the result follows.

Now suppose that γ_p is a base branch for the g_n^1, then $\phi(p) = \phi'(p) = 0$ and we have that $\phi(x(t), y(t)) = t^i u(t)$ and $\phi'(x(t), y(t)) = t^j v(t)$, for some $i, j \geq 1$ and $\{u(t), v(t)\}$ units in $L[[t]]$. Again, we divide the proof into the following cases;

$i > j$. In this case, by definition, $val_{\gamma_p}(\frac{\phi}{\phi'}) = 0$. We compute;

$ord_t(\phi(x(t), y(t)) - \lambda\phi'(x(t), y(t))) = ord_t(t^i u(t) - \lambda t^j v(t))$

When $\lambda = 0$, we obtain, by Theorem 6.1 of [9], that $I_{\gamma_p}(C, \phi) = i$ and, for $\lambda \neq 0$, that $I_{\gamma_p}(C, \phi - \lambda\phi') = j$. Using Lemma 5.27 of [9], we obtain that $I_{\gamma_p}^{\Sigma, mobile}(C, \phi) = i - j = ord_{\gamma_p}(\frac{\phi}{\phi'})$, as required.

$i < j$. In this case, by definition, $val_{\gamma_p}(\frac{\phi}{\phi'}) = \infty$. The computation for ord_{γ_p} is similar, with the critical value being $\lambda = \infty$.

$i = j$. We compute;

$ord_t(\phi(x(t), y(t)) - \lambda\phi'(x(t), y(t))) = ord_t[t^i(u(t) - \lambda v(t))]$

Again, there exists a unique value of $\lambda = \frac{u(0)}{v(0)} = val_{\gamma_p}(\frac{\phi}{\phi'}) \neq \{0, \infty\}$ such that $ord_t(u(t) - \lambda v(t)) = k \geq 1$. By the same calculation as above, we have that $I_{\gamma_p}^{\Sigma, mobile}(C, \phi - \lambda\phi') = k$, for this critical value of λ. By a similar algebraic calculation to the above, using the fact that $v(t)$ is a unit, we also compute $ord_{\gamma_p}(\frac{\phi}{\phi'}) = k$, hence the result follows.

□

We now show the following;

Lemma 2.2. *Given any algebraic curve $C \subset P^w$, with function field $L(C)$, for a non-constant rational function $f \in L(C)$ and a branch γ_p, we can unambiguously define $ord_{\gamma_p}(f)$ and $val_{\gamma_p}(f)$.*

Proof. The proof is similar to the above. We may, without loss of generality, assume that p corresponds to the origin of a coordinate system $(x_1, \ldots, x_w)$. Using Theorem 6.1 of [9], we can find algebraic power series $(x_1(t), \ldots, x_w(t))$ parametrising the branch γ_p. By the assumption that f is non-constant, we can find a representation of f as a rational function $\frac{\phi(x_1,\ldots,x_w)}{\phi'(x_1,\ldots,x_w)}$ in this coordinate system, such that the pencil Σ defined by $\{\phi,\phi'\}$ has finite intersection with C, hence defines a g_n^1. Using the method above, we can define $ord_{\gamma_p}(\frac{\phi}{\phi'})$ and $val_{\gamma_p}(\frac{\phi}{\phi'})$ for this representation. The proof of Lemma 2.1 shows that these are defined independently of the particular power series parametrising the branch. We need to check that they are also defined independently of the particular representation of f. Suppose that $\{\phi_1,\phi_2,\phi_3,\phi_4\}$ are algebraic forms with the property that $\frac{\phi_1}{\phi_2}=\frac{\phi_3}{\phi_4}$ as rational functions on C. We claim that, for any branch γ_p of C, $ord_{\gamma_p}(\frac{\phi_1}{\phi_2})=ord_{\gamma_p}(\frac{\phi_3}{\phi_4})$ and $val_{\gamma_p}(\frac{\phi_1}{\phi_2})=val_{\gamma_p}(\frac{\phi_3}{\phi_4})$, $(*)$. In order to see this, let $U\subset NonSing(C)$ be an open subset of C, on which $\frac{\phi_1}{\phi_2}$ and $\frac{\phi_3}{\phi_4}$ are defined and equal. Let g_n^1 and g_m^1 on C be defined by the pencils $\Sigma_1=\{\phi_1-\lambda\phi_2\}_{\lambda\in P^1}$ and $\Sigma_2=\{\phi_3-\lambda\phi_4\}_{\lambda\in P^1}$. Let $V=U\setminus Base(\Sigma_1)\cup Base(\Sigma_2)$. Then $V\subset U$ is also an open subset of C, which we will refer to as the canonical set. Now, suppose that $\gamma_p\subset V$. We will prove $(*)$ for this branch. As both $\frac{\phi_1}{\phi_2}$ and $\frac{\phi_3}{\phi_4}$ are defined and equal at p, using the argument in Lemma 2.1, we have that $val_{\gamma_p}(\frac{\phi_1}{\phi_2})=val_{\gamma_p}(\frac{\phi_3}{\phi_4})$. It is therefore sufficient, again by Lemma 2.1, to show that;

$$I_{\gamma_p}(C,\phi_1-\lambda\phi_2)=I_{\gamma_p}(C,\phi_3-\lambda\phi_4),\text{ for }\frac{\phi_1}{\phi_2}(p)=\frac{\phi_3}{\phi_4}(p)=\lambda\ (\dagger)$$

Suppose that $I_{\gamma_p}(\phi_1-\lambda\phi_2)=m$, then, by Lemma 5.25 of [9], we can find $\lambda'\in\mathcal{V}_\lambda\cap P^1$ and $\{p_1,\ldots,p_m\}=V\cap\mathcal{V}_p\cap(\phi_1-\lambda'\phi_2)=0$ witnessing this multiplicity. As $\{p,p_1,\ldots,p_m\}$ lie inside V, we also have that $\{p_1,\ldots,p_m\}\subset V\cap\mathcal{V}_p\cap(\phi_3-\lambda'\phi_4)=0$, hence $I_{\gamma_p}(C,\phi_3-\lambda\phi_4)\geq m$. The result $(\dagger)$ then follows from the converse argument.

Now, suppose that γ_p is one of the finitely many branches of C, not lying inside V. We will just consider the case when γ_p is a base branch for both the g_n^1 and the g_m^1 defined above, the other cases being similar. In order to prove $(*)$ for this branch, it is sufficient, by Lemma 2.1, to show that;

$$I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\lambda\phi_2)=I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\mu\phi_4),\text{ for the critical values }\{\lambda,\mu\}$$

and that the critical values $\{\lambda,\mu\}$ coincide, $(\dagger\dagger)$.

Using the argument to prove (†), witnessing the corresponding multiplicities in the canonical set V, it follows that for *any* $\nu \in P^1$;

$$I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\nu\phi_2)=I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\nu\phi_4),\ (\dagger\dagger\dagger)$$

If the critical values $\{\lambda,\mu\}$ were distinct, we would have that;

$$\begin{array}{ccc} I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\lambda\phi_2) & > & I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\mu\phi_2) \\ || & & || \\ I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\lambda\phi_4) & < & I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\mu\phi_4) \end{array}$$

which is clearly a contradiction. Hence, $\lambda=\mu$ and the result (††) follows from (†††). The lemma is shown. □

Lemma 2.3. *Birational Invariance of ord_{γ_p} and val_{γ_p}*

Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map between projective algebraic curves with corresponding isomorphisms $\Phi^* : L(C_2) \rightarrow L(C_1)$ and $[\Phi]^* : \bigcup_{p\in C_2}\gamma_p \rightarrow \bigcup_{q\in C_1}\gamma_q$. Then, for non-constant $f \in L(C_2)$ and γ_p a branch of C_2, $ord_{\gamma_p}(f) = ord_{[\Phi]^*\gamma_p}(\Phi^* f)$ and $val_{\gamma_p}(f) = val_{[\Phi]^*\gamma_p}(\Phi^* f)$.

Proof. Let f be represented as a rational function by $\frac{\phi_1}{\phi_2}$, as in Lemma 2.2, and consider the g_n^1 on C_2, defined by the linear system $\Sigma = \{\phi_1-\lambda\phi_2\}_{\lambda\in P^1}$. Let Φ_{Σ_1} be a presentation of the birational map Φ. Using this presentation, we may lift the system Σ to a corresponding linear system $\{\overline{\phi_1}-\lambda\overline{\phi_2}\}_{\lambda\in P^1}$. It is trivial to check that $\Phi^* f$ is represented by the rational function $\frac{\overline{\phi_1}}{\overline{\phi_2}}$. The proof of Theorem 1.14 shows that, for a branch γ_p of C_2;

$$I_{\gamma_p}^{\Sigma,mobile}(C_2,\phi_1-\lambda\phi_2)=I_{[\Phi]^*\gamma_p}^{\Sigma,mobile}(C_1,\overline{\phi_1}-\lambda\overline{\phi_2}),\ (*)$$

We now need to consider the following cases;

Case 1. γ_p and $[\Phi]^*\gamma_p$ are not base branches for Σ on C_2 and C_1.

Case 2. γ_p is not a base branch, but $[\Phi]^*\gamma_p$ is a base branch for Σ on C_2 and C_1.

Case 3. γ_p is a base branch and $[\Phi]^*\gamma_p$ is a base branch for Σ on C_2 and C_1.

For Case 1, we have, by Lemma 2.1 and $(*)$;

$$ord_{\gamma_p}(\frac{\phi_1}{\phi_2}) = I_{\gamma_p}(C_2, \phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}(C_1, \overline{\phi_1} - \lambda\overline{\phi_2}) = ord_{[\Phi]^*\gamma_p}(\overline{\frac{\phi_1}{\phi_2}})$$

where $\frac{\phi_1}{\phi_2}(p) = \overline{\frac{\phi_1}{\phi_2}}(q) = val_{\gamma_p}(\frac{\phi_1}{\phi_2}) = val_{\gamma_q}(\overline{\frac{\phi_1}{\phi_2}}) = \lambda$ and $[\Phi]^*\gamma_p = \gamma_q$.

For Case 3, we have, by Lemma 2.1, $(*)$ and a similar argument to the previous lemma, to show that the critical value $\lambda = val_{\gamma_p}(\frac{\phi_1}{\phi_2})$ is also the critical value $val_{\gamma_q}(\overline{\frac{\phi_1}{\phi_2}})$ for the lifted system at the corresponding branch $[\Phi]^*\gamma_p$, that;

$$ord_{\gamma_p}(\frac{\phi_1}{\phi_2}) = I^{\Sigma,mobile}_{\gamma_p}(C_2, \phi_1 - \lambda\phi_2) = I^{\Sigma,mobile}_{[\Phi]^*\gamma_p}(C_1, \overline{\phi_1} - \lambda\overline{\phi_2}) = ord_{[\Phi]^*\gamma_p}(\overline{\frac{\phi_1}{\phi_2}})$$

Case 2 is similar, we leave the details to the reader.

The lemma now follows from the previous lemma, that the definitions of $ord_{\gamma_p}(f)$, $ord_{[\Phi]^*\gamma_p}(\Phi^* f)$,$val_{\gamma_p}(f)$ and $val_{[\Phi]^*\gamma_p}(\Phi^* f)$ are independent of their particular representations.

□

We now show;

Lemma 2.4. *Let C be a projective algebraic curve, then, to any non-constant rational function f on C, we can associate a g^1_n on C, which we will denote by (f), where $n = deg(f)$, (flatness?).*

Proof. We define the weighted set $(f = \lambda)$ as follows;

$$(f = \lambda) := \{n_{\gamma_1}, \ldots, n_{\gamma_r}\}$$

where $\{\gamma_1, \ldots, \gamma_r\} = \{\gamma : val_\gamma(f) = \lambda\}$ and $n_\gamma = ord_\gamma(f)$.

As λ varies over P^1, we obtain a series of weighted sets W_λ on C. We claim that this series does in fact define a g^1_n. In order to see this, let f be represented as a rational function by $\frac{\phi}{\phi'}$. As before, we consider the pencil Σ of forms defined by $(\phi - \lambda\phi')_{\lambda \in P^1}$. We claim that the series is defined by this system Σ, after removing its fixed branch contribution, $(*)$. In order to see this, we compare the weighted sets $(f = \lambda)$ and $C \sqcap (\phi - \lambda\phi')$. For a branch γ_p which is not a fixed branch of the system Σ, we have, using Lemmas 2.1 and 2.2, that;

$\gamma_p \in (f=\lambda)$ iff $val_{\gamma_p}(f)=\lambda$ iff $\frac{\phi}{\phi'}(p)=\lambda$ iff $p \in C \cap (\phi-\lambda\phi')$

In this case, by Lemmas 2.1 and 2.2, we have that;

$n_{\gamma_p} = ord_{\gamma_p}(f) = ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}(C, \phi-\lambda\phi')$

For a branch γ_p which is a fixed branch of the system Σ, we have, by Lemmas 2.1 and 2.2, that;

$\gamma_p \in (f=\lambda)$ iff $val_{\gamma_p}(\frac{\phi}{\phi'})=\lambda$ iff $p \in C \cap (\phi-\lambda\phi')$ and λ is a critical value for the system Σ at γ_p.

In this case, by Lemmas 2.1 and 2.2, we have that;

$n_{\gamma_p} = ord_{\gamma_p}(f) = ord_{\gamma_p}(\frac{\phi}{\phi'}) = I^{\Sigma,mobile}_{\gamma_p}(C, \phi-\lambda\phi')$ (1)

Let $I_{\gamma_p} = min_{\mu \in P^1} I_{\gamma_p}(C, \phi-\mu\phi')$ be the fixed branch contribution of Σ at γ_p. Then, at the critical value λ for the system Σ;

$I^{\Sigma,mobile}_{\gamma_p}(\phi-\lambda\phi') = I_{\gamma_p}(C, \phi-\lambda\phi') - I_{\gamma_p}$ (2)

Hence, the result $(*)$ follows from $(1), (2)$ and the definition of $C \sqcap (\phi-\lambda\phi')$.

Finally, we show that $n = deg(f)$. Let Γ_f be the correspondence determined by the rational map $f : C \rightsquigarrow P^1$. By classical arguments, $deg(f)$ is equal to the cardinality of the generic fibre $\Gamma_f(\lambda)$, for $\lambda \in P^1$. Fixing a presentation $\frac{\phi}{\phi'}$ for f, if $U \subset NonSing(C)$ is the canonical set for this presentation, one may assume that the generic fibre $\Gamma_f(\lambda)$ lies inside U. By Lemma 2.17 of [9], one may also assume that the corresponding weighted set of the g^1_n defined by $(f=\lambda)$ consists of n distinct branches, centred at the points of the generic fibre $\Gamma_f(\lambda)$. Therefore, the result follows.

□

Remarks 2.5. *By convention, for a non-zero rational function $c \in L \setminus \{0\}$, we define $(c=0)$ and $(c=\infty)$ to be the empty weighted sets. The notion of a weighted set in a g^1_n, generalises the classical notion of the divisor on a non-singular curve. Using the above theorem, we can make sense of the notion of linear equivalence of weighted sets.*

We make the following definition;

Definition 2.6. *Linear equivalence of weighted sets*

Let C be an algebraic curve and let A and B be weighted sets on C of the same total multiplicity. We define $A \equiv B$ if there exists a g_n^r on C such that A and B belong to this g_n^r as weighted sets.

Theorem 2.7. *Let hypotheses be as in the previous definition. If $A \equiv B$, then there exists a rational function g on C, such that A is defined by $(g = 0)$ and B is defined by $(g = \infty)$, possibly after adding some fixed branch contribution.*

Proof. If $r = 0$ in the definition, then we must have that $A = B$. Hence, we obtain the statement of the theorem by adding the fixed branch contribution A to the empty g_0^0, defined by $(c = 0) = (c = \infty)$, for a non-constant $c \in L^*$. Otherwise, by the definition of a g_n^r, we may, without loss of generality, find a pencil Σ of algebraic forms, $\{\phi - \lambda\phi'\}_{\lambda \in P^1}$, having finite intersection with C, such that;

$$A = C \sqcap (\phi - \lambda_1\phi),$$

$$B = C \sqcap (\phi - \lambda_2\phi') \qquad (\lambda_1 \neq \lambda_2)$$

Let f be the rational function on C defined by $\frac{\phi}{\phi'}$. If A and B have no branches in common (with multiplicity), (†), then the pencil Σ can have no fixed branches and, by Lemma 2.4, we have that;

$$A = (f = \lambda_1)$$

$$B = (f = \lambda_2) \qquad (\lambda_1 \neq \lambda_2)$$

Now we can find an algebraic automorphism α of P^1, taking λ_1 to 0 and λ_2 to ∞. We will assume that $\{\lambda_1, \lambda_2\} \neq \infty$, in which case α can be given, for a coordinate z on P^1, by the Mobius transformation $\frac{z-\lambda_1}{z-\lambda_2}$. The other cases are left to the reader. Let g be the rational function on C defined by $\alpha \circ f$. Now, suppose that γ is a branch of C, with $val_\gamma(f) = \lambda$ and $ord_\gamma(f) = m$. Then, we claim that $val_\gamma(g) = \alpha(\lambda)$ and $ord_\gamma(g) = m$, $(*)$. If $\lambda \neq \{\lambda_2, \infty\}$, using the method before Lemma 2.1, we obtain the following power series representation of g at γ;

$$\begin{aligned}\tfrac{(\lambda+\mu t^m+o(t^m))-\lambda_1}{(\lambda+\mu t^m+o(t^m))-\lambda_2} &= [(\lambda-\lambda_1)+\mu t^m+o(t^m)]\centerdot\tfrac{1}{(\lambda-\lambda_2)}[1-\tfrac{\mu}{(\lambda-\lambda_2)}t^m+o(t^m)] \\ &= \tfrac{\lambda-\lambda_1}{\lambda-\lambda_2}+t^m[\tfrac{\mu(\lambda-\lambda_2)-\mu(\lambda-\lambda_1)}{(\lambda-\lambda_2)^2}]+o(t^m)\end{aligned}$$

$$= \frac{\lambda-\lambda_1}{\lambda-\lambda_2} + t^m[\frac{\mu(\lambda_1-\lambda_2)}{(\lambda-\lambda_2)^2}] + o(t^m)$$

and the claim $(*)$ follows from the assumption that $\lambda_1 \neq \lambda_2$. If $\lambda = \lambda_2$, we obtain the following power series representation of g at γ;

$$\frac{(\lambda+\mu t^m+o(t^m))-\lambda_1}{(\mu t^m+o(t^m))} = \frac{1}{t^m} \cdot [(\lambda - \lambda_1) + \mu t^m + o(t^m)] \cdot [\mu + o(1)]^{-1}$$

which gives that $val_\gamma(g) = \infty = \alpha(\lambda_2)$ and $ord_\gamma(g) = m$, using the fact that $\lambda \neq \lambda_1$. Finally, if $\lambda = \infty$, the Mobius transformation at ∞ is given by $\frac{\frac{1}{z}-\lambda_1}{\frac{1}{z}-\lambda_2} = \frac{1-\lambda_1 z}{1-\lambda_2 z}$ and g may be represented at γ by $\frac{\phi-\lambda_1\phi'}{\phi-\lambda_2\phi'}$. We then obtain the power series representation of g at γ;

$$\frac{(t^i u(t)-\lambda_1 t^{i+m}v(t))}{(t^i u(t)-\lambda_2 t^{i+m}v(t))} = \frac{(u(t)-\lambda_1 t^m v(t))}{(u(t)-\lambda_2 t^m v(t))} = \frac{[1-\lambda_1 t^m \frac{v(t)}{u(t)}]}{[1-\lambda_2 t^m \frac{v(t)}{u(t)}]}$$
$$= 1 + (\lambda_2 - \lambda_1)t^m w(t) + o(t^m), \text{ for } \{u(t), v(t), w(t)\} \text{ units in } L[[t]]$$

which gives that $val_\gamma(g) = 1 = \alpha(\infty)$ and $ord_\gamma(g) = m$, using the fact that $\lambda_1 \neq \lambda_2$ again. This gives the claim $(*)$. It follows that the weighted sets $(f = \lambda)$ correspond exactly to the weighted sets $(g = \alpha(\lambda))$, in particularly the g_n^1 defined by (f) and (g), as in Lemma 2.4, is the same. With this new parametrisation of the g_n^1, we then have that;

$A = (g = 0)$

$B = (g = \infty)$

Hence, the result follows, with the assumption (†). If A and B have branches in common, with multiplicity, we let $A \cap B$ denote the weighted set consisting of these common branches (with multiplicity). Then, the same argument holds, replacing A by $A \setminus B = A - (A \cap B)$ and B by $B \setminus A = B - (A \cap B)$. After adding the fixed branch contribution $(A \cap B)$ to the g_n^1 defined by (g), we then obtain the result. Note that, by Lemma 1.13, this addition defines a $g_{n+n'}^1$, where n' is the total multiplicity of $(A \cap B)$.

□

Remarks 2.8. *The definition we have given of linear equivalence of weighted sets on a projective algebraic curve C generalises the modern* *definition of linear equivalence for effective divisors on a smooth projective algebraic curve. More precisely we have;*

Modern Definition; Let A and B be effective divisors on a smooth projective algebraic curve C, then $A \equiv B$ iff $A - B = div(g)$, for some $g \in L(C)^*$.

See, for example, p161 of [15] *for relevant definitions and notation. We now show that our definition is the same in this case. First, observe that there exists a natural bijection between the set of effective divisors on C, in the sense of* [15]*, and the collection of weighted sets on C, $(*)$. This follows immediately from the fact, given in Lemma 5.29 of* [9]*, that, for each point $p \in C$, there exists a unique branch γ_p, centred at p. Secondly, observe that the notion of $div(g)$, for $g \in L(C)$, as given in* [15]*, is the same as the notion of $div(g)$ which we give in Definition 2.9 below, (taking into account the identification $(*)$), $(\dagger)$. This amounts to checking that, for a point $p \in C$, with corresponding branch γ_p;*

$v_p(g) = ord_{\gamma_p}(g)$ $(\dagger\dagger)$

where $v_p(g)$ is defined in p152 of [15]*, and we temporarily adopt the convention that $ord_{\gamma_p}(g) = 0$ if $val_\gamma(g) \neq \{0, \infty\}$ and $ord_{\gamma_p}(g)$ is counted negatively if $val_{\gamma_p}(g) = \infty$. First, one can use the fact, given in Lemma 4.9 of* [9]*, together with remarks from the final section of this paper, that there exists a birational map $\phi : C \leftrightsquigarrow C'$, such that C' is a plane projective algebraic curve, and p corresponds to a non-singular point $p' \in C'$ with $\{p, p'\}$ lying inside the canonical sets associated to ϕ. Using the calculation given below, in Lemma 2.10, for ord_{γ_p}, and the definition of v_p, one can assume that $v_p(g) \geq 0$ and $g \in O_{p,C}$. Let $g' \in L(C')$ denote the corresponding rational function to g on $L(C)$. It is then a trivial algebraic calculation, using the fact that the local rings $O_{p,C}$ and $O_{p',C'}$ are isomorphic, to show that $v_p(g) = v_p(g')$. It also follows from Lemma 2.3 that $ord_{\gamma_p}(g) = ord_{\gamma_{p'}}(g')$. Hence, it is sufficient to check $(\dagger)$ for the plane projective curve C'. We may, without loss of generality, assume that $v_p(g') \geq 1$ and that g' is represented in some choice of affine coordinates $\{x, y\}$ by the polynomial $q(x, y)$. If $Q(X, Y, Z)$ denotes the projective equation of this polynomial and p corresponds to the origin of this coordinate system, then;*

$v_p(g') = I_p(C,Q) = length(\frac{L[x,y]}{<h,q>})$

where h is a defining equation for C' in the coordinate system $\{x,y\}$ and I_p is the algebraic intersection multiplicity. It also follows from Lemma 2.1, that;

$ord_{\gamma_p}(g') = I_{\gamma_p}(C,Q)$

Hence, it is sufficient to check that;

$I_p(C,Q) = I_{\gamma_p}(C,Q)$

This calculation was done in the paper [8]*, hence* $(\dagger\dagger)$ *and therefore* $(\dagger)$ *is shown. Thirdly, it remains to check that the definitions of linear equivalence are the same. In order to see this, observe that we can write (for effective divisors or weighted sets A and B);*

$A-B = (A\setminus B)+(A\cap B)]-[(B\setminus A)+(A\cap B)] = (A\setminus B)-(B\setminus A)$, $(\dagger\dagger\dagger)$

If $A \equiv B$ in the sense of weighted sets (Definition 2.6), then the calculation $(\dagger\dagger\dagger)$ *(which removes the fixed branch contribution) and Theorem 2.7 shows that $A - B = div(g)$, for some rational function $g \in L(C)$, where, here, $div(g)$ is as defined in Definition 2.9. By* $(\dagger)$*, it then follows that $A \equiv B$ as effective divisors. Conversely, if $A \equiv B$ as effective divisors, then there exists a rational function $g \in L(C)$ such that $A - B = div(g)$, in the sense of the modern definition given above. The above calculations* $(\dagger\dagger\dagger)$ *and* $(\dagger)$ *then show that $div(g) = (A \setminus B) - (B \setminus A)$, in the sense of Definition 2.9 below. It follows, by Lemma 2.4, that there exists a g^1_n to which $(A \setminus B)$ and $(B \setminus A)$ belong as weighted sets. Adding the fixed branch contribution $(A\cap B)$ to this g^1_n, we then obtain that $A \equiv B$ in the sense of Definition 2.6, as required.*

Definition 2.9. *Let C be a projective algebraic curve and let f be a non-zero rational function on C. Then we define $div(f)$ to be the weighted set $A - B$ where;*

$A = (f = 0), \quad B = (f = \infty)$

We now require the following lemma;

Lemma 2.10. *Let C be a projective algebraic curve, and let f and g be non-zero rational functions on C. Then;*

$div(\frac{1}{f}) = -div(f)$

$div(fg) = div(f) + div(g)$

$div(\frac{f}{g}) = div(f) - div(g)$

Proof. In order to prove the first claim, it is sufficient to show that, for a branch γ of C;

$val_\gamma(f) = 0$ iff $val_\gamma(\frac{1}{f}) = \infty$

$val_\gamma(f) = \infty$ iff $val_\gamma(\frac{1}{f}) = 0$

and ord_γ is preserved in both cases. This follows trivially from the relevant power series calculation at a branch. Namely, we can represent f by $\frac{\phi}{\phi'}$ and $\frac{1}{f}$ by $\frac{\phi'}{\phi}$. Substituting the branch parametrisation, we obtain that;

$$\begin{aligned} val_\gamma(f) = 0, ord_\gamma(f) = m &\text{ iff } f \sim t^m u(t),\ \ m \geq 1, u(t) \in L[[t]] \text{ a unit.} \\ &\text{ iff } \tfrac{1}{f} \sim t^{-m} u(t)^{-1} \\ &\text{ iff } val_\gamma(f) = \infty, ord_\gamma(f) = m \end{aligned}$$

and the calculation for $val_\gamma(f) = \infty, ord_\gamma(f) = m$ is similar.

In order to prove the second claim, we need to verify the following cases at a branch γ of C;

Case 1. If $val_\gamma(f) = val_\gamma(g) \in \{0, \infty\}$, $ord_\gamma(f) = m$ and $ord_\gamma(g) = n$

then $val_\gamma(fg) \in \{0, \infty\}$ and $ord_\gamma(fg) = m + n$

Case 2. If $val_\gamma(f) \neq val_\gamma(g) \in \{0, \infty\}$, $ord_\gamma(f) = m$ and $ord_\gamma(g) = n$

then $val_\gamma(fg) \in \{0, \infty\}$ and $ord_\gamma(fg) = |m - n|$

Case 3. If exactly one of $val_\gamma(f)$ and $val_\gamma(g)$ is in $\{0, \infty\}$, with $ord_\gamma(f)$ or $ord_\gamma(g) = m$

then $val_\gamma(fg) \in \{0,\infty\}$, with $ord_\gamma(fg) = m$.

Case 4. If neither of $val_\gamma(f)$ and $val_\gamma(g)$ are in $\{0,\infty\}$

then $val_\gamma(fg)$ is not in $\{0,\infty\}$

If f is represented by $\frac{\phi}{\phi'}$ and g is represented by $\frac{\psi}{\psi'}$, then we can represent fg by $\frac{\phi\psi}{\phi'\psi'}$. The proof of these cases then follow by elementary power series calculations at the branch γ. For example, for Case 2, if $val_\gamma(f) = 0$ and $ord_\gamma(f) = m$, $val_\gamma(g) = \infty$ and $ord_\gamma(g) = n$, then we have;

$$f \sim t^n u(t),\ g \sim t^{-m} v(t),\ fg \sim t^n t^{-m} u(t)v(t) = t^{n-m} w(t),$$
$$\text{for } \{u(t), v(t), w(t)\} \text{ units in } L[[t]].$$

The third claim follows from the first two claims.

□

We now claim the following;

Theorem 2.11. *Transitivity of Linear Equivalence*

Let C' be an algebraic curve. If A, B, C are weighted sets on C' of the same total multiplicity, then, if $A \equiv B$ and $B \equiv C$, we must have that $A \equiv C$.

Proof. By Theorem 2.7, we can find rational functions f and g on C', such that;

$$(A \setminus B) - (B \setminus A) = div(f)$$

$$(B \setminus C) - (C \setminus B) = div(g)$$

By Lemma 2.9, we have that;

$$div(fg) = (A \setminus B) - (B \setminus A) + (B \setminus C) - (C \setminus B)$$

By drawing a Venn diagram, one easily checks that;

$$(A \setminus B) - (B \setminus A) = (A \cap B^c \cap C^c) + (A \cap B^c \cap C) - (A^c \cap B \cap C^c) - (A^c \cap B \cap C)$$

$$+$$

$$(B \setminus C) - (C \setminus B) = (A \cap B \cap C^c) + (A^c \cap B \cap C^c) - (A^c \cap B^c \cap C) - (A \cap B^c \cap C)$$

$$||$$

$$(A \setminus C) - (C \setminus A) = (A \cap B^c \cap C^c) + (A \cap B \cap C^c) - (A^c \cap B^c \cap C) - (A^c \cap B \cap C)$$

Hence, $div(fg) = (A \setminus C) - (C \setminus A)$. Now, given the g_n^1 defined by the rational function fg, as in Lemma 2.4, it follows that $(A \setminus C)$ and $(C \setminus A)$ belong to this g_n^1 as weighted sets. We can now add the fixed branch contribution $A \cap C$ to this g_n^1, giving a $g_{n+n'}^1$, to which A and C belong as weighted sets. Therefore, the result follows.

□

As an immediate corollary, we have;

Theorem 2.12. *Let C be a projective algebraic curve, then $\equiv$ is an equivalence relation on weighted sets for C of a given multiplicity.*

We also have;

Theorem 2.13. *Linear Equivalence preserved by Addition*

Let C' be a projective algebraic curve and suppose that $\{A, B, C, D\}$ are weighted sets on C' with;

$A \equiv B$ *and* $C \equiv D$

then;

$A + C \equiv B + D$

Proof. By Definition 2.6, we can find a g_n^r containing C and D as weighted sets. If s is the total multiplicity of A, then, by Lemma 1.13, we can add the weighted set A as a fixed branch contribution to this g_n^r and obtain a g_{n+s}^r, containing $A + C$ and $A + D$ as weighted sets. Hence, by Definition 2.6 again, we have that;

$A + C \equiv A + D$ (1)

Similarily, one shows, by adding D as a fixed branch contribution to the $g_{n'}^{r'}$ containing A and B as weighted sets, that;

$A + D \equiv B + D$ (2)

The result then follows immediately by combining (1), (2) and using Theorem 2.11.

□

We now develop further the theory of g_n^r on a projective algebraic curve C. We begin with the following definition;

Definition 2.14. *Subordinate* g_n^r

Let $\{g_n^r, g_n^t\}$ *be given on* C *with the* same *order* n. *Then we say that;*

$g_n^r \subseteq g_n^t$

if every *weighted set in* g_n^r *is included in the weighted sets of the* g_n^t.

We now claim the following;

Theorem 2.15. *Amalgamation of* g_n^r

Let $\{g_n^r, g_n^s\}$ *be given on* C, *having a common weighted set* G, *then there exists* t *with* $r \leq t, s \leq t$ *and a* g_n^t *such that* $g_n^r \subseteq g_n^t$ *and* $g_n^s \subseteq g_n^t$.

Proof. Assume first that $\{g_n^r, g_n^s\}$ have no fixed branch contribution and are defined exactly by linear systems. Then we can find algebraic forms $\{\phi_0, \psi_0\}$ such that;

$G = (C \sqcap \phi_0 = 0) = (C \sqcap \psi_0 = 0)$

and;

g_n^r is defined by $C \sqcap (\epsilon_0\phi_0 + \epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r = 0)$

g_n^s is defined by $C \sqcap (\eta_0\psi_0 + \eta_1\psi_1 + \ldots + \eta_s\psi_s = 0)$

Now consider the linear system Σ defined by;

$\epsilon\phi_0\psi_0 + \psi_0(\epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r) + \phi_0(\eta_1\psi_1 + \ldots + \eta_s\psi_s) = 0$

and let g_m^t be defined by Σ. As $deg(\psi_0\phi_0) = deg(\psi_0) + deg(\phi_0)$, we have that $m = 2n$. We claim that the fixed branch contribution of g_{2n}^t

is exactly G, $(*)$. In order to see this, observe that we can write an algebraic form in Σ as;

$$\psi_0\phi_{\bar{\epsilon}} + \phi_0\psi_{\bar{\eta}}$$

If γ is a branch counted w-times in G, then, using the proof at the end of Lemma 1.13 and linearity of multiplicity at a branch, see [9];

$$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}}) = I_\gamma(C, \psi_0) + I_\gamma(C, \phi_{\bar{\epsilon}}) \geq w$$

$$I_\gamma(C, \phi_0\psi_{\bar{\eta}}) = I_\gamma(C, \phi_0) + I_\gamma(C, \psi_{\bar{\eta}}) \geq w$$

$$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}} + \phi_0\psi_{\bar{\eta}}) = min\{I_\gamma(C, \psi_0\phi_{\bar{\epsilon}}), I_\gamma(C, \phi_0\psi_{\bar{\eta}})\} \geq w \ (\dagger)$$

Hence, γ is w-fold for the g^t_{2n} and G is contained in the fixed branch contribution of the g^t_{2n}. In order to obtain the exactness statement, $(*)$, first observe that, if γ is a fixed branch of the g^t_{2n}, then, in particular, it belongs to $(C \sqcap \phi_0\psi_0 = 0)$. Hence, it belongs either to $(C \sqcap \phi_0 = 0)$ or $(C \sqcap \psi_0 = 0)$. Hence, it belongs to G. Now, using the fact that the original $\{g^r_n, g^s_n\}$ had no fixed branch contribution, we can easily find $\phi_{\bar{\epsilon}_0}$ and $\psi_{\bar{\eta}_0}$ with G disjoint from both $(C \sqcap \phi_{\bar{\epsilon}_0} = 0)$ and $(C \sqcap \psi_{\bar{\eta}_0} = 0)$. Then, by the same argument $(\dagger)$, we obtain, for a branch γ of G;

$$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}_0} + \phi_0\psi_{\bar{\eta}_0}) = w$$

hence, γ is counted w-times in $C \sqcap (\psi_0\phi_{\bar{\epsilon}_0} + \phi_0\psi_{\bar{\eta}_0} = 0)$ and, therefore, $(*)$ holds, as required. Now, as G had total multiplicity n, removing this fixed branch contribution from the g^t_{2n}, we obtain a g^t_n. We then claim that $g^r_n \subseteq g^t_n$ and $g^s_n \subseteq g^t_n$, $(**)$. By Definition 2.14, it is sufficient to check that, if $\{W_1, W_2\}$ are weighted sets appearing in $\{g^r_n, g^s_n\}$, defined by $(C \sqcap \phi_{\bar{\epsilon}} = 0)$ and $(C \sqcap \psi_{\bar{\eta}} = 0)$, then they appear in the g^t_n. We clearly have that both $\psi_0\phi_{\bar{\epsilon}}$ and $\phi_0\psi_{\bar{\eta}}$ belong to Σ and the calculation $(\dagger)$ shows that;

$$C \sqcap (\psi_0\phi_{\bar{\epsilon}} = 0) = W_1 + G$$

$$C \sqcap (\phi_0\psi_{\bar{\eta}} = 0) = W_2 + G$$

Hence, the result $(**)$ follows after removing the fixing branch contribution G. The fact that $r \leq t$ and $s \leq t$ then follows easily from the definition of the dimension of a g^r_n and Theorem 1.3.

Now consider the case when the $\{g_n^r, g_n^s\}$ are defined exactly by linear systems and *have* a fixed branch contribution. Let $G_1 \subseteq G$ and $G_2 \subseteq G$ be these fixed branch contributions and let $G_3 = G_1 \cap G_2$. We claim that the fixed branch contribution of the g_{2n}^t defined by Σ, as given above, in this case is exactly $G_3 + G$. The proof is similar to the above and left to the reader. Now, removing the fixed branch contribution G, we obtain a series g_n^t with fixed branch contribution G_3. A similar proof to the above, left to the reader, shows that this g_n^t contains the original series $\{g_n^r, g_n^s\}$. Finally, we need to consider the case when the $\{g_n^r, g_n^s\}$ are defined, after removing some fixed branch contribution from linear series. Let G_1 and G_2, with total multiplicity r_1 and r_2, be these fixed branch contributions and let $\{g_{n+r_1}^r, g_{n+r_2}^s\}$ be the series obtained from adding these fixed branch contributions to $\{g_n^r, g_n^s\}$. In this case, the linear system Σ, as given above, defines a $g_{2n+r_1+r_2}^t$. We claim that the weighted set $G \cup G_1 \cup G_2$, of total multiplicity $(n+r_1+r_2)$, is contained in the fixed branch contribution of this series. This follows from a similar calculation, using the method above, the details are left to the reader. Removing this weighted set from the $g_{2n+r_1+r_2}^t$, we obtain a g_n^t and a similar calculation shows that this contains the original $\{g_n^r, g_n^s\}$, again the details are left to the reader. □

As a corollary, we have;

Theorem 2.16. *Let a g_n^r be given on C, then there exists a* unique g_n^t *on C, with $r \leq t \leq n$, such that;*

$$g_n^r \subseteq g_n^t$$

and, for any g_n^s *such that $g_n^r \subseteq g_n^s$, we have that;*

$$g_n^s \subseteq g_n^t$$

Proof. By Lemma 1.16, we can find $r \leq t \leq n$ and a g_n^t on C, with $g_n^r \subseteq g_n^t$ and t maximal with this property. If $g_n^r \subseteq g_n^s$, then $\{g_n^s, g_n^t\}$ would contain a common weighted set. By Theorem 2.15, we could then find $t' \leq n$ such that $s \leq t'$, $t \leq t'$ and $g_n^s \subseteq g_n^{t'}$, $g_n^t \subseteq g_n^{t'}$. If $g_n^s \subsetneq g_n^t$, then, by elementary dimension considerations, we would have that $t < t' \leq n$ and $g_n^r \subset g_n^{t'}$, contradicting maximality of t. Hence, $g_n^s \subseteq g_n^t$. The uniqueness statement also follows from a similar amalgamation argument, using Theorem 2.15.

□

We can then make the following definition;

Definition 2.17. *We call a g_n^r on C complete if it cannot be strictly contained in a g_n^t of greater dimension. If G is any weighted set on C of total multiplicity n, then we define $|G|$ to be the unique complete g_n^t to which G belongs.*

We then have that;

Theorem 2.18. *Let G be a weighted set on C, then, $G \equiv G'$ if and only if G' belongs to $|G|$. In particular, $G \equiv G'$ if and only if $|G| = |G'|$.*

Proof. The proof of the first part of the theorem is quite straightforward. By definition, if G' belongs to $|G|$, then $G \equiv G'$. Conversely, if $G' \equiv G$, then, by Definition 2.6, we can find a g_n^1, containing the given weighted sets G and G'. By Theorem 2.16, we can find a unique complete g_n^t on C, with $1 \leq t \leq n$, such that $g_n^1 \subseteq g_n^t$. As G belongs to this g_n^t as a weighted set, it follows by Definition 2.17 that $|G| = g_n^t$. Hence, G' belongs to $|G|$ as required. For the second part, if $G \equiv G'$, then, by the first part, G' belongs to $|G|$. It follows immediately from Definition 2.17 and Theorem 2.16, that $|G| \subseteq |G'|$. Reversing this argument, we have that $|G'| \subseteq |G|$, hence $|G| = |G'|$ as required. Conversely, if $|G| = |G'|$, then clearly $G \equiv G'$ by Definition 2.6. □

We now make the following definition;

Definition 2.19. *Linear System of a Weighted Set*

Let G be a weighted set on a projective algebraic curve C, then we define the Riemann-Roch space $\mathcal{L}(C, G)$ or $\mathcal{L}(G)$ to be the vector space defined as;

$$\{g \in L(C)^* : div(g) + G \geq 0\} \cup \{0\}$$

where $div(g)$ was defined in Definition 2.9.

Remarks 2.20. *That $\mathcal{L}(G)$ defines a vector space follows easily from Lemma 2.10, the fact that, for non-constant rational functions $\{f, g, f+g\} \subset L(C)$ and a branch γ of C, we have that;*

$$ord_\gamma(f+g) \geq min\{ord_\gamma(f), ord_\gamma(g)\}, \ (*)$$

where, for this remark only, ord_γ *is counted* negatively *if* val_γ *is infinite, and an argument on constants, (**). We now give a brief proof of (*);*

We just consider the following 2 cases;

Case 1. $val_\gamma(f) < \infty$ *and* $val_\gamma(g) < \infty$

We then have, substituting the relative parametrisations, that;

$f \sim c + c_1 t^m + \ldots$ *and* $g \sim d + d_1 t^n + \ldots$, *where* $ord_\gamma(f) = m \geq 1$, $ord_\gamma(g) = n \geq 1$ *and* $\{c_1, d_1\} \subset L$ *are non-zero. Then;*

$f + g \sim (c + d) + c_1 t^m + d_1 t^n + \ldots$

If $(f+g)-(c+d) \equiv 0$, *as an algebraic power series in* $L[[t]]$, *then* $(f+g) = (c+d)$ *as a rational function on* C, *contradicting the assumption. Hence, we obtain that* $ord_\gamma(f+g) = min\{ord_\gamma(f), ord_\gamma(g)\}$, *if* $m \neq n$ *or* $m = n$ *and* $c_1 + d_1 \neq 0$, *and* $ord_\gamma(f+g) > min\{ord_\gamma(f), ord_\gamma(g)\}$ *otherwise. Hence, (*) is shown in this case.*

Case 2. $val_\gamma(f) = val_\gamma(g) = \infty$

We then have that;

$f \sim c_1 t^{-m} + \ldots$ *and* $g \sim d_1 t^{-n} + \ldots$, *where* $ord_\gamma(f) = -m \leq -1$, $ord_\gamma(g) = -n \leq -1$ *and* $\{c_1, d_1\} \subset L$ *are non-zero. Then;*

$f + g \sim c_1 t^{-m} + d_1 t^{-n} + \ldots$

By the assumption that $f + g$ *is not a constant, if* $m = n$ *and* $c_1 + d_1 = 0$, *we must have higher order terms in* t *in the Cauchy series for* $(f + g)$, *hence* $ord_\gamma(f + g) > min\{ord_\gamma(f), ord_\gamma(g)\}$. *Otherwise, we have that* $ord_\gamma(f + g) = min\{ord_\gamma(f), ord_\gamma(g)\}$, *hence (*) is shown in this case as well.*

The remaining cases are left to the reader. One should also consider the case of constants, (**). Technically, one cannot define ord_γ *for a constant in* L. *However, we did, by convention, define* $div(c) = 0$, *for* $c \in L^*$, *in Remarks 2.5.*

We now show the following;

Lemma 2.21. *For a weighted set G, $dim(\mathcal{L}(G)) = t+1$, where t is given in Definition 2.17. In particular, $\mathcal{L}(G)$ is finite dimensional.*

Proof. Let t be given by Definition 2.17. If $t = 0$, then $G = (0)$ and $\mathcal{L}(G) = L$. This follows easily from the well known fact that the only regular functions on a projective algebraic curve are the constants (see, for example, [15], p59). In this case, we then have that $dim(\mathcal{L}(G)) = 1$, as required. Otherwise, let $t \geq 1$ be given as in Definition 2.17, with the unique complete g_n^t containing G. After adding some fixed branch contribution W, we can find a linear system Σ, having finite intersection with C, with basis $\{\phi_0, \ldots, \phi_j, \ldots, \phi_t\}$ defining this g_n^t. Moreover, we may assume that $C \sqcap \phi_0 = G \cup W$, $(*)$. Let $\{f_1, \ldots, f_j, \ldots, f_t\}$ be the sequence of rational functions on C defined by $f_j = \frac{\phi_j}{\phi_0}$. We claim that;

$div(f_j) + G \geq 0$, for $1 \leq j \leq t$ $(**)$

In order to show $(**)$, it is sufficient to prove that, for a branch γ with $val_\gamma(f_j) = \infty$, we have that γ belong to G and, moreover, that γ is counted at least $ord_\gamma(f_j)$ times in G. Let Σ_j be the pencil of forms defined by $(\phi_j - \lambda\phi_0)_{\lambda \in P^1}$. By the proof of Lemma 2.4, we have that $(f_j = \infty)$ is defined by $(C \sqcap \phi_0)$, after removing the fixed branch contribution of this pencil. By $(*)$ and the fact that the fixed branch contribution of Σ_j includes W, we have that $(f_j = \infty) \subseteq G$. Hence, $(**)$ is shown as required. By Definition 2.19, we then have that f_j belongs to $\mathcal{L}(G)$. We now claim that there do *not* exist constants $\{c_0, \ldots, c_j, \ldots, c_t\} \subset L$ such that;

$c_0 + c_1 f_1 + \ldots + c_j f_j + \ldots + c_t f_t = 0$ $(***)$

as rational functions on C. If so, we would have that;

$c_0\phi_0 + c_1\phi_1 + \ldots + c_j\phi_j + \ldots + c_t\phi_t$

vanished identically on C, contradicting the fact that Σ has finite intersection with C. Hence, by $(***)$, $\{1, f_1, \ldots, f_t\} \subset \mathcal{L}(G)$ are linearly independent and $dim(\mathcal{L}(G)) \geq t+1$. Conversely, suppose that $dim(\mathcal{L}(G)) \geq k+1$, then we can find $\{1, f_1, \ldots, f_j, \ldots, f_k\} \subset \mathcal{L}(G)$ which are linearly independent, $(\dagger)$. By the usual method of equating denominators, we can find algebraic forms $\{\phi_0, \ldots, \phi_k\}$ of the same degree, such that f_j is represented by $\frac{\phi_j}{\phi_0}$, for $1 \leq j \leq k$. Let Σ be the linear system defined by this sequence of forms. By (†), Σ has finite intersection with C. Let W, having total multiplicity n', be the fixed branch contribution of this system and let $(C \sqcap \phi_0) = G_0 \cup W$. We claim that $G_0 \subseteq G$, (††). Suppose not, then there exists a branch γ with $I_\gamma^{\Sigma,mobile}(C, \phi_0) = s$, where γ is counted strictly less than s-times in G. By the definition of $I_\gamma^{\Sigma,mobile}$, we can find a form ϕ_λ belonging to Σ, distinct from ϕ_0, witnessing this multiplicity. Consider the pencil Σ_λ defined by $(\phi_\lambda - \mu\phi_0)_{\mu \in P^1}$. We then clearly have that $I_\gamma^{\Sigma_\lambda,mobile}(C, \phi_0) = s$ as well, (†††). Let $f_\lambda = \frac{\phi_\lambda}{\phi_0}$. By the proof of Lemma 2.4, we have that $(f_\lambda = \infty)$ is defined by $(C \sqcap \phi_0)$, after removing the fixed branch contribution of Σ_λ. By (†††), it follows that the branch γ is counted s-times in $(f_\lambda = \infty)$ and therefore $div(f_\lambda) + G \not\geq 0$. However, f_λ is a linear combination of $\{1, \ldots, f_k\}$, hence $f_\lambda \in \mathcal{L}(G)$, which is a contradiction. Hence, (††) is shown. Now, consider the g_n^k defined by Σ. Let W' be the weighted set $G \setminus G_0$ of total multiplicity n''. By Lemma 1.13, we can add the weighted set W' to the g_n^k and obtain a $g_{n+n''}^k$ with fixed branch contribution $W' \cup W$. Now, removing the fixed branch contribution W from this $g_{n+n'}^k$, we obtain a $g_{n+n''-n'}^k$ containing G exactly as a weighted set. It follows, from Definition 2.17, that $k \leq t$. Hence, in particular, $dim(\mathcal{L}(G))$ is finite and $dim(\mathcal{L}(G) \leq t + 1$. Therefore, the lemma is proved.

□

We now extend the notion of linear equivalence to include virtual, or non-effective, weighted sets.

Definition 2.22. *We define a generalised weighted set G on C to be a linear combination of branches;*

$$n_1\gamma_{p_1}^{j_1} + \ldots + n_r\gamma_{p_r}^{j_r}$$

where $\{n_1, \ldots, n_r\}$ belong to $\mathcal{Z}$. If $\{n_1, \ldots, n_r\}$ belong to $\mathcal{Z}_{\geq 0}$, we call the weighted set effective. Otherwise, we call the weighted set virtual. We define $n = n_1 + \ldots + n_r$ to be the total multiplicity or degree of G.

Remarks 2.23. *It is an easy exercise to see that there exist well defined operations of addition and subtraction on generalised weighted sets. It is also easy to check that any generalised weighted set G may be written uniquely as $G_1 - G_2$, where $\{G_1, G_2\}$ are* disjoint effective *weighted sets.*

Definition 2.24. *Let A and B be generalised weighted sets on C of the same total multiplicity. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be the unique* *effective weighted sets, as given by the previous remark. Then we define;*

$(A_1 - A_2) \equiv (B_1 - B_2)$ *iff* $(A_1 + B_2) \equiv (B_1 + A_2)$

and;

$A \equiv B$ *iff* $(A_1 - A_2) \equiv (B_1 - B_2)$

Remarks 2.25. *Note that if* $\{A_1', A_2'\}$ *and* $\{B_1', B_2'\}$ *are* any *effective weighted sets such that;*

$A = A_1' - A_2'$ *and* $B = B_1' - B_2'$

then $A \equiv B$ *iff* $A_1' + B_2' \equiv B_1' + A_2'$

The proof is just manipulation of effective weighted sets. We clearly have that;

$A_1 + A_2' = A_1' + A_2$ *and* $B_1 + B_2' = B_1' + B_2$ $(*)$

We then have;

$$
\begin{array}{lll}
A \equiv B & \textit{iff} & A_1 + B_2 \equiv B_1 + A_2 \textit{(Definition 2.20)} \\
 & \textit{iff} & A_1 + A_2' + B_2 \equiv B_1 + A_2 + A_2' \textit{ (Theorem 2.12)} \\
 & \textit{iff} & A_1' + A_2 + B_2 \equiv B_1 + A_2 + A_2' \textit{ (by (*))} \\
 & \textit{iff} & A_1' + B_2 \equiv B_1 + A_2' \textit{ (Theorem 2.12)} \\
 & \textit{iff} & A_1' + B_2 + B_1' \equiv B_1 + B_1' + A_2' \textit{ (Theorem 2.12)} \\
 & \textit{iff} & A_1' + B_1 + B_2' \equiv B_1 + B_1' + A_2' \textit{ (by (*))} \\
 & \textit{iff} & A_1' + B_2' \equiv B_1' + A_2' \textit{ (Theorem 2.12)}
\end{array}
$$

We then have;

Theorem 2.26. *Transitivity of Linear Equivalence*

Let C' be an algebraic curve. If A, B, C are generalised weighted sets on C' of the same total multiplicity, then, if $A \equiv B$ and $B \equiv C$, we must have that $A \equiv C$.

Proof. Let $\{A_1, A_2\}$, $\{B_1, B_2\}$ and $\{C_1, C_2\}$ be the effective weighted sets as given by Remarks 2.23. Then, by Definition 2.24, we have that;

$(A_1 + B_2) \equiv (B_1 + A_2)$ and $(B_1 + C_2) \equiv (C_1 + B_2)$

By Theorem 2.12, we have that;

$(A_1 + B_1 + B_2 + C_2) \equiv (C_1 + B_1 + B_2 + A_2)$

It then follows, by Definition 2.6, that there exists a g_n^1, containing $(A_1 + B_1 + B_2 + C_2)$ and $(C_1 + B_1 + B_2 + A_2)$ as weighted sets. Clearly $(B_1 + B_2)$ is contained in the fixed branch contribution of this g_n^1. Removing this fixed branch contribution, we obtain;

$A_1 + C_2 \equiv C_1 + A_2$

By Definition 2.20, we then have that $A \equiv C$ as required.

□

It follows immediately from Theorem 2.11 and Theorem 2.22 that;

Theorem 2.27. *Let C be a projective algebraic curve, then $\equiv$ is an equivalence relation on generalised weighted sets for C of a given total multiplicity.*

Remarks 2.28. *Again, the definition of linear equivalence that we have given for generalised weighted sets on a smooth projective algebraic curve C is equivalent to the modern definition for divisors. More precisely, we have;*

Modern Definition; Let A and B be divisors on a smooth projective algebraic curve C, then $A \equiv B$ iff $A - B = div(g)$, for some $g \in L(C)^*$.

See, for example, p161 of [15] *for relevant definitions and notation. In order to show that our definition is the same, use Remarks 2.8 and the following simple argument;*

$A \equiv B$ as generalised weighted sets iff $A_1 + B_2 \equiv B_1 + A_2$

where $\{A_1, A_2, B_1, B_2\}$ are the effective weighted sets given by Definition 2.24. Then;

$A_1 + B_2 \equiv B_1 + A_2$ iff $(A_1 + B_2) - (B_1 + A_2) = div(g)$ $(g \in L(C)^*)$

by Remarks 2.8, where $div(g)$ is the modern definition. By a straightforward calculation, we have that;

$(A_1 + B_2) - (B_1 + A_2) = A - B$ *as divisors or generalised weighted sets.*

Hence, the notions of equivalence coincide.

We also have;

Theorem 2.29. *Linear Equivalence Preserved by Addition*

Let C' be a projective algebraic curve and suppose that $\{A, B, C, D\}$ are generalised weighted sets on C' with;

$A \equiv B$ *and* $C \equiv D$

then;

$A + C \equiv B + D$

Proof. Let $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$ and $\{D_1, D_2\}$ be effective weighted sets as given by Remarks 2.19. Then, by Definition 2.20, we have that;

$A_1 + B_2 \equiv B_1 + A_2$ and $C_1 + D_2 \equiv D_1 + C_2$

Hence, by Theorem 2.20;

$A_1 + B_2 + C_1 + D_2 \equiv B_1 + A_2 + D_1 + C_2 \; (*)$

We clearly have that;

$A + C = (A_1 + C_1) - (A_2 + C_2)$ and $B + D = (B_1 + D_1) - (B_2 + D_2)$

as an identity of generalised weighted sets. Moreover, as $(A_1 + C_1), (A_2 + C_2), (B_1 + D_1)$ and $(B_2 + D_2)$ are all effective, we can apply Remarks 2.23 and $(*)$ to obtain the result. □

We now make the following definition;

Definition 2.30. *Let G be a generalised weighted set on a projective algebraic curve C, then we define $|G|$ to be the collection of generalised* *weighted sets G' with $G' \equiv G$. We define order$(|G|)$ to be the total multiplicity (possibly negative) of any generalised weighted set in $|G|$.*

Remarks 2.31. *If G is an* effective *weighted set, the collection defined by Definition 2.30 is* not *the same as the collection given by Definition 2.17, as it includes virtual weighted sets. Unless otherwise stated, we will use Definition 2.17 for* effective *weighted sets. This convention is in accordance with the Italian terminology, we hope that this will not cause too much confusion for the reader.*

We now show that the notions of linear equivalence introduced in this section are birationally invariant;

Theorem 2.32. *Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map. Let A and B be generalised weighted sets on C_2, with corresponding generalised weighted sets $[\Phi]^*A$ and $[\Phi]^*B$ on C_1. Then $A \equiv B$, in the sense of either Definition 2.6 or 2.24, iff $[\Phi]^*A \equiv [\Phi]^*B$.*

Proof. Suppose that $A \equiv B$ in the sense of Definition 2.6. Then, there exists a g_n^r on C_2 containing A and B as weighted sets. By Theorem 1.14, there exists a corresponding g_n^r on C_1, containing $[\Phi]^*A$ and $[\Phi]^*B$ as weighted sets. Hence, again by Definition 2.6, $[\Phi]^*A \equiv [\Phi]^*B$. The converse is similar, using $[\Phi^{-1}]^*$. If $A \equiv B$ in the sense of Definition 2.24, then the same argument works.

□

As a result of this theorem, we introduce the following definition;

Definition 2.33. *Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map. Then, given a generalised weighted set A on C_2, we define;*

$$[\Phi]^*|A| = |[\Phi]^*A|$$

where, in the case that A is effective, $|A|$ can be taken either in the sense of Definition 2.17 or Definition 2.30.

Remarks 2.34. *The definition depends only on the complete series $|A|$, rather than its particular representative A. This follows immediately from Definition 2.17, Definition 2.30 and Theorem 2.32.*

We finally introduce the following definition;

Definition 2.35. *Summation of Complete Series*

Let A and B be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 2.30. Then, we define the sum;

$|A| + |B|$

to be the complete series, in the sense of Definition 2.30, containing all generalised weighted sets of the form $A' + B'$ with $A' \in |A|$ and $B' \in |B|$. If A and B are effective *weighted sets with $|A|$, $|B|$ taken in the sense of Definition 2.17, then we make the same definition for the sum in the sense of Definition 2.17.*

Remarks 2.36. *This is a good definition by Theorem 2.13 and Theorem 2.29.*

Definition 2.37. *Difference of Complete Series*

Let A and B be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 2.30. Then, we define the difference;

$|A| - |B|$

to be the complete series, in the sense of Definition 2.30, containing all generalised weighted sets of the form $A' - B'$ with $A' \in |A|$ and $B' \in |B|$. If A and B are effective *weighted sets with $|A|$, $|B|$ taken in the sense of Definition 2.17, then we can in certain cases define a difference in the sense of Definition 2.17. (This is called the residual series, the reader can look at* [14] *for more details)*

Remarks 2.38. *This is again a good definition, for generalised weighted sets $\{A, B\}$, it follows trivially from the previous definition and the fact that $\{A, -B\}$ are also generalised weighted sets.*

3. A geometrical definition of the genus of an algebraic curve

The purpose of this section is to give a geometrical definition of the genus of an algebraic curve. In the case of a singular curve, this cannot be achieved using purely algebraic methods. Our treatment follows the presentation of Severi in [14].

We begin with the following lemma;

Lemma 3.1. *Let C be a projective algebraic curve, and suppose that a g_n^1 is given on C, with no fixed branch contribution. Then there exist a finite number of weighted sets W_λ in the g_n^1, possessing multiple branches.*

Proof. We may assume that the g_n^1 is defined by a pencil Σ, having finite intersection with C. Let $\theta(\lambda)$ be the statement;

$$\theta(\lambda) \equiv \forall y[(y \in \phi_\lambda \cap C) \rightarrow y \in NonSing(C) \wedge R.Mult_y(C, \phi_\lambda) = 1]$$

See also Lemma 2.17 of [9]. Then θ defines a constructible condition on Par_Σ and moreover, using Lemma 2.17 of [9], we have that θ holds on an open subset $U \subset Par_\Sigma$. For $\lambda \in U$, we have that the intersection $(C \cap \phi_\lambda)$ is transverse, in the sense of Lemma 2.4 of [9], and is contained in W. Now, using Lemma 5.29 of [9] and the definition of $(C \sqcap \phi_\lambda)$, it follows that each branch of the corresponding weighted set W_λ is counted once and lies inside W. Hence, if W_λ is a weighted set, possessing multiple branches, we must have that $\lambda \in (Par_\Sigma \setminus U)$. As Par_Σ has dimension 1, this is a finite set, hence the result follows. □

We now make the following definition;

Definition 3.2. *Let C be a projective algebraic curve and suppose that a g_n^1 is given on C, with no fixed branch contribution. Then we define the Jacobian of this g_n^1 to be the weighted set;*

$$Jac(g_n^1) = \{\alpha_{\gamma_1}, \ldots, \alpha_{\gamma_j}, \ldots, \alpha_{\gamma_r}\}$$

where $\{\gamma_1, \ldots, \gamma_j, \ldots, \gamma_r\}$ consists of the finitely many branches which are multiple for some weighted set W_{λ_j} of the g_n^1 and γ_j appears with multiplicity $\alpha_{\gamma_j} + 1$ in W_{λ_j}.

Remarks 3.3. *This is a good definition by Lemma 3.1 and the fact that any branch γ can only appear in one weighted set W_λ, using the hypothesis that the g_n^1 has no fixed branches.*

We now analyse further the Jacobian of a g_n^1, with *no* fixed branch contribution. Using Theorem 1.14 of this paper and Theorem 4.16 of [9], we will derive general results for projective algebraic curves from consideration of the case where C is a plane projective curve, having at most nodes as singularities, (†). Until the end of Theorem 3.21, this assumption will be in force.

Lemma 3.4. *Let the g_n^1, without fixed branch contribution, be given on C. Then there exists a rational function h on C, defining this g_n^1, such that the weighted set;*

$$G = (h = \infty)$$

consists of n distinct branches, each counted once, lying inside $NonSing(C)$.

Proof. By Theorem 2.7, we may assume that the given g_n^1 is defined by (g) for some rational function g on C. Using the proof of Lemma 3.1, we may assume, that, for generic $\lambda \in P^1$, the weighted set $(g = \lambda)$ consists of n distinct branches, lying inside $NonSing(C)$, each counted once. Using the proof of Theorem 2.7, we can find a Mobius transformation α of P^1, taking λ to ∞, such that $h = \alpha \circ g$ also defines the given g_n^1 and such that $G = (h = \infty) = (g = \lambda)$. The result follows.

□

We now show the following, the reader should refer to [9] for the relevant notation;

Lemma 3.5. *Suppose that $deg(C) = m$, then there exists a homographic change of variables of P^2, such that, in this new coordinate system (x', y');*

(i). The line at ∞ cuts C transversely in m distinct non-singular points.

(ii). The tangent lines to C parallel to the y'-axis all have 2-fold contact (contatto), and are based at non-singular points of C.

(iii). The branches of $J = Jac(g_n^1)$ and $G = (h = \infty)$ are all in finite position, with base points distinct from the points of contact in (ii).

Proof. We use the fact that a generic point of C has character $(1, 1)$. The proof of this result requires duality arguments, which may be found later in the paper. It follows, by Remark 6.6 of [9], that there exist only finitely many non-ordinary branches. Hence, there exist finitely many tangent lines $\{l_{\gamma_1}, \ldots, l_{\gamma_r}\}$, based at $\{p_1, \ldots, p_r\}$, (possibly with repetitions), such that;

$I_{\gamma_j}(C, p_j, l_{\gamma_j}) \geq 3$, $(for\ 1 \leq j \leq r)$

By assumption, C has at most finitely many nodes as singularities. Let $\{q_1, \ldots, q_s\}$ be the base points of these nodes and suppose that $\{l_{\gamma_{q_1^1}}, l_{\gamma_{q_1^2}}, \ldots, l_{\gamma_{q_s^1}}, l_{\gamma_{q_s}^2}\}$ are the $2s$ tangent lines (possibly with repetitions) corresponding to these nodes. Let $\{l_{\gamma'_1}, \ldots, l_{\gamma'_t}\}$ define the tangent lines to each of the branches appearing in $(Jac(g_n^1) \cup G)$. Now choose a point P not lying on C or any of the above defined tangent lines. Let $\Sigma = \{l_\lambda^P\}_{\lambda \in P^1}$ be the pencil defined by all lines passing through the point P. Then Σ defines a g_m^1 on C without fixed branches. By the proof of Lemma 3.1, for generic λ, l_λ^P intersects C transversely in m distinct branches, based at non-singular points of C. Moreover, we may assume, using the fact that the pencil has no base branches, that $(C \sqcap l_\lambda^P)$ is disjoint from $Jac(g_n^1)$ and G $(*)$. By construction, we also have that, if l_{Px} belongs to Σ and defines the tangent line l_{γ_x} to a branch γ_x based at x, in the sense of Definition 6.3 of [9], then $x \in NonSing(C)$, l_{Px} has 2-fold contact (contatto) with the branch γ_x, $(**)$, and the branch γ_x does not appear in $(Jac(g_n^1) \cup G)$,(***). Now choose a homography, sending the point $[0:1:0]$ and the line $Z = 0$, in the original coordinates $[X : Y : Z]$, to P and l_λ^P. Let $[X' : Y' : Z']$ be the new coordinate system defined by this homography. For the affine coordinate system (x', y'), defined by $x' = \frac{X'}{Z'}$ and $y' = \frac{Y'}{Z'}$, we have that the line at ∞ has the property (i), and, by $(*)$, the branches of $Jac(g_n^1)$ and G are all in finite position. The lines parallel to the y'-axis correspond to the lines, exluding l_λ^P, in the pencil defined by Σ, in this new coordinate system. Hence, (ii) follows immediately from $(**)$ and (iii) then follows from $(***)$. The lemma is proved.

□

We now claim the following;

Lemma 3.6. *Let C be given in the coordinate system defined by Lemma 3.5, henceforth denoted by (x, y). Then the g_m^1 on C, given by the lines parallel to the y-axis, and the line at ∞, is defined by (x), as in Lemma 2.4. Moreover, $J' = Jac(g_m^1)$ consists exactly of the branches of contact between C and the lines parallel to the y-axis, while $G' = (x = \infty)$ consists of m distinct branches, centred at non-singular points of C.*

Proof. In order to prove the first claim, first observe that, by its construction, the given g_m^1 has no fixed branch contribution. Moreover, it is generated by the lines $X = 0$ and $Z = 0$, that is defined by the pencil of lines $(X - \lambda Z)_{\lambda \in P^1}$. By Lemma 2.4, because x is represented by $(\frac{X}{Z})$, as a rational function on C, we have that the series (x) is defined by $(X - \lambda Z)_{\lambda \in P^1}$, after removing its fixed branch contribution. As this series has no fixed branch contribution, the result follows. In order to compute $J' = Jac(g^1_m)$, we need to determine;

$$\{\lambda \in P^1 : C \sqcap (X - \lambda Z) \text{ contains a multiple branch}\}$$

This corresponds to the set;

$$\{\lambda \in P^1 : I_{\gamma_p}(C, p, (X - \lambda Z)) \geq 2\}, \text{ for some } p \in C \cap (X - \lambda Z),\ \gamma_p \text{ a branch at } p.$$

As C has at most nodes as singularities, the order of each branch γ on C is 1, see Definition 6.3 of [9]. Using Theorem 6.2 of [9], we then have that $I_{\gamma_p}(C, p, (X - \lambda Z)) \geq 2$ iff $(X - \lambda Z)$ is the tangent line l_{γ_p} of γ_p. By construction of the g^1_m, this can only occur if $(X - \lambda Z)$ is the tangent line to an ordinary branch, see Definition 6.3 of [9] again, centred at a non-singular point of C, $(*)$. When $\lambda = \infty$, the corresponding line in the pencil is given by the line at ∞, which cuts C transversely, hence this possibility is excluded. Therefore, the only possible values of λ occur for lines parallel to the y-axis. By $(*)$, for such a branch γ_p of contact, we have;

$$I_{\gamma_p}(C, p, (X - \lambda Z)) = 2$$

Now, by definition, $mult_{\gamma_p}(Jac(g^1_m)) = 2 - 1 = 1$. Hence, the result follows. For the final part of the lemma, it is easy to verify that $\{\gamma \in C : val_\gamma(x) = \infty\}$ correspond to the branches of intersection between C and the line at ∞. The result then follows by (i) of Lemma 3.5.

□

We now claim;

Lemma 3.7. *Let γ be a branch of C, in finite position, with coordinates (a, b), which does* not *belong to $J' = Jac(g^1_m)$. Then one can find a power series representation of γ of the form;*

$$y(x) = b + c_1(x - a) + c_2(x - a)^2 + \ldots \ (*)$$

Remarks 3.8. *In the representation* $(*)$, *given in Lemma 3.7, we have slightly abused the terminology of Theorem 6.2 in* [9]. *We mean, here, that* $(x, y(x))$ *should parametrise the branch* γ, *in the sense that, for any algebraic function* $F(x, y)$, $F(x, y(x)) \equiv 0$ *iff* F *vanishes on* C, *otherwise* F *has finite intersection with* C *and;*

$$\begin{aligned} ord_{(x-a)}F(x, y(x)) &= ord_{(x-a)}F(a + (x - a), b + c_1(x - a) + \ldots) \\ &= I_{\gamma_{(a,b)}}(C, F) \end{aligned}$$

Proof. (Lemma 3.7)

First make the linear change of coordinates $x' = x-a$ and $y' = y-b$, so that, in this new coordinate system, the branch γ is centred at $(0, 0)$. Using Theorem 6.2 of [9], we can find algebraic power series $\{x'(t), y'(t)\}$, parametrising the branch γ in the coordinate system (x', y'), of the form;

$x'(t) = a_1t + a_2t^2 + \ldots$

$y'(t) = b_1t + b_2t^2 + \ldots$

It is then a trivial calculation to check that;

$x(t) = a + a_1t + a_2t^2 + \ldots$

$y(t) = b + b_1t + b_2t^2 + \ldots$

parametrises the branch γ in the coordinate system (x, y), with the terminology similar to the previous remarks.

Using the fact that the branch has order 1, which is preserved by the homographic change of coordinates, the vector $(a_1, b_1) \neq 0$. If $a_1 = 0$, then the tangent line l_γ to the branch, in the coordinate system (x', y'), would be parallel to the y'-axis, hence the translation of l_γ by (a, b), which is the tangent line to γ in the coordinate system (x, y) would be parallel to the y-axis. Therefore, by Lemma 3.6, γ would belong to $J' = Jac(g^1_m)$, contradicting the assumption of the lemma. Hence, we can assume that $a_1 \neq 0$. As $\frac{dx'}{dt}|_{t=0} \neq 0$, we can apply the inverse function theorem to the power series $x'(t)$ and find an algebraic power series $t(x')$, with $\frac{dt}{dx'}|_{x'=0} \neq 0$, such that $x'(t(x')) = x'$. Then we can write;

$y'(t(x')) = b_1 t(x') + b_2 t(x')^2 + \ldots$

We claim that the sequence $(x', y'(t(x')))$ parametrises the branch γ in the coordinate system (x', y'), $(*)$. We clearly have that, for any algebraic function $F(x', y')$;

$F(x'(t), y'(t)) \equiv 0$ iff $F(x'(t(x')), y'(t(x'))) \equiv 0$ iff $F(x', y'(t(x'))) \equiv 0$

If $ord_t F(x'(t), y'(t)) = m < \infty$, then we have;

$F(x'(t), y'(t)) = t^m u(t)$, for a unit $u(t) \in L[[t]]$.

We then have that;

$F(x', y'(t(x'))) = t(x')^m u(t(x'))$, (1)

As $\frac{dt}{dx'}|_{x'=0} \neq 0$, we have that;

$t(x') = x' v(x')$, for a unit $v(x') \in L[[x']]$, (2)

Combining (1) and (2) gives that;

$F(x', y'(t(x'))) = (x'v(x'))^m u(x'v(x')) = (x')^m v(x')^m u(x'v(x'))$,

where $v(x')^m u(x'v(x'))$ is a unit in $L[[x']]$

This implies that $ord_{x'} F(x', y'(t(x'))) = m$ as well, hence $(*)$ is shown. It follows easily that the sequence $(a+x', b+y'(t(x')))$ parametrises the branch γ in the coordinate system (x, y), with the above extension of terminology. Hence, if we let $y(x) = b + y'(t(x-a))$, then so does the sequence $(x, y(x))$, with the convention of Remarks 3.8. The lemma is then shown.

$\square$

Now let C be defined in the coordinate system (x, y) by $f = 0$ and let h be the rational function, given by Lemma 3.4, in this coordinate system. With hypotheses as in Lemma 3.7, any rational function θ formally determines a Cauchy series $\theta(x, y(x))$, in $(x-a)$, see the explanation at the beginning of Section 2, which we will also denote by θ.

We have that;

$f(x,y(x))\equiv 0$

and, hence;

$f_x+f_y\frac{dy}{dx}=0,\ \frac{dy}{dx}=-\frac{f_x}{f_y}$

$\frac{dh}{dx}=h_x+h_y\frac{dy}{dx}=h_x+h_y\cdot-\frac{f_x}{f_y}=\frac{h_xf_y-h_yf_x}{f_y}$ $(*)$

The calculation $(*)$ should be justified carefully at the level of Cauchy series. The first part and the case when h belongs to the polynomial ring $L[x,y]$ was considered in Lemma 2.10 of [9], $(**)$. In general, we can find $\{h_1,h_2\}$ in $L[x,y]$ such that $h=\frac{h_1}{h_2}$. Using the result $(**)$ and the quotient rule for differentiating rational functions, it is sufficient to check that for the formal (algebraic) power series in $(x-a)$, determined by h_2;

$\frac{d(1/h_2)}{dx}=\frac{dh_2}{dx}\cdot\frac{-1}{(h_2)^2}$ $(***)$

This can be shown by a similar calculation to that in Lemma 2.10 of [9]. Namely, we can find a sequence of polynomials $\{h_2^m\}_{m\geq 0}$ in $(x-a)$, converging to h_2 in the power series ring $L[[x-a]]$. The result $(***)$ holds for each $\{h_2^m\}$, hence, by general continuity results for multiplication in $L[[x-a]]$, it is sufficient to show that;

$\frac{d(1/h_2^m)}{dx}\rightarrow\frac{d(1/h_2)}{dx}$ and $\frac{dh_2^m}{dx}\rightarrow\frac{dh_2}{dx}$

The second part of this calculation was done in Lemma 2.10 of [9], (even in non-zero characteristic). The first part follows from the second part by representing $(1/h_2)$ as $(x-a)^{-n}u_2(x-a)$, for some $n\geq 0$ and a unit $u_2(x-a)\in L[[x-a]]$, and finding a sequence of units $\{u_2^m(x-a)\}_{m\geq 1}$ in $L[[x-a]]$ such that $u_2^m\rightarrow u_2$ and $(1/h_2^m)=(x-a)^{-n}u_2^m$. One can then use the product rule for an algebraic power series and the function $(x-a)^{-n}$, along with continuity of addition in $L[[x-a]]$.

Using $(*)$, we can consider $\frac{dh}{dx}$ as a rational function on the curve C and, for a branch γ, define $val_\gamma(\frac{dh}{dx})$ and $ord_\gamma(\frac{dh}{dx})$. For convenience, we will use the notation $ord_\gamma(\frac{dh}{dx})=-k$, for a positive integer k, to mean that $val_{\gamma}(\frac{dh}{dx})=\infty$ and $ord_{\gamma}(\frac{dh}{dx})=k$. We now claim the following;

Lemma 3.9. *Let γ be a branch of C, distinct from $G\cup G'\cup J'$, then, for the g_n^1 defined by Lemma 3.1, γ is counted (contato) s-times in some weighted set iff $ord_{\gamma}(\frac{dh}{dx})=s-1$. If γ belongs to G, then $ord_{\gamma}(\frac{dh}{dx})=-2$.*

Proof. For the first part of the lemma, suppose that γ is counted s-times in some weighted set. Then, by Lemma 3.4, $val_{\gamma}(h)<\infty$ and $ord_{\gamma}(h)=s$. By Lemma 3.7 and the construction before Lemma 2.1, h determines an algebraic power series, at the branch γ, of the form;

$h=\lambda+(x-a)^s\psi(x-a)$, with $\psi(0)\neq 0,\lambda<\infty$

We then have that;

$\frac{dh}{dx}=(x-a)^{s-1}[s\psi(x-a)+(x-a)\psi'(x-a)]$

At $x=a$, the expression in brackets reduces to $s\psi(0)\neq 0$, hence $ord_{\gamma}(\frac{dh}{dx})=s-1$ as required. The converse statement is also clear by this calculation. If γ belongs to G, then, by Lemma 3.4, $val_{\gamma}(h)=\infty$ and $ord_{\gamma}(h)=-1$. Then h determines an algebraic power series at γ of the form;

$h=(x-a)^{-1}\psi(x-a)$, with $\psi(0)\neq 0$

We then have that;

$\frac{dh}{dx}=-(x-a)^{-2}[\psi(x-a)-(x-a)\psi'(x-a)]$

At $x=a$, the expression in brackets reduces to $\psi(0)\neq 0$, hence $ord_{\gamma}(\frac{dh}{dx})=-2$ as required. □

It remains to consider the branches $G'\cup J'$. We achieve this by the following geometric constructions;

(i). Construction for G';

We use the change of variables $x=\frac{1}{x'}$ and $y=\frac{y'}{x'}$. This is a homography as the map;

$\Theta:(x',y')\mapsto(\frac{1}{x'},\frac{y'}{x'})$

is the restriction to affine coordinates of the map;

$\Theta : [X' : Y' : Z'] \mapsto [Z' : Y' : X']$

As Θ is a homography, the character of corresponding branches is preserved. Let $F(x', y')$ define the equation of C in this new coordinate system. The point P, given by $[0 : 1 : 0]$, relative to the coordinate system (x, y), is fixed by this homography, hence the weighted set G'_1, corresponding to G', consists of branches in finite position relative to the coordinate system (x', y'), and is defined by $C \sqcap (x' = 0)$. As the branches of G'_1 are simple, they cannot coincide with any of the branches of contact of tangents to C parallel to the y'-axis. Hence, we can apply the power series method given by Lemma 3.7. Let $H(x', y')$ be the rational function corresponding to $h(x, y)$ from Lemma 3.4. We claim the following;

Lemma 3.10. *Let hypotheses be as in* (i). *Then, for every branch* γ *of* G'_1, *we have that* $ord_\gamma(\frac{dH}{dx'}) = 0$.

Proof. By (i) and (iii) of Lemma 3.5, we have that $G = (h = \infty)$ and G' are disjoint. Hence, for every branch γ of G', $val_\gamma(h) < \infty$. For a given branch γ of G', let $val_\gamma(h) = c$ and consider the rational function $h - c$. We clearly have that $val_\gamma(h - c) = 0$. If $ord_\gamma(h - c) \geq 2$, then γ would be multiple for $(h = c)$, hence, by Lemma 3.4, would belong to $J = Jac(g^1_n)$. This contradicts the fact, from (i) of Lemma 3.5, that J and G' are disjoint. Therefore, we must have that $ord_\gamma(h - c) = 1$. It follows that, for any branch γ of G'_1, we can find a constant c such that $ord_\gamma(H - c) = 1$ as well. Now, if γ is centred at (a, b) in the coordinate system (x', y'), then, using the method of Lemma 3.7, H determines an algebraic power series at γ of the form;

$H(x', y'(x')) = c + c_1(x' - a) + \ldots \; (c_1 \neq 0)$

It follows, from differentiating this expression, that $ord_\gamma(\frac{dH}{dx'}) = 0$ as required.

□

We now claim;

Lemma 3.11. *Let hypotheses be as in* (i). *Then;*

$\frac{dH}{dx'} = \frac{dh}{dx} \cdot \frac{dx}{dx'}$

as an identity of Cauchy series, for corresponding branches satisfying the requirements that they are in finite position and are not branches of contact for tangent lines parallel to the y'-axis or y-axis respectively.

Proof. Let γ' and γ be such corresponding branches, centred at $p' = (a,b)$ and $p = (\frac{1}{a}, \frac{b}{a})$, of $F = 0$ and $f = 0$ respectively. Let γ' be parametrised, as in Lemma 3.7, by the sequence $(x', y'(x'))$. Then, we claim that the sequence;

$$(x(x', y'(x')), y(x', y'(x'))) = (x(x'), y(x', y'(x'))) = (\tfrac{1}{x'}, \tfrac{y'(x')}{x'})$$

parametrises the corresponding branch γ, in the sense of Remarks 3.8, $(*)$. This follows easily from the fact that the morphism Θ, given in (i), is a homography of P^2, hence, in particular, one has that, for any algebraic function θ;

$$I_{\gamma'}(p', C, \Theta^*(\theta)) = I_\gamma(p, C, \theta)$$

We now claim that;

$$(x(x'), y(x', y'(x'))) = (x(x'), y(x(x'))),\ (**)$$

as an identity of algebraic power series in $(x'-a)$. By $(*)$ and Lemma 3.7, both sequences define valid parametrisations, $(\theta_1(x-a), \theta_2(x-a))$ and $(\theta_1(x-a), \theta_3(x-a))$ in the sense of Remarks 3.8, of γ. Moreover, we clearly have that the initial terms of both sequences are identically equal. Now, observe that $ord_{(x'-a)}(x(x') - \frac{1}{a}) = 1$, hence we may apply the method of Lemma 3.7 (Inverse Function Theorem), in order to find an algebraic power series $t(z) \in zL[[z]]$ such that;

$$\theta_1(t(x'-a)) = \tfrac{1}{a} + (x'-a)$$

and

$$(\tfrac{1}{a} + (x'-a), \theta_2(t(x'-a))),\ (\tfrac{1}{a} + (x'-a), \theta_3(t(x'-a)))$$

both define valid parametrisations of γ in the sense of Remarks 3.8, $(***)$. Now, supposing that $\theta_2(t(x'-a)) = \theta_3(t(x'-a))$, then $(\theta_2 - \theta_3)(t(x'-a)) \equiv 0$, hence, it follows straightforwardly that $(\theta_2 - \theta_3)(x-a) \equiv 0$ and $\theta_2(x-a) = \theta_3(x-a)$. Then $(**)$ is shown.

We may, therefore, assume that;

$\theta_2(t(x'-a)) = \sum_{n=0}^{\infty} a_n(x'-a)^n$, $\theta_3(t(x'-a)) = \sum_{n=0}^{\infty} b_n(x'-a)^n$

and $a_n \neq b_n$ for some $n \geq 1$. Consider the rational function $\psi_n = \frac{1}{n!}\frac{d^n y}{dx^n}$ on C, given by the explanation after the proof of Lemma 3.7. By $(***)$ and Lemma 2.1, we would then have that both $val_\gamma(\psi_n) = a_n$ and $val_\gamma(\psi_n) = b_n$, which is clearly a contradiction. Hence, the claim $(**)$ is shown.

Now, we have, by $(**)$, for the corresponding branches γ' and γ;

$H(x', y'(x')) = h(x(x'), y(x', y'(x'))) = h(x(x'), y(x(x')))$, $(\dagger)$

Applying the chain rule for differentiating Cauchy series, $(\dagger\dagger)$, to the composition;

$H : x' \mapsto x(x') \mapsto h(x(x'), y(x(x')))$

and using $(\dagger)$, the lemma follows. However, we should still justify the use of $(\dagger\dagger)$ in the following form;

A Chain Rule for Cauchy Series.

Let $q(x') \in L(x')$ be a rational function, such that $q(a)$ is defined as an element of L, and let $h(x)$ define a Cauchy series in $Frac(L[[x]])$. Then $H(x') = h(q(x')-q(a))$ defines a Cauchy series in $Frac(L[[x'-a]])$ and;

$\frac{dH}{dx'} = \frac{dh}{dx}|_{(q(x')-q(a))} \cdot \frac{dq}{dx'}$, $(*)$

as an identity of Cauchy series in $Frac(L[[x'-a]])$.

In order to see the first part of the claim, observe that $q(x')-q(a)$ can be expanded as an algebraic power series in $(x'-a)L[[x'-a]]$, hence the formal substitution of $q(x')-q(a)$ in $h(x)$ determines a Cauchy series in $Frac(L[[x'-a]])$. For the second part of the claim, choose a sequence $\{h_m\}_{m\geq 0}$ of *polynomials* in $L[x, 1/x]$ such that $\{h_m\} \rightarrow h$ in the non-archimidean topology induced by the standard valuation on the field $Frac(L[[x]])$. Let $H_m(x') = h_m(q(x')-q(a))$ be the corresponding rational function, which also determines a Cauchy series in the field $Frac(L[[x'-a]])$. By the chain rule for rational functions, $(*)$ holds, replacing H by H_m and h by h_m. Hence, the identity $(*)$ also holds at the level of Cauchy series in $Frac(L[[x'-a]])$. We have that the sequence $\{H_m\}_{m\geq 0} \rightarrow H$, in the non-archimidean topology induced by the standard valuation on the field $Frac(L[[x'-a]])$. This follows from the fact that, for sufficently large m;

$$ord_{(x'-a)}(h-h_m)(q(x')-q(a)) \geq ord_x(h-h_m)$$

By calculations shown above (even in non-zero characteristic), we have that $(\frac{dh_m}{dx})_{m\geq 0} \rightarrow \frac{dh}{dx}$ and $(\frac{dH_m}{dx'})_{m\geq 0} \rightarrow \frac{dH}{dx'}$. The result $(*)$ then follows by continuity of multiplication in $L[[x'-a]]$ and uniqueness of limits.

□

We now combine Lemmas 3.10 and Lemmas 3.11, in order to obtain;

Lemma 3.12. *Let C be given in the original coordinate system (x,y), then, for every branch γ of G', we have that $ord_\gamma(\frac{dh}{dx}) = 2$.*

Proof. By Lemma 3.11 and the fact that $x(x') = \frac{1}{x'}$, we have;

$$\frac{dH}{dx'} = -\frac{1}{x'^2}\frac{dh}{dx} = -x^2\frac{dh}{dx}, \ (*)$$

This identity holds at the level of Cauchy series in $L[[x'-a]]$ for corresponding branches satisfying the conditions of Lemma 3.10. However, using Lemma 3.10, we can also consider $G = \frac{dH}{dx'} + \frac{1}{x'^2}\frac{dh}{dx}$ as a rational function on the curve C. By $(*)$, for an appropriate branch γ, satisfying the conditions of Lemma 3.10, we have that the Cauchy series expansion of G at γ is identically zero. By Lemma 2.1, this can only occur if G vanishes identically on C. Hence, we can assume that $(*)$ holds at the level of rational functions on C as well. By Lemma 3.10, we found that, for a branch γ of G'_1, $ord_\gamma(\frac{dH}{dx'}) = 0$. Using the fact that the branches of G'_1 are centred at $x' = 0$, we have, straightforwardly, that, for such a branch γ, $ord_\gamma(-x'^2\frac{dH}{dx'}) = 2$. Using Lemma 2.3, applied to the homography Θ given in (i), we have that $ord_\gamma(\frac{dh}{dx}) = 2$ for any given branch of G', as required.

□

(ii). Construction for J';

We use the change of variables $x = y'$ and $y = x'$. This is a homography as the map;

$$\Theta : (x', y') \rightarrow (y', x')$$

is the restriction to affine coordinates of the map;

$$\Theta : [X' : Y' : Z'] \mapsto [Y' : X' : Z']$$

Again, as Θ is a homography, the character of corresponding branches is preserved. Let $F(x', y')$ denote the equation of C in this new coordinate system. The point P, given by $[0 : 1 : 0]$, relative to the coordinate system (x, y), corresponds to the point $[1 : 0 : 0]$ in the coordinate system (x', y'). Hence, the weighted set J'_1, corresponding to J', consists of branches in finite position relative to the coordinate system (x', y'), defined by the branches of contact between C and tangents parallel to the x'-axis. In particular, they *cannot* be branches of contact between C and tangents parallel to the y'-axis. Hence, we can again apply the power series method given by Lemma 3.7. Let $H(x', y')$ be the rational function corresponding to $h(x, y)$ from Lemma 3.4. We claim the following;

Lemma 3.13. *Let hypotheses be as in (ii). Then, for every branch γ of J'_1, we have that $ord_\gamma(\frac{dH}{dx'}) = 0$.*

Proof. The proof is similar to Lemma 3.10. By (iii) of Lemma 3.5, we have that $G = (h = \infty)$ and J' are disjoint. Hence, for every branch γ of J', $val_\gamma(h) < \infty$. For a given branch γ of J', let $val_\gamma(h) = c$ and consider the rational function $h - c$. We then have that $val_\gamma(h - c) = 0$. If $ord_\gamma(h - c) \geq 2$, then γ would be multiple for $(h = c)$, hence, by Lemma 3.4, would belong to $J = Jac(g^1_n)$. This contradicts the fact, from (iii) of Lemma 3.5, that J and J' are disjoint. Therefore, we must have that $ord_\gamma(h - c) = 1$. It follows that, for any branch γ of J'_1, we can find a constant c such that $ord_\gamma(H - c) = 1$ as well. Now, if γ is centred at (a, b) in the coordinate system (x', y'), then, using the method of Lemma 3.7, H determines an algebraic power series at γ of the form;

$$H(x', y'(x')) = c + c_1(x' - a) + \ldots \ (c_1 \neq 0)$$

By differentiating this expression, we have that $ord_\gamma(\frac{dH}{dx'}) = 0$ as required.

□

We now make an obvious extension to Lemma 3.7;

Lemma 3.14. *Let γ be a branch of C, in finite position, with coordinates (a, b), such that the tangent line of γ is not parallel to the x-axis. Then one can find a power series representation of γ of the form;*

$$x(y) = a + c_1(y - b) + c_2(y - b)^2 + \ldots \ (*)$$

Proof. The proof is the same as Lemma 3.7. □

We can also make a similar extension to the remarks made between the proof of Lemma 3.7 and Lemma 3.9. Namely, we can define $\frac{dx}{dy}$, $\frac{dh}{dy}$ (as rational functions on C), and we have that;

$$\frac{dx}{dy} = -\frac{f_y}{f_x}, \ \frac{dh}{dy} = \frac{h_y f_x - h_x f_y}{f_x}$$

We now claim the following;

Lemma 3.15. *Let C be given in the original coordinate system (x, y) and let γ be a branch of C, centred at (a, b), with the property that its tangent line l_γ is neither parallel to the x-axis nor to the y-axis. Then the sequences $(x(y), y(x(y))$ and $(x(y), y)$ both parametrise the branch γ and moreover;*

$$(x(y), y(x(y))) = (x(y), y)$$

as an identity of sequences of algebraic power series in $L[[y - b]]$.

Proof. By Lemmas 3.7 and 3.14, the sequences $(x, y(x))$ and $(x(y), y)$ both parametrise the branch γ, as algebraic power series, in $L[[x - a]]$ and $L[[y - b]]$ respectively. Moreover, we claim that;

$$ord_{(y-b)}(x(y) - a) = 1, \ (*)$$

This follows from a close inspection of the proof of Lemma 3.7, using the fact that $\{a_1, b_1\}$ given there are both non-zero, by the hypotheses on the branch γ. Now, by $(*)$ and the fact that $y(x)$ defines an algebraic power series in $L[[x - a]]$, the substitution of $x(y)$ for x in this power series defines an algebraic power series in $L[[y - b]]$. We need to show that the sequence $(x(y), y(x(y)))$ still parametrises γ, $(**)$. The proof is very similar to Lemma 3.7. Suppose that $F(x,y)$ is an algebraic function. Then we have that;

$F(x,y(x))\equiv 0$ iff $F(x(y),y(x(y)))\equiv 0$ iff F vanishes on C.

If $ord_{(x-a)}F(x,y(x))=m<\infty$, then we have;

$F(x,y(x))=(x-a)^m u(x,a)$, where $u(x,a)$ is a unit in $L[[x-a]]$.

We then have that;

$F(x(y),y(x(y)))=(x(y)-a)^m u(x(y),a)$, (1)

By $(*)$, we have that;

$(x(y)-a)=(y-b)v(y,b)$, where $v(y,b)$ is a unit in $L[[y-b]]$, (2)

Combining (1) and (2) gives that;

$F(x(y),y(x(y)))=(y-b)^m v(y,b)^m u((y-b)v(y,b))$, where $u(x)$ is a unit in $L[[x]]$

It is easily checked that $v(y,b)^m u((y-b)v(y,b))$ defines a unit in $L[[y-b]]$. Hence;

$ord_{(y-b)}F(x(y),y(x(y)))=m$

as well. This proves $(**)$ as required. In order to show the last part of the lemma, we use $(*)$ above and the method of Lemma 3.11, (uniqueness of parametrisation of a branch with given first term in the sequence, this was the proof of $(**)$ in that Lemma). The details are left to the reader.

□

Lemma 3.16. *Let C be given in the original coordinate system (x,y). Then;*

$\frac{dh}{dy}=\frac{dh}{dx}\cdot\frac{dx}{dy}$ $(*)$

where the identity can be taken either at the level of Cauchy series, for branches γ of C with the property that they are in finite position *and their tangent line is neither parallel to the x-axis nor to the y-axis, or at the level of rational functions on C.*

Proof. For a branch γ, centred at (a,b), satisfying the requirements of the lemma, we consider the composition;

$$h : y \mapsto x(y) \mapsto h(x,y(x))|_{x(y)} = h(x(y),y(x(y))) = h(x(y),y)\ (\dagger)$$

where the final identity comes from Lemma 3.15. Applying a chain rule for Cauchy series, $(\dagger\dagger)$, to $(\dagger)$, we obtain the result $(*)$ at the level of Cauchy series. Again, we justify $(\dagger\dagger)$ in the following form, one should compare the result with the version given in Lemma 3.11;

Another Chain Rule for Cauchy Series.

Let $x(y)$ define an algebraic power series in $L[[y-b]]$, with constant term $x(b)=a$, and let $h(x)$ define a Cauchy series in $Frac(L[[x-a]])$. Then $g(y)=h(x(y))$ defines a Cauchy series in $Frac(L[[y-b]])$ and;

$$\frac{dg}{dy} = \frac{dh}{dx}|_{x(y)} \cdot \frac{dx}{dy}\ (**)$$

as an identity of Cauchy series in $Frac(L[[y-b]])$.

The first part of the claim follows easily from the obvious fact that $x(y)-a$ belongs to $(y-b)L[[y-b]]$, hence, as $h(x)$ is a Cauchy series in $Frac(L[[x-a]])$, the formal substitution of $x(y)$ in $h(x)$ (and $\frac{dh}{dx}$) defines a Cauchy series in $Frac(L[[y-b]])$. In order to show $(**)$, let $\{x_n(y)\}_{n\geq 1}$ define a sequence of *polynomials* in $L[y-b]$ with the property that $\{x_n(y)\}_{n\geq 1} \rightarrow x(y)$ in the non-archimidean topology on $L[[y-b]]$ and $x_n(b)=a$. Let $g_n(y)=h(x_n(y))$ be the corresponding Cauchy series in $Frac(L[[y-b]])$. Then we have that;

$$\frac{dg_n}{dy} = \frac{dh}{dx}|_{x_n(y)} \cdot \frac{dx_n}{dy}$$

as an identity of Cauchy series in $Frac(L[[y-b]])$, by the proof of the previous version of the Chain Rule in Lemma 3.11. Now, by proofs given in the paper [9], we have that $(\frac{dx_n}{dy})_{n\geq 1} \rightarrow \frac{dx}{dy}$ as algebraic power series in $L[[y-b]]$. We now claim that $\frac{dh}{dx}|_{x_n(y)} \rightarrow \frac{dh}{dx}|_{x(y)}$ as Cauchy series in $Frac(L[[y-b]])$, $(***)$. In order to see this, observe that we can write $\frac{dh}{dx} = h_0 + h_1$, where h_1 belongs to $(x-a)L[[x-a]]$, and $h_0 = \frac{dh}{dx} - h_1$ is a finite sum of terms of order at most 0. We then have, by explicit calculation, that;

$h_1 = \sum_{m\geq 1} a_m(x-a)^m$

$h_1(x(y)) - h_1(x_n(y)) = \sum_{m\geq 1} a_m[(x(y)-a)^m - (x_n(y)-a)^m]$

$= \sum_{m\geq 1} a_m[[(x_n(y)-a)+(x-x_n)(y)]^m - (x_n(y)-a)^m]$

$= \sum_{m\geq 1} a_m(x-x_n)(y)r_m(y)$, with $ord_{(y-b)}r_m(y) \geq m-1$

Using the definition of convergence, given any $s \geq 1$, we can find $n(s)$ such that $ord_{(y-b)}(x-x_{n(s)})(y) \geq s$. The above calculation then shows that $ord_{(y-b)}[h_1(x(y)) - h_1(x_{n(s)}(y))] \geq s$ as well. Hence, we must have that $\{h_1(x_n(y))\}_{n\geq 1} \rightarrow h_1(x(y))$ in $L[[y-b]]$. As h_0 is a finite sum of terms, it follows easily, by continuity of the basic operations $\{+,.\}$, that $\{h_0(x_n(y))\}_{n\geq 1} \rightarrow h_0(x(y))$ in $Frac(L[[y-b]])$. Hence, the result $(***)$ follows. A similar argument shows that $\{g_n(y)\}_{n\geq 1} \rightarrow g(y)$ in $Frac(L[[y-b]])$. Hence, using arguments in [9], we have that $(\frac{dg_n}{dy})_{n\geq 1} \rightarrow g$ in $Frac(L[[y-b]])$, as well. Now, $(**)$ follows from uniqueness of limits and continuity of multiplication for the non-archimidean topology on $Frac(L[[y-b]])$.

In order to complete the proof of Lemma 3.16, it remains to prove that the identity $(*)$ may be taken at the level of rational functions on C. This follows from the same argument given at the beginning of Lemma 3.12.

□

We now show the following;

Lemma 3.17. *Let hypotheses be as in* (ii)*, then;*

$\frac{dH}{dx'} = \frac{dh}{dy} = \frac{dh}{dx}\frac{dx}{dy} = -\frac{f_y}{f_x}\frac{dh}{dx}$

where the identities may be taken either at the level of Cauchy series, for corresponding branches satisfying the requirement that they are in finite position and are not branches of contact for tangent lines parallel to the x'*-axis or* y'*-axis and tangent lines parallel to the* x*-axis or* y*-axis respectively, or at the level of rational functions on* C*.*

Proof. Let γ and γ' be such corresponding branches, centred at (a,b) and (b,a) respectively, in the coordinate systems (x',y') and (x,y). By

Lemma 3.7, we can find a parametrisation $(x', y'(x'))$ of γ such that;

$\frac{dH}{dx'} = \frac{d}{dx'}H(x', y'(x'))$, (1)

as a Cauchy series in $Frac(L[[x'-a]])$. As the morphism Θ given in (ii) is a homography, we have that the sequence;

$(x(x', y'(x')), y(x', y'(x'))) = (y'(x'), x')$

also parametrises γ' in the sense of Remarks 3.8, see the corresponding proof of Lemma 3.11. Moreover, we have that;

$H(x', y'(x')) = h(y'(x'), x')$, (2)

as a Cauchy series in $Frac(L[[x'-a]])$, by definition of H and h. By the hypotheses on γ', we can find a parametrisation of γ', given by Lemma 3.14, of the form $(x(y), y)$. Making the substitution $y = x'$, gives an identical parametrisation $(x(x'), x')$ in the variable x'. Using the uniqueness result for such parametrisations, see the proof of Lemma 3.11, we have that;

$(y'(x'), x') = (x(x'), x')$, (3)

as sequences of algebraic power series in $Frac(L[[x'-a]])$. We, therefore, have, combining $(1), (2)$ and (3), that;

$\frac{dH}{dx'} = \frac{d}{dx'}h(y'(x'), x') = \frac{d}{dx'}h(x(x'), x') = \frac{dh}{dy}|_{y=x'}$

This gives the first identity at the level of Cauchy series in $Frac(L[[x'-a]])$. The second identity, at the branch γ', for Cauchy series, comes from Lemma 3.16. The third identity, for Cauchy series, follows from the remarks made between Lemmas 3.14 and 3.15. In order to obtain the identities at the level of rational functions on C, it is sufficient to consider the case $\frac{dH}{dx'} = \frac{dh}{dy}$, the other identities having already been considered previously. Let G be the rational function corresponding to $\frac{dh}{dy}$ in the coordinate system (x', y'). We may consider $\frac{dH}{dx'} - G$ as a rational function on the curve C, in this coordinate system. For a parametrisation $(x', y'(x'))$ of the branch γ, we then have that;

$\frac{dH}{dx'}(x', y'(x')) = \frac{dH}{dx'} = \frac{dh}{dy}|_{y=x'} = \frac{dh}{dy}(y'(x'), x') = G(x', y'(x'))$

Hence, $(\frac{dH}{dx'}-G)(x',y(x'))\equiv 0$. This implies, by arguments already shown in this paper, that $\frac{dH}{dx'}-G$ vanishes identically on C, hence $\frac{dH}{dx'}=\frac{dh}{dy}$ as rational functions on C, as required.

□

We can now show the following;

Lemma 3.18. *Let C be given in the original coordinate system (x,y), then, for each branch γ of J', $ord_\gamma(\frac{dh}{dx})=-1$.*

Proof. By Lemma 3.17, we have that;

$\frac{dH}{dx'}=-\frac{f_y}{f_x}\frac{dh}{dx}$, $(*)$

as an identity of rational functions on C. We now claim that, for a branch γ of J', $ord_\gamma(-\frac{f_y}{f_x})=1$ $(**)$. In order to see this, observe that, by (ii) of Lemma 3.5, such a branch is based at a non-singular point q of C. Hence, the vector (f_x,f_y), evaluated at q, is not identically zero. By hypotheses on γ, that the tangent line is parallel to the y-axis, we have that $f_y(q)=0$, hence $f_x(q)\neq 0$. In particular, this implies that $0<val_\gamma(f_x)<\infty$ and, therefore, that $ord_\gamma(-f_x)=0$. It also implies that $val_\gamma(f_y)=0$. In order to show $(**)$, it is therefore sufficient to prove that $ord_\gamma(f_y)=1$. By the result of Lemma 2.1, this is equivalent to showing that;

$I_\gamma(C,f_y=0)=1$, $(\dagger)$

We achieve this by the following method;

Let $q=(a,b)$ and let l_γ be the tangent line to the branch γ. By the assumption that γ belongs to J' ,(ii) of Lemma 3.5 and the fact that γ is based at the non-singular point q, we have;

$I_\gamma(C,l_\gamma)=I_q(C,l_\gamma)=I_q(f=0,l_\gamma)=2$

As was shown in the paper [8], I_q is symmetric for plane algebraic curves. Hence, we must have that;

$I_q(l_\gamma,C)=I_q(l_\gamma,f=0)=2$, $(\dagger\dagger)$

as well. By the assumption that l_γ is parallel to the y-axis, we clearly have that the sequence $(a,b+t)$ parametrises this tangent line. Hence, using Lemma 2.1 and (††), we have that;

$ord_t f(a, b+t) = 2$ and $f(a, b+t) = t^2 u(t)$ for a unit $u(t) \in L[[t]]$

Applying the Rule for differentiating algebraic power series, given in Lemma 3.7, we have that;

$f_y(a, b+t) = 2tu(t) + t^2 u'(t)$, ([1])

Hence;

$ord_t f_y(a, b+t) = I_q(l_\gamma, f_y = 0) = I_q(f_y = 0, l_\gamma) = 1$

This clearly implies that $f_y = 0$ is non-singular at q and that the tangent line $l_{(f_y=0)}$ is distinct from l_γ. Hence, the intersection between C and $f_y = 0$ is algebraically transverse at q, therefore, as was shown in [8], we obtain (†) and then $(**)$ follows as well.

In order to finish the proof of the lemma, we use the fact from Lemma 3.13, that, for a branch γ of J', $ord_\gamma(\frac{dH}{dx'}) = 0$. Combined with $(*)$ and $(**)$, it follows easily that $ord_\gamma(\frac{dh}{dx}) = -1$ as required.

□

We can now summarise what we have shown in the following theorem;

Theorem 3.19. *Theorem on Differentials*

Let hypotheses and notation be as in the remarks immediately before Lemma 3.9, then;

$(\frac{dh}{dx} = 0) = J \cup 2G'$ *and* $(\frac{dh}{dx} = \infty) = J' \cup 2G$

In particular, one has that $J + 2G' \equiv J' + 2G$.

Proof. By Lemma 3.9, a branch γ, distinct from $G \cup G' \cup J'$, is counted in $(\frac{dh}{dx} = 0)$ exactly if it appears in $J = Jac(g^1_m)$, and is, moreover, also counted with the same multiplicity. By Lemma 3.12, each branch γ of G' is counted twice in $(\frac{dh}{dx} = 0)$. Again, by Lemma 3.9, each branch γ of G is counted twice in $(\frac{dh}{dx} = \infty)$. Finally, by Lemma 3.18, each

[1] An alternative argument is required when $char(L) = 2$, we leave the details to the reader

branch γ of $J'=Jac(g_n^1)$ is counted once in $(\frac{dh}{dx}=\infty)$. Hence, the first part of the lemma is shown. As $\frac{dh}{dx}$ defines a rational function on C, if $s=deg(\frac{dh}{dx})$, then, by Lemma 2.4, we can associate a g_s^1, defined by $(\frac{dh}{dx})$, to which $J+2G'$ and $J'+2G$ belong as weighted sets. Hence, by the definition of linear equivalence given in Definition 2.6, we have that $J+2G'\equiv J'+2G$ as required.

□

Remarks 3.20. *In the above theorem, one may assume that G and G' denote* any 2 *weighted sets of the given g_n^1 and g_m^1 respectively. In order to see this, suppose that G_1 and G_1' are any two such weighted sets. Then we claim that;*

$$J'+2G_1\equiv J'+2G \text{ and } J+2G_1'\equiv J+2G' \ (*)$$

In order to show $(*)$, by Theorem 2.7 we may assume that $G-G_1=div(f_1)$ and $G'-G_1'=div(f_2)$ for rational functions $\{f_1,f_2\}$ on C. By Lemma 2.9, we then have that;

$$2G-2G_1=div(f_1^2) \text{ and } 2G'-2G_1'=div(f_2^2)$$

Hence, in particular, by Lemma 2.4 and Definition 2.6;

$$2G\equiv 2G_1 \text{ and } 2G'\equiv 2G_1' \ (**)$$

The claim $(*)$ then follows by adding the fixed branch contributions J' and J to the series defining the equivalences in $(**)$ and using Lemma 1.13. From $(*)$, we deduce the equivalence;

$$J'+2G_1\equiv J+2G_1'$$

immediately from Theorem 3.19 and Theorem 2.10, as required.

We now improve Theorem 3.19 as follows;

Theorem 3.21. *Let a $g_{n_1}^1$ and $g_{n_2}^1$ be given on C, with no fixed branch branch contribution. Then, if $J_1=Jac(g_{n_1}^1)$, $J_2=Jac(g_{n_2}^1)$ and $\{G_1,G_2\}$ denote any 2 weighted sets appearing in $\{g_{n_1}^1,g_{n_2}^1\}$, then;*

$$J_1+2G_2\equiv J_2+2G_1$$

Proof. By Lemma 3.4, we can find rational functions $\{h_1, h_2\}$, defining the $\{g^1_{n_1}, g^1_{n_2}\}$ on C, satisfying the conclusion of the lemma with respect to $G_1^\infty = (h_1 = \infty)$ and $G_2^\infty = (h_2 = \infty)$. An inspection of Lemma 3.5 shows that we can also obtain the conclusion there with J_1 replacing J and G_1^∞ replacing G. Now one can follow through the proof up to Theorem 3.19, in order to obtain;

$J_1 + 2G' \equiv J' + 2G_1^\infty$ and $J_2 + 2G' \equiv J' + 2G_2^\infty$

By Remarks 3.20, we can replace G_1^∞ by G_1 and G_2^∞ by G_2, in order to obtain;

$J_1 + 2G' \equiv J' + 2G_1$ and $J_2 + 2G' \equiv J' + 2G_2$

We then obtain, by Theorem 2.24;

$$J_1 - 2G_1 \equiv J' - 2G' \equiv J_2 - 2G_2$$

Hence, by Theorem 2.23;

$$J_1 - 2G_1 \equiv J_2 - 2G_2$$

and, therefore, the lemma follows from Theorem 2.24 again.

□

We now improve Definition 3.2.

Definition 3.22. *Let C be a projective algebraic curve and let an arbitrary g^1_n be given on C. Let K be the fixed branch contribution of this g^1_n, with total multiplicity m and let g^1_{n-m} be obtained by removing this fixed branch contribution. Then we define;*

$$Jac(g^1_n) = 2K + Jac(g^1_{n-m})$$

Remarks 3.23. *Note that, as a consequence of the definition, if a g^1_n is given on C, then a branch γ, which is counted β times for the g^1_n and $\beta+\alpha$ times in a particular weighted set of the g^1_n, is counted $2\beta+\alpha-1$ times in $Jac(g^1_n)$.*

We then have an improved version of Theorem 3.21;

Theorem 3.24. *Let C be a plane projective algebraic curve and let a g_n^1 be given on C. Then, if G is any weighted set in this g_n^1 and $J = Jac(g_n^1)$, then the series $|J - 2G|$ is* independent *of the particular g_n^1.*

Proof. Suppose that a $g_{n_1}^1$ and a $g_{n_2}^1$ are given on C and let $\{K_1, K_2\}$ be their fixed branch contributions of multiplicity $\{m_1, m_2\}$ respectively. Removing these contributions, we obtain a $g_{n_1-m_1}^1$ and a $g_{n_2-m_2}^1$ on C with no fixed branch contributions. Let $\{G_1, G_2\}$ be weighted sets of $\{g_{n_1}^1, g_{n_2}^1\}$, then we can find weighted sets $\{G_1', G_2'\}$ of $\{g_{n_1-m_1}^1, g_{n_2-m_2}^1\}$ such that;

$G_1 = K_1 + G_1'$ and $G_2 = K_2 + G_2'$ $(*)$

Let $J_1' = Jac(g_{n_1-m_1}^1)$ and $J_2' = Jac(g_{n_2-m_2}^1)$, then we have, by Theorem 3.21;

$J_1' + 2G_2' \equiv J_2' + 2G_1'$

Therefore, by Theorem 2.12;

$(J_1' + 2K_1) + 2(K_2 + G_2') \equiv (J_2' + 2K_2) + 2(K_1 + G_1')$

If $J_1 = Jac(g_{n_1}^1)$ and $J_2 = Jac(g_{n_2}^1)$, we then have, by Definition 3.22 and $(*)$, that;

$J_1 + 2G_2 \equiv J_2 + 2G_1$

Hence, the result follows by Theorem 2.24 and Definition 2.25. □

We now transfer this result to an arbitrary projective algebraic curve C, using birationality. We first require;

Lemma 3.25. *Let $[\Phi] : C_1 \leftrightsquigarrow C_2$ be a birational map. Let a g_n^1 be given on C_2, with corresponding $[\Phi]^*(g_n^1)$ on C_1. Then we have that;*

$[\Phi]^*(Jac(g_n^1)) = Jac([\Phi]^*(g_n^1))$

Proof. The result follows trivially from the fact that $[\Phi]^*$ defines a bijection on branches, see Lemma 5.7 of [9], the definition of $Jac(g_n^1)$ given in Definition 3.22 and the definition of $[\Phi]^*(g_n^1)$ given in Theorem 1.14.

□

We can now extend Theorem 3.24;

Theorem 3.26. *Let C be an arbitrary projective algebraic curve and let a g_n^1 be given on C. Then, if G is any weighted set in this g_n^1 and $J = Jac(g_n^1)$, then the series $|J - 2G|$ is* independent *of the particular g_n^1.*

Proof. By Theorem 1.33 of [9], we find a birational $\Phi : C \leftrightsquigarrow C'$, with C' a plane projective curve. Suppose that a $g_{n_1}^1$ and a $g_{n_2}^1$ are given on C. Let $\{G_1, G_2\}$ be weighted sets of $\{g_{n_1}^1, g_{n_2}^1\}$ and let $\{J_1, J_2\}$ be the Jacobians of $\{g_{n_1}^1, g_{n_2}^1\}$. Using Lemma 3.25, we, therefore, obtain that $\{[\Phi^{-1}]^*(G_1), [\Phi^{-1}]^*(G_2)\}$ are weighted sets of $\{[\Phi^{-1}]^*(g_{n_1}^1), [\Phi^{-1}]^*(g_{n_2}^1)\}$ and $\{[\Phi^{-1}]^*(J_1), [\Phi^{-1}]^*(J_2)\}$ are Jacobians of $\{[\Phi^{-1}]^*(g_{n_1}^1), [\Phi^{-1}]^*(g_{n_2}^1)\}$. By the result of Theorem 3.24, we obtain;

$$[\Phi^{-1}]^*(J_1) - 2[\Phi^{-1}]^*(G_1) \equiv [\Phi^{-1}]^*(J_2) - 2[\Phi^{-1}]^*(G_2)$$

Applying $[\Phi]^*$ to this equivalence, using Theorem 2.32 and the fact that this map is linear on branches, we obtain that;

$$J_1 - 2G_1 \equiv J_2 - 2G_2$$

Hence, by Definition 2.30, $|J_1 - 2G_1| = |J_2 - 2G_2|$ as required. □

Using this result, we are able to show the following;

Theorem 3.27. *Let C be a projective algebraic curve and let a g_n^r be given on C. Then, if $\{g_n^1, g_n^{1'}\}$ are any* two *series with;*

$$g_n^1 \subseteq g_n^r \text{ and } g_n^{1'} \subseteq g_n^r$$

then;

$$Jac(g_n^1) \equiv Jac(g_n^{1'})$$

Proof. Let $\{G, G'\}$ be weighted sets of $\{g_n^1, g_n^{1'}\}$ and let $\{J, J'\}$ be their Jacobians. By Theorem 3.26, we have that;

$$J - 2G \equiv J' - 2G'$$

As $\{G, G'\}$ both belong to the same g_n^r, we also have that $G \equiv G'$. By Theorem 2.29, we then obtain that $J \equiv J'$ as required.

□

As a result of this theorem, we can make the following definition;

Definition 3.28. *Jacobian Series*

Let C be a projective algebraic curve and let a g_n^r be given on C, with $r \geq 1$, then we define $Jacob(g_n^r)$ to be the complete *series containing the Jacobians $Jac(g_n^1)$ for* any $g_n^1 \subseteq g_n^r$.

Remarks 3.29. *If we choose any two subordinate $g_n^1 \subseteq g_n^r$, $g_n^{1'} \subseteq g_n^r$ then, we have, by Theorem 2.18 and Theorem 3.27, that $|Jac(g_n^1)| = |Jac(g_n^{1'})|$. Hence, there* does *exist a complete linear series with the property required of the definition.*

We now make the further;

Definition 3.30. *Let C be a projective algebraic curve and let G be an effective weighted set, defining a complete series $|G|$, with $dim(|G|) \geq 1$. Then, we define;*

$$|G_j| = Jacob(|G|)$$

We then have;

Theorem 3.31. *Fundamental Theorem of Jacobian Series*

Let C be a projective algebraic curve and let $\{A, B\}$ be effective weighted sets on C, with multiplicity $\{m, n\}$ respectively, defining complete series $|A|$ and $|B|$, with $dim(|A|) \geq 1$ and $dim(|B|) \geq 1$. Then;

$$|(A + B)_j| = |A_j| + |2B| = |B_j| + |2A|$$

Proof. Consider first the complete series $|A| + |B| = |A + B|$. By the hypotheses, we can find a $g_m^1 \subseteq |A|$, $(*)$. Adding B as a fixed branch contribution to this g_m^1, we obtain a $g_{m+n}^1 \subseteq |A + B|$. We then have, by Definition 3.22;

$$Jac(g_{m+n}^1) = Jac(g_m^1) + 2B$$

It then follows immediately from Remarks 3.29 and Definition 3.30 that;

$|(A+B)_j| = |Jac(g^1_m)+2B| = |Jac(g^1_m)| + |2B|$

Again, by Remarks 3.29, $(*)$ and Definition 3.30, we have that;

$|Jac(g^1_m)| = |A_j|$

Hence,

$|(A+B)_j| = |A_j| + |2B|$

as required. The remaining equality is similar.

□

We now have;

Theorem 3.32. *Canonical Series*

Let C be a projective algebraic curve and let A be an effective weighted set, such that $dim(|A|) \geq 1$, $(*)$. Then the series;

$|A_j| - |2A|$

depends only on C and not *on the choice of A with the requirement $(*)$.*

Proof. Suppose that $\{A, B\}$ are effective weighted sets satisfying $(*)$. By Theorem 3.31, we have that;

$|A_j| + |2B| = |B_j| + |2A|$

Subtracting the series $|2A| + |2B|$ from both sides, we obtain that;

$|A_j| - |2A| = |B_j| - |2B|$

as required.

□

We now make the following;

Definition 3.33. *Genus of a Projective Algebraic Curve*

Let C be a projective algebraic curve, then we define the genus ρ of C by the formula;

$$2\rho - 2 = r - 2n$$

where $n = order(|A|)$ and $r = order(|A_j|)$, for any effective weighted set with the property that $dim(|A|) \geq 1$.

Remarks 3.34. *This is a good definition by Theorem 3.32.*

It is clear from the definition that the genus ρ is rational. We also have;

Theorem 3.35. *The genus ρ is a birational invariant.*

Proof. Let $\Phi : C \leftrightsquigarrow C'$ be a birational map. If A is an effective weighted set on C' with $dim(|A|) \geq 1$, then $[\Phi]^*A$ is an effective weighted set on C with $dim(|[\Phi]^*A|) \geq 1$. We clearly have that $order(|A|) = order(|[\Phi]^*A|)$. We also have that $Jacob(|[\Phi^*]A|) = [\Phi]^*(Jacob(|A|))$, using Lemma 3.25. In particular, we then have that $order(|A_j|) = order(|([\Phi^*]A)_j|)$. The result then follows immediately from Definition 3.33. □

Theorem 3.36. *Let C be a plane projective algebraic curve of order n, having d nodes as singularities, then;*

$$\rho = \frac{(n-1)(n-2)}{2} - d$$

In particular, the genus ρ of any projective algebraic curve is an integer.

Proof. Fix a generic point P in the plane, and consider the pencil of lines passing through P. This defines a g_n^1 on C, where $n = order(C)$. Using Lemma 3.5 and the corresponding coordinate system (x, y), we may assume that $Jac(g_n^1)$ consists exactly of the non-singular branches of C (each counted once) in finite position, whose tangent lines are parallel to the y-axis. (Here, the branch formulation of a g_n^1 is critical, the lines in the given pencil passing through the nodes of C make no contribution to $Jac(g_n^1)$ as they pass through each *branch* of the node transversely.) It follows that r in Definition 3.33 is exactly the number of these branches. By Remarks 4.2 below, this is exactly the class of C, which gives the following formula relating the class and the genus;

$2\rho - 2 = r - 2n$, $(\dagger)$

Now, applying the Plucker formula, given in Section 4, we have;

$r = n(n-1) - 2d$, $(*)$

Combining $(\dagger)$ and $(*)$, we obtain;

$\rho = \frac{n(n-1)-2d}{2} - (n-1) = \frac{(n-1)(n-2)}{2} - d$

as required. The remaining part of the theorem then follows immediately from Theorem 3.35 and the fact that any projective algebraic curve is birational to a plane projective curve, having at most nodes as singularities. (Note when C has no nodes, the genus gives the number of alcoves of n lines in general position).

□

Theorem 3.37. *Let C be a plane projective curve of order n, having d nodes as singularities, then;*

$d \leq \frac{(n-1)(n-2)}{2}$

In particular, $\rho \geq 0$, for any projective algebraic curve.

Proof. By $(*)$ of Theorem 3.36, we have;

$d \leq \frac{n(n-1)}{2} \leq \frac{(n-1)(n+2)}{2}$, $(**)$

As the parameter space Par_{n-1} of plane curves, having dimension $n-1$, has dimension $\frac{(n-1)(n+2)}{2}$, by $(**)$, we can find a plane curve C' of order $n-1$, passing through the d nodes of C and through $\frac{(n-1)(n+2)}{2} - d$ further non-singular points of C. As C is irreducible, the curve C' cannot contain C as a component, therefore, by Bezout's Theorem, must intersect C in exactly $n(n-1)$ points, counted with multiplicity. Using Lemma 1.11 of [11], we have the total intersection multiplicity contributed by the nodes is at least $2d$, and, clearly, the total intersection multiplicity contributed by the further non-singular points is at least $\frac{(n-1)(n+2)}{2} - d$. Therefore,

$n(n-1) \geq (2d + \frac{(n-1)(n+2)}{2} - d)$

which gives that;

$\frac{(n-1)(n-2)}{2} \geq d$

as required. Again, using the fact that any projective algebraic curve is birational to a plane projective curve, having at most nodes as singularities, and combining the first part of this theorem with Theorem 3.36, we obtain that $\rho \geq 0$.

□

We now require the following definition;

Definition 3.38. *Let C be a projective algebraic curve. We say that C is rational if it is birational to a line.*

We now show the following result;

Theorem 3.39. *Let C be a projective algebraic curve, then C is rational if and only if its genus $\rho = 0$.*

Proof. By Theorem 3.35, it is sufficient to show that the genus of a line is zero. As a line in the plane has no nodes and is of order 1, this follows immediately from Theorem 3.36. Conversely, suppose that the genus ρ of C is zero. We can assume that C is a plane projective curve, having just d nodes as singularities. By Theorem 3.36, we then have that;

$d = \frac{(n-1)(n-2)}{2}$ (1)

Now, consider the linear system Σ of curves of degree $(n-1)$, passing through the d nodes of C. As Par_{n-1} has dimension $\frac{(n-1)(n+2)}{2}$, using (1), this system has dimension;

$h \geq \frac{(n-1)(n+2)}{2} - \frac{(n-1)(n-2)}{2} = 2(n-1)$ (2)

In particular, as we can assume that that $n \geq 2$, otherwise the result is proved, we have that $h \geq 2$. Now, choose a further $(2n-3)$ non-singular points on C. By (2), there exists a linear system Σ_1 of plane curves of degree $(n-1)$, passing through the d nodes of C and these further non-singular points, and this system has dimension;

$h_1 \geq (2n-2) - (2n-3) = 1$ (3)

Now, as C is irreducible, any form F_λ in the linear system Σ_1 has finite intersection with C, hence, using the Branched Version of Bezout's

Theorem, given in Theorem 5.13 of [9], given a curve F_λ belonging to Σ_1, $C \sqcap F_\lambda$ defines an effective weighted set W_λ of total multiplicity $n(n-1)$. We, therefore, obtain a $g^{h_1}_{n(n-1)}$ on C. By construction, the number of base branches for this $g^{h_1}_{n(n-1)}$ is at least;

$$2d + (2n-3) = (n-1)(n-2) + (2n-3) = n(n-1) - 1$$

We consider the following cases;

(i). One of these $n(n-1)-1$ base branches is 2-fold for the $g^{h_1}_{n(n-1)}$.

(ii). There exists a further base branch for the $g^{h_1}_{n(n-1)}$. (†)

Using the Branched Version of Bezout's theorem again, we would then have that the total intersection multiplicity of intersection is contributed by the base branches. By Theorem 1.3, this implies that some form contains all of C, contradicting the fact that C is irreducible. Hence, (†) cannot occur. It follows that each base branch is 1-fold, for the $g^{h_1}_{n(n-1)}$ and there are exactly $n(n-1)-1$ base branches. By Definition 1.4, we can remove this base branch contribution, in order to obtain a $g^{h_1}_1$ with no base branches. By Lemma 1.16 and (3), we then have that $h_1 = 1$. The g^1_1 then obtained is simple, by Definition 2.29 of [9], hence, by Lemma 2.30 of [9], defines a birational map to P^1. This proves the theorem.

□

Remarks 3.40. *The characterisation of rational curves in terms of the vanishing of their genus may lead to applications in the field of Zariski structures, see the remarks immediately after Theorem 1.3 of* [9]*. As the technique we have used to define the genus of an algebraic curve is geometric rather than algebraic, one would hope to extend the definition to Zariski curves, and therefore, to find examples of non-algebraic Zariski curves which exhibit properties of lines.*

4. A Plucker Formula for Plane Algebraic Curves

The purpose of this section is to give a geometric proof of an elementary Plucker formula for plane projective algebraic curves. The proof that we give depends on the method of the Italian school, however the underlying geometrical idea is due to Plucker. One can find other modern *algebraic* proofs, see, for example [2]. In order to state the theorem, we first require the following definition;

Definition 4.1. *Class of a Plane Algebraic Curve*

Let $C \subset P^2$ be a plane projective algebraic curve. Then we define the class of C to be the number of its tangent lines passing through a generic point $Q \in P^2$.

Remarks 4.2. *In order to see that this is a good definition, let $F(X,Y,Z)$ be a defining equation for C. For a non-singular point p of C, the equation of the tangent line l_p is given by;*

$$F_X(p)U + F_Y(p)V + F_Z(p)W = 0$$

For the finitely many singular points $\{p_1, \ldots p_j, \ldots, p_r\}$, which are the origins of branches $\{\gamma_1^1, \ldots, \gamma_1^{t(1)}, \ldots, \gamma_r^1, \ldots, \gamma_r^{t(r)}\}$, we obtain finitely many tangent lines $\{l_{\gamma_1^1}, \ldots, l_{\gamma_1^{t(1)}}, \ldots, l_{\gamma_r^1}, \ldots, l_{\gamma_r^{t(r)}}\}$. It is easily checked that the union V of these finitely many tangent lines is defined over the field of definition of C. Hence, we may assume that Q does not lie on any of these tangents. It also follows from duality arguments, see Section 5, that there exist finitely many bitangent lines $\{l_1, \ldots, l_s\}$ (Check this). Again, it is easily checked that the union W of these finitely many tangent lines is defined over the field of definition of C. Hence, we can assume that Q does not lie on any of these bitangents. Now, the condition $Cl_{\geq m}(Q)$ that there exist at least m tangent lines, centred at non-singular points of C, passing through Q, is given by;

$$\exists_{x_1 \neq \ldots \neq x_m} [\textstyle\bigwedge_{1 \leq j \leq m} (NS(x_j) \wedge F_X(x_j)Q_0 + F_Y(x_j)Q_1 + F_Z(x_j)Q_2 = 0)]$$

The condition $Cl_{=m}(Q)$ that there exist exactly m tangent lines, centred at non-singular points of C, passing through Q, is given by;

$$Cl_{\geq m}(Q) \wedge \neg Cl_{\geq m+1}(Q)$$

It is clear that each predicate Cl_m is defined over the field of definition of C, hence, if it holds for some *generic $Q \in P^2$, it holds for* any *generic $Q \in P^2$. Finally, observe that for* generic *Q, there can only exist finitely many tangent lines passing through Q. (We leave the reader to check this result.) Hence, the class does define a non-negative integer $m \geq 0$. Observe that it is possible for $m = 0$, for example if C is a line. Also, observe that it is possible for $\neg Cl_m(P)$ $(m \geq 0)$ to hold, if P is* not *generic, for example if C is strange. Excluding exceptional cases, one can show, using duality arguments, that a generic point of a projective algebraic curve is an ordinary simple point. In this case,* *it follows that the union W of the tangent lines to non-ordinary simple points is defined over the field of definition of C, hence that the class is witnessed by ordinary simple points.*

We now give a statement of the elementary Plucker formula;

Theorem 4.3. *Let C be a plane projective algebraic curve of order n and class m, with d nodes. Then;*

$$n + m + 2d = n^2$$

Remarks 4.4. *We refer the reader to the papers* [9] *and* [11] *for some relevant terminology. The reader should observe that we are using the* weaker *definition of a node as the origin of exactly* 2 *non-singular branches with distinct tangent directions. The original formula is usually stated with cusps, however we will not require this stronger version.*

Proof. Theorem 4.3

Let the class of m of C be witnessed by a generic point Q of P^2. Let $\{\tau_1, \ldots, \tau_m\}$ be the finitely many tangents of C passing through Q. We can assume that these tangent lines are all based at non-singular points of C and do not coincide with the tangent directions of any of the finitely many nodes of C. Choose a line l, passing though Q, which does not coincide with any of these tangent lines and does not intersect C in any of the finitely many nodes. Let $\{X, Y, Z\}$ be a choice of coordinates such that the line l corresponds to $Z = 0$, hence defines the line at infinity in the affine coordinate system $\{x = \frac{X}{Z}, y = \frac{Y}{Z}\}$. Let $F(x, y) = 0$ define C in this coordinate system. By construction, we have that the tangent lines $\{\tau_1, \ldots, \tau_m\}$ are all parallel in the coordinate system $\{x, y\}$. Let (α, β) be the gradient vector of each of these lines. We define the translation C_t of C as follows;

$$C_t = \{(x, y) : (x - t\alpha, y - t\beta) \in C\} = \{(x, y) : F(x - t\alpha, y - t\beta) = 0\}$$

Now observe that we can find polynomials $\lambda_{(i,j)}(t)$, for $0 \leq (i+j) \leq n$, such that;

$$F(x - t\alpha, y - t\beta) = F(x, y; t) = \textstyle\sum_{0 \leq (i+j) \leq n} \lambda_{(i,j)}(t) x^i y^j \ (*)$$

Let $\{T_0, T_1\}$ be coordinates for P^1 such that $t = \frac{T_0}{T_1}$. Making the substitutions $\{t = \frac{T_0}{T_1}, x = \frac{X}{Z}, y = \frac{Y}{Z}\}$ in $(*)$, and projectivising the resulting equation, by multiplying through by a suitable denominator, we obtain an *algebraic* (not necessarily linear) family of curves, parametrised by P^1. Let $Par_t \subset P^{\frac{n(n+3)}{2}} = Par_n$ be the image of the map;

$$\Phi : P^1 \rightarrow P^{\frac{n(n+3)}{2}}, \ \tfrac{T_0}{T_1} \mapsto (\lambda_{(0,0)}(\tfrac{T_0}{T_1}) : \ldots : \lambda_{(i,j)}(\tfrac{T_0}{T_1}) : \ldots)$$

Then Par_t defines a projective algebraic curve, parametrised by P^1, whose points determine each curve in the algebraic family $\{C_t\}_{t \in P^1}$. For a given $t_0 \in Par_t$, we can define a tangent line l_{t_0} to Par_t by;

$$(\lambda_{(0,0)}(t_0) + \mu\lambda'_{(0,0)}(t_0) : \ldots : \lambda_{(i,j)}(t_0) + \mu\lambda'_{(i,j)}(t_0), \ldots)$$

Remarks 4.5. *Observe that, in the case when t_0 is a smooth point of Par_t, l_{t_0} defines* the *tangent line of Par_t at t_0 in the sense of Section 1 of* [9], $(*)$. *This follows easily from the fact that, for any homogeneous polynomial $G(X_{(0,0)}, \ldots, X_{(i,j)}, \ldots)$ vanishing on Par_t, we have from;*

$$G(\lambda_{(0,0)}(t), \ldots, \lambda_{(i,j)}(t), \ldots) = 0$$

that;

$$\tfrac{\partial G}{\partial X_{(0,0)}}|_{\lambda_{(0,0)}(t)}\lambda'_{(0,0)}(t) + \ldots + \tfrac{\partial G}{\partial X_{(i,j)}}|_{\lambda_{(i,j)}(t)}\lambda'_{(i,j)}(t) + \ldots = 0 \ (1)$$

and from;

$$G(s\lambda_{(0,0)}(t_0), \ldots, s\lambda_{(i,j)}(t_0), \ldots) = 0$$

that;

$$\tfrac{\partial G}{\partial X_{(0,0)}}|_{\lambda_{(0,0)}(t_0)}\lambda_{(0,0)}(t_0) + \ldots + \tfrac{\partial G}{\partial X_{(i,j)}}|_{\lambda_{(i,j)}(t_0)}\lambda_{(i,j)}(t_0) + \ldots = 0 \ (2)$$

In order to extend these considerations to singular points, let $U = \Phi^{-1}(NonSing(Par_t))$ and consider the covers $V \subset U \times P^{\frac{n(n+3)}{2}}$ and $V^* \subset U \times P^{\frac{n(n+3)}{2}}$ defined by;

$$V = \{(t, y) : y \in l_{\Phi(t)}\}$$

$$V^* = \{(t, \lambda) : H_\lambda \supset l_{\Phi(t)}\}$$

Let $\bar{V}$ *and* $\bar{V}^*$ *define the Zariski closure of these two covers inside* $P^1 \times P^{\frac{(n+1)(n+2)}{2}}$. *Let* Par_t^{ns} *be a nonsingular model of* Par_t *with birational map* $\Phi' : Par_t^{ns} \leftrightsquigarrow Par_t$ *and let* $\Phi'' = \Phi'^{-1} \circ \Phi$. *Using the method of Lemma 6.13 in* [9], *one can show that, for* $t_0 \in P^1 \setminus U$, *the fibre* $\bar{V}^*(t_0)$ *consists of parameters for forms* H_λ *containing the tangent line* $l_{\gamma_p^j}$, *where* $\Phi(t_0) = p$ *and the branch* γ_p^j *corresponds to* $p_j \in Par_t^{ns}$, *with* $\Phi''(t_0) = p_j$. *Using the duality argument in Lemma 5.3 below, one then deduces that the fibre* $\bar{V}(t_0)$ *defines the tangent line* $l_{\gamma_p^j}$, $(**)$.

It then follows, by combining the arguments $(*)$ *and* $(**)$, *that, even for a singular point* t_0 *of* Par_t, l_{t_0} *defines a tangent line to the branch corresponding to* $\Phi''(t_0)$.

Following the Italian terminology, we refer to the pencil *of curves defined by* l_{t_0}, *for given* $t_0 \in P^1$, *as the curves* infinitamente vicine *to* C_{t_0}, *and we refer to* any *curve in the pencil, distinct from* C_{t_0}, *by;*

$F(x, y; t_0 + dt_0)$

This notation is motivated by the following fact;

Let $\Phi : P^1 \rightarrow Par_t$ *be the parametrisation of* Par_t, *then, for* any *polynomial representation of* Φ *of the form;*

$\Phi : t \mapsto (\lambda_0(t) : \lambda_1(t) : \ldots)$

l_{t_0} *is generated by* $(\lambda_0(t_0) : \lambda_1(t_0) : \ldots)$ *and* $(\lambda'_0(t_0) : \lambda'_1(t_0) : \ldots)$

The proof of this fact follows easily from the above remarks or from the observation that, if $h(t)$ *is a polynomial with;*

$(\lambda_0(t) : \lambda_1(t) : \ldots) = h(t)(\mu_0(t) : \mu_1(t) : \ldots)$

then $(\lambda'_0(t_0) : \lambda'_1(t_0) : \ldots)$ *corresponds to;*

$h(t_0)(\mu'_0(t_0) : \mu'_1(t_0) : \ldots) + h'(t_0)(\mu_0(t_0) : \mu_1(t_0) : \ldots)$

and hence belongs to the tangent line generated by;

$\{(\mu_0(t_0) : \mu_1(t_0) : \ldots), (\mu'_0(t_0) : \mu'_1(t_0) : \ldots)\}$

We now continue the proof of Theorem 4.3;

Claim 1: Let $t_0 = 0$, then the algebraic curve C_{dt_0}, infinitamente vicine to $C_{t_0} = C$, passes through the finitely many points $\{p_1, \ldots, p_m\}$ which witness the class of C and, moreover;

$I_{p_j}(C_{t_0}, C_{dt_0}) = 1$, for $1 \leq j \leq m$

In order to prove Claim 1, let $p_j = (a_j, b_j)$ be such a point, then we obtain a parametrisation of the tangent line l_{p_j} of the form;

$$(x(s), y(s)) = (a_j + \alpha s, b_j + \beta s)$$

We may assume, see Remarks 4.2, that p_j is an ordinary simple point, hence that $I_{p_j}(C_{t_0}, l_{p_j}) = 2$. This implies that;

$$F(x(s), y(s); t) = F(x(s) - \alpha t, y(s) - \beta t) = (s - t)^2 u(s - t), \quad u(s-t) \text{ a unit in } L[[s-t]] \ (*)$$

Now, differentiating both sides of the equation $(*)$ with respect to t, we obtain that;

$$\begin{aligned} \frac{\partial F}{\partial t}|_{(x(s),y(s),t)} &= -2(s - t)u(s - t) - (s - t)^2 u'(s - t) \\ &= -(s - t)[2u(s - t) + (s - t)u'(s - t)] \end{aligned}$$

Setting $t = 0$, we obtain that;

$$F_{dt_0}(x(s), y(s)) = -s[2u(s) + su'(s)]$$

Hence, $I_{p_j}(C_{dt_0}, l_{p_j}) = 1$. This implies, by arguments given in [8], that $I_{p_j}(C_{t_0}, C_{dt_0}) = 1$ as well.

Claim 2: Let p be a non-singular point of C, in finite position, then C_{dt_0} passes through p iff p is one of the finitely many points $\{p_1, \ldots, p_m\}$ witnessing the class of C.

One direction of the claim follows immediately from Claim 1. Conversely, suppose that $p = (p_1, p_2)$ lies in finite position and $C_{dt_0}(p_1, p_2)$, $(\dagger)$. We have a parametrisation of the line L_p of the form;

$(x(t), y(t)) = (p_1 + \alpha t, p_2 + \beta t)$

where L_p denotes any of the tangent lines witnessing the class of C, translated to p. We then have that;

$F(x(t), y(t); t) = F(p_1, p_2) = 0$ $(**)$

Differentiating $(**)$ with respect to t, we obtain that;

$F_x|_{(x(t),y(t);t)}x'(t) + F_y|_{(x(t),y(t);t)}y'(t) + F_t|_{(x(t),y(t);t)} = 0$

Setting $t = 0$, we obtain that;

$F_x|_p\alpha + F_y|_p\beta + F_{dt_0}|_p = 0$, $(***)$

By (†), $(**)$ and the fact that p is non-singular, we obtain that L_p is the tangent line to C at p. Hence, p must witness the class of C.

Claim 3. Let $\{q_1, \ldots, q_n\}$ be the finitely many points lying at infinity. Then C_{dt_0} passes through q_j, for $1 \leq j \leq n$, and, moreover;

$I_{q_j}(C_{t_0}, C_{dt_0}) = 1$

The points at infinity are given by the intersections of C with the line l passing through Q. By the choice of l, given at the beginning of the proof, these intersections all define non-singular points of C. If $q \in (C \cap l)$ and $I_q(C, l) \geq 2$, then l would define the tangent line of C at q. This contradicts the fact that l was chosen to avoid the finitely many tangent lines $\{\tau_1, \ldots, \tau_m\}$ witnessing the class of C. Hence, $I_q(C, l) = 1$, (†), and the fact that there exist n distinct points at infinity then follows by an application of Bezout's theorem, using the assumption that $deg(C) = n$. We now claim that, if $q \in (C \cap l)$, then q belongs to $\{C_t\}_{t \in P^1}$, $(*)$. In order to see this, first observe that the line l at infinity (defined by $Z = 0$ in the choice of coordinates $\{X, Y, Z\}$) is *fixed* by the translation θ_t along the tangent lines witnessing the class of C;

$\theta_t : (x, y) \mapsto (x - t\alpha, y - t\beta) : (X : Y : Z) \mapsto (X - t\alpha Z : Y - t\alpha Z : Z)$

The claim $(*)$ then follows from the fact that C_t is defined as $Zero(F \circ \theta_t)$ and the defining equation F of C vanishes on q. We further claim that q belongs to C_{dt_0}, for $t_0 = 0$, $(**)$. In order to see this, choose a system of coordinates $\{x', y'\}$ such that $q = (q_1, q_2)$ is in finite position.

It is a simple algebraic calculation to show that we can find a polynomial $G(x', y'; \bar{z})$ such that the family of curves $\{C_t\}_{t\in P^1}$ is represented by $G(x', y'; \Phi^{new}(t))$, for a choice of morphism $\Phi^{new} : P^1 \rightarrow P^{\frac{n(n+3)}{2}}$. As before, let Par_t^{new} be the image of P^1 under the morphism Φ^{new}. It is a straightforward algebraic calculation to show that, if $\theta : P^2 \rightarrow P^2$ is chosen to be a *homographic* change of variables, then the corresponding induced morphism $\Theta : P^{\frac{n(n+3)}{2}} \rightarrow P^{\frac{n(n+3)}{2}}$, on the parameter space for projective algebraic curves of degree n, is also a homography. By construction, we have that $\Theta \circ \Phi = \Phi^{new}$, $(***)$, where Φ denoted the old parametrisation of Par_t. As Θ is a homography, using the identity $(***)$ and the chain rule, we obtain that, for corresponding points $\{\Phi(t), \Phi^{new}(t)\}$ of $\{Par_t, Par_t^{new}\}$, the tangent line $l_{\Phi(t)}$ of Par_t is mapped by Θ to the tangent line $l_{\Phi^{new}(t)}$ of Par_t^{new}. It therefore follows that, for given $t_0 \in P^1$, the curves infinitamente vicine to C_{t_0} (see Remarks 4.5), can be computed by differentiating with respect to the parameter t in *either* of the coordinate systems $\{\{x, y\}, \{x', y'\}\}$. By $(*)$, we have that;

$G(q_1, q_2; t) = 0$, for $t \in P^1$

Therefore, differentiating with respect to t, we have that;

$\frac{\partial G}{\partial t}(q_1, q_2; t) = 0$

In particular, setting $t = t_0 = 0$, we obtain the claim $(**)$. We now consider the pencil of curves defined by $\{C_{t_0}, C_{dt_0}\}$, which clearly has finite intersection with l, hence defines a g_n^1. By $(**)$, we have that the set of intersections $(C \cap l)$ are base points (branches) for this g_n^1. By $(\dagger)$ and results of [9], we have that the base branch contribution of any of these intersections is 1. In particular, it follows that, for a *generic* choice of C_{dt_0}, that $I_{q_j}(l, C_{dt_0}) = 1$, for $1 \leq j \leq n$. Now applying the usual argument on tangent lines, we obtain that $I_q(C_{t_0}, C_{dt_0}) = 1$, for $1 \leq j \leq n$ as well. The result of Claim 3 then follows.

Claim 4. Let $\{p_1, \ldots, p_d\}$ be the d nodes of C. Then C_{dt_0} passes through p_j, for $1 \leq j \leq d$, and, moreover;

$I_{p_j}(C_{t_0}, C_{dt_0}) = 2$

Let $p_j = (c_j, d_j)$ be such a point, and let;

$(x(t), y(t)) = (c_j + \alpha t, d_j + \beta t)$

be a parametrisation of any of the tangent lines $\{\tau_1, \ldots, \tau_m\}$, translated to p_j. We have that;

$F(x(t), y(t); t) = F(c_j, d_j) = 0 \; (*)$

Moreover, as $(x(t), y(t))$ defines a node of the translated curve C_t, we have that $F_x|_{(x(t),y(t);t)} = 0$ and $F_y|_{(x(t),y(t);t)} = 0$. Hence, differentiating $(*)$ with respect to t, we obtain that;

$F_t|_{(x(t),y(t);t)} = 0$

Setting $t = 0$, we obtain that $F_{dt_0}|_{p_j} = 0$, that is C_{dt_0} passes through p_j. Hence, the first part of the claim is shown. The second part of the claim depends heavily on a geometric argument. Let l_{t_0} be the tangent line to Par_t, at t_0, as given immediately before Remarks 4.5. As explained in Remarks 4.5, the curve C_{dt_0} corresponds to a point Q on l_{t_0}, and we denote by O, the point corresponding to C_{t_0} in Par_t, hence $l_{t_0} = l_{OQ}$. By Remarks 4.5, the line l_{t_0} defines the tangent line to the branch $\gamma_0^{t_0}$, corresponding to t_0, in the parametrisation $\Phi : P^1 \rightarrow Par_t$.

Similarly to the argument in Remarks 4.5, let;

$U = (\Phi^{-1}(NonSing(Par_t)) \setminus \{t_0\}) \subset P^1$

and consider the covers $V \subset U \times P^{\frac{(n+1)(n+2)}{2}}$ and $V^* \subset U \times P^{\frac{(n+1)(n+2)}{2}}$ defined by;

$V = \{(t, y) : y \in l_{O\Phi(t)}\}$

$V^* = \{(t, \lambda) : H_\lambda \supset l_{O\Phi(t)}\}$

Let $\bar{V}$ and $\bar{V}^*$ define the Zariski closure of these two covers inside $P^1 \times P^{\frac{n(n+3)}{2}}$. Using the method of Lemma 6.14 in [9], one can show that the fibre $\bar{V}^*(t_0)$ consists of parameters for forms H_λ containing the tangent line l_{t_0}. Using the duality argument in Lemma 5.4 below, one then deduces that the fibre $\bar{V}(t_0)$ defines the tangent line l_{t_0}. The details are left to the reader. We now have that $\bar{V}(t_0, Q)$ holds, hence, as t_0 is regular for the cover $(\bar{V}/P^1)$, given generic $t_0' \in \mathcal{V}_{t_0} \cap P^1$, we can find $Q' \in \mathcal{V}_Q \cap P^{\frac{n(n+3)}{2}}$, such that $\bar{V}(t_0', Q')$, hence, Q' belongs to

$l_{O\Phi(t'_0)}$. Denote by $D_{Q'}$ the plane curve of order n corresponding to the point Q' in Par_n We now consider the pencil of curves $\mathcal{P}$ defined by the line $l_{O\Phi(t'_0)} \subset Par_n$. By construction, this pencil contains the plane curves $\{C_{t_0}, C_{t'_0}, D_{Q'}\}$. As $t'_0 \in Par_t$, the plane curve $C_{t'_0}$ is an infinitesimal translation of C_{t_0} in the direction defined by the tangent lines $\{\tau_1, \ldots, \tau_m\}$. By the assumption at the beginning of the proof, for a given node p_j, this direction does not coincide with either of its two distinct tangent lines $\{l_{\gamma^1_{p_j}}, l_{\gamma^2_{p_j}}\}$. We are, therefore, able to apply Theorem 1.13 of [11], to conclude that $C_{t_0} \cap C_{t'_0} \cap \mathcal{V}_{p_j}$ consists of exactly two distinct points $\{p^1_j, p^2_j\}$, situated on the branches $\{\gamma^1_{p_j}, \gamma^2_{p_j}\}$ of C_{t_0}, and moreover these intersections are transverse. By elementary facts on linear systems, $\{p^1_j, p^2_j\}$ are base points for the pencil defined by $\mathcal{P}$, and, moreover, $C_{t_0} \cap C_{t'_0} = C_{t_0} \cap D_{Q'}$. It follows immediately that $C_{t_0} \cap D_{Q'} \cap \mathcal{V}_{p_j}$ also consists of exactly the two points $\{p^1_j, p^2_j\}$. Again, by elementary facts on linear systems, see [9] for more details, these intersections are also transverse. We have, therefore, shown, using the non-standard definition of intersection multiplicity, given in [8] or [9], that $I_{p_j}(C_{t_0}, C_{dt_0}) = 2$. Hence, Claim 4 is shown.

We now complete the proof of Theorem 4.3. Combining Claims 1,2,3 and 4, the total multiplicity of intersection between C and C_{dt_0} is $n + m + 2d$. By construction, $deg(C)deg(C_{dt_0}) = n.n = n^2$. Hence, Bezout's Theorem gives that $n + m + 2d = n^2$, as required. By the results of [8], Plucker's formula also holds for the algebraic definition of intersection multiplicity.

□

Remarks 4.6. *The proof that we have given follows Plucker's original geometrical idea and the presentation of Severi in* [14]. *Although long, the methods used are almost entirely geometrical, and adapt easily to handle cases where the singularities of C are more complicated. The reader is invited to extend the approach to these situations. One can find algebraic proofs in the literature, for example in* [2]. *Unfortunately, these proofs generally fail to give a precise calculation of intersection multiplicity for C and some other curve C', a problem that we observed in Remarks 1.12 of* [11].

5. The Transformation of Branches by Duality

The purpose of this section is to give a general account of the theory of duality and to develop the connection with the theory of branches given in [9]. We first give a brief account of Grassmannians on P^w. We define;

$G_{w,j} = \{P_j \subset P^w\}$, $(0 \leq j \leq w)$

where P_j is a plane of dimension j. It follows from classical well known arguments that $G_{w,j}$ may be given the structure of a smooth algebraic variety, see for example [2] p193. (These arguments begin with the observation that $G_{w,j}$ can be identified with the set of $j+1$-dimensional subspaces of a $w+1$-dimensional vector space over L.) The fact that $G_{w,j}$ defines a projective algebraic variety follows from the Plucker embedding;

$\rho : G_{w,j} \rightarrow P(\wedge^{j+1}L^{w+1}) = P^{C^{w+1}_{j+1}-1}$

given by sending a $j+1$ dimensional subspace of L^{w+1}, with basis $\{v_1, \ldots, v_{j+1}\}$, to the multivector $v_1 \wedge \ldots \wedge v_{j+1}$. It is easily checked that the Plucker map ρ defines a morphism of algebraic varieties, is injective on points and its differential $d\rho_x$ has maximal rank, for $x \in G_{w,j}$. In the case when the underlying field $L = \mathcal{C}$, one can then use the Immersion Theorem and Chow's Theorem to show that $Image(\rho)$ has the structure of a projective algebraic variety. In general, one can show;

The image of the Grassmannian $G_{w,j}$ under the Plucker embedding ρ is defined by a linear system of quadrics.

For want of a convenience reference, we leave the reader to check that the proof given of this result in [2] holds for arbitrary characteristic. We now observe the following duality between $G_{w,j}$ and $G_{w,w-(j+1)}$, for $0 \leq j \leq w-1$;

Lemma 5.1. *If $P_j \subset P^w$ is a j-dimensional plane, then;*

$\{\lambda \in P^{w*} : H_\lambda \supset P_j\}$ (1)

determines a $w-(j+1)$-dimensional plane $P^*_{w-(j+1)}$ in P^{w*}. Conversely, if $P^*_{w-(j+1)} \subset P^{w*}$ is a $w-(j+1)$-dimensional plane, then;

$\{x \in P^w : x \in \bigcap_{\lambda \in P^*_{w-(j+1)}} H_\lambda\}$ (2)

determines a j-dimensional plane P_j in P^w. Moreover, these correspondences are inverse and determine a closed projective variety;

$I \subset G_{w,j} \times G_{w,w-(j+1)}$

Proof. The proof is quite elementary. If $P_j \subset P^w$ is a j-dimensional plane, then one can find an independent sequence $\{\bar{a}_0, \ldots, \bar{a}_j\}$ defining it,$(*)$, see Section 1 of [9]. The condition that a hyperplane H_λ contains P_j is then given by the conditions;

$a_{0i}\lambda_0 + a_{1i}\lambda_1 + \ldots + a_{wi}\lambda_w = 0$, $(0 \leq i \leq j)$

It is elementary linear algebra, using $(*)$, to see that these conditions determine a plane of codimension $(j+1)$ in P^{w*}. For the converse direction, if $P^*_{w-(j+1)} \subset P^{w*}$ is a $w-(j+1)$-dimensional plane, then one can find an independent sequence $\{\bar{b}_0, \ldots, \bar{b}_{w-(j+1)}\}$ defining it, $(**)$. The condition that $x \in P^w$ is contained in the intersection of the planes defined by $P^*_{w-(j+1)}$ is then given by the conditions;

$x_0 b_{0i'} + x_1 b_{1i'} + \ldots + x_w b_{wi'} = 0$, $(0 \leq i' \leq w-(j+1))$

By the same elementary linear algebra argument, using $(**)$, these conditions determine a plane of codimension $(w-j)$ in P^w, that is a plane of dimension j. The fact that these correspondences are inverse follows immediately from the relations;

$a_{0i}b_{0i'} + a_{1i}b_{1i'} + \ldots + a_{wi}b_{wi'} = 0$, $(0 \leq i \leq j, 0 \leq i' \leq w-(j+1))$ $(\dagger)$

and an elementary dimension argument. The final part of the lemma follows by checking that the relations $(\dagger)$ can be defined by matrix multiplication on representatives of $G_{w,j}$ and $G_{w,w-(j+1)}$. If $\{U_I\}$ and $\{U_J\}$ define the standard open affine covers of these Grassmannians, as given on p193 of [2], then these relations clearly define closed algebraic subvarieties $I_{I,J} \subset U_I \times U_J$. Using standard patching arguments, we then obtain a closed algebraic subvariety $I \subset G_{w,j} \times G_{w,w-(j+1)}$ determining the duality correspondence.

□

Remarks 5.2. *The duality correspondence I clearly induces bijective maps, in the sense of model theory;*

$* : G_{w,j} \rightarrow G_{w,w-(j+1)}$

$*^{-1} : G_{w,w-(j+1)} \rightarrow G_{w,j}$

When the underlying field L has characteristic 0*, it follows that they define isomorphisms in the sense of algebraic geometry. However, in non-zero characteristic, it is difficult to determine whether these maps define morphisms or are seperable, therefore, whether the maps are inverse in the sense of algebraic geometry. As we will be concerned with the behaviour of algebraic curves under duality, we will be able to deal with this problem using arguments on Frobenius that we have already seen in* [9]*. The reader should also note that* $*$ *is* not *canonical, even as a set theoretic map, for it depends on a particular identification of* P^w *and* P^{w*}*. We will, henceforth, denote the (set theoretic) inverse* $*^{-1}$ *by* $*$*. This is motivated by the fact that we can identify* P^w *naturally with* P^{w**}*, in which case the relation* (2)*, defining* $*^{-1}$ *in the previous lemma, becomes an instance of the relation* (1)*.*

As an application of the above, we have;

Lemma 5.3. *Tangent Variety of a Projective Algebraic Curve*

Let $C \subset P^w$ *be* any *projective algebraic curve and let;*

$V \subset NonSing(C) \times P^w$ *be* $\{(x,y) : x \in NonSing(C) \wedge y \in l_x\}$

Then V *defines an irreducible algebraic variety and, if* $\overline{V} \subset C \times P^w$ *defines its Zariski closure, then, for a singular point* p*, which is the origin of branches* $\{\gamma_p^1, \ldots, \gamma_p^m\}$*, the fibre* $\overline{V}(p)$ *consists exactly of the finite union* $\bigcup_{1 \leq j \leq m} l_{\gamma_p^j}$ *of the tangent lines to the* m *branches at* p*.*

Proof. The fact that V defines an irreducible algebraic variety follows easily from arguments given in Section 1 of [9]. Let $G_{w,1}$ be the Grassmannian of lines in P^w, let $G_{w,w-2}$ be the Grassmannian of planes of codimension 2 in P^{w*} and let I be the duality correspondence between $G_{w,1}$ and $G_{w,w-2}$, as defined in Lemma 5.1. Without loss of generality, we can find algebraic forms $\{G_1, \ldots, G_{w-1}\}$ defining $NonSing(C)$, see Section 1 of [9]. For each G_j, with $1 \leq j \leq w-1$, the differential dG_j determines a morphism;

$dG_j : NonSing(C) \rightarrow P^{w*} : x \mapsto dG_j(x) = (\frac{\partial G_j}{\partial X_0}(x) : \ldots : \frac{\partial G_j}{\partial X_n}(x))$

We then obtain a morphism;

$\Phi_1 : NonSing(C) \rightarrow G_{w,w-2} : x \mapsto (dG_1(x), \ldots, dG_{w-1}(x)) = P_x$

Using the duality correspondence I and the observation that, for $x \in NonSing(C)$, the tangent line l_x is determined by the intersection of the hyperplanes determined by $dG_j(x)$, for $1 \leq j \leq w-1$, see Section 1 of [9], we obtain a morphism;

$\Phi_2 : NonSing(C) \rightarrow G_{w,1} : x \mapsto l_x$

By construction, we must clearly have the duality $I(l_x, P_x)$, whenever $x \in NonSing(C)$, (†). Let $C_1 = \overline{Im(\Phi_1)}$ and $C_2 = \overline{Im(\Phi_2)}$. Let $\Gamma_{\Phi_1} \subset C \times C_1$ and $\Gamma_{\Phi_2} \subset C \times C_2$ be the irreducible correspondences defined by $\overline{Graph(\Phi_1)}$ and $\overline{Graph(\Phi_2)}$. Let $\Gamma^*_{\Phi_2} \subset C \times G_{w,w-2}$ be the dual correspondence to Γ_{Φ_2} defined by;

$\Gamma^*_{\Phi_2}(x, y) \equiv \exists z(\Gamma_{\Phi_2}(x, z) \wedge I(z, y))$

We then have that $\Gamma^*_{\Phi_2}$ defines a closed irreducible projective variety. By (†), $\Gamma^*_{\Phi_2}$ agrees with Γ_{Φ_1} on the open subset obtained by restricting to $NonSing(C)$. Hence, we have that Γ_{Φ_1} defines the dual correspondence to Γ_{Φ_2}. Let $W_{\Phi_1} \subset C \times C_1 \times P^{w*}$ and $W_{\Phi_2} \subset C \times C_2 \times P^w$ be defined by;

$W_{\Phi_1} = \{(y, P, z) : \Gamma_{\Phi_1}(y, P) \wedge z \in P\}$

$W_{\Phi_2} = \{(y, l, z) : \Gamma_{\Phi_2}(y, l) \wedge z \in l\}$

We have that W_{Φ_1} and W_{Φ_2} are closed irreducible projective varieties. Let $p_{13} : C \times C_{1,2} \times P^w \rightarrow C \times P^w$ be the projection map. Let V^* be defined as in Lemma 6.13 of [9]. We have that $p_{13}(W_{\Phi_1})$ and $p_{13}(W_{\Phi_2})$ agree with V^* and V, restricted to $NonSing(C)$, hence $p_{13}(W_{\Phi_1}) = \bar{V}^*$ and $p_{13}(W_{\Phi_2}) = \bar{V}$. It follows that, for a singular point p of C, the fibre $\bar{V}(p)$ consists of a finite number of lines determined by the planes appearing in the fibre $\bar{V}^*(p)$. The result of the theorem then follows from the description of the fibre $\bar{V}^*(p)$ given in Lemma 6.13 of [9]. □

One can also formulate the following "desingularised version" of Lemma 5.3;

Lemma 5.4. *Let $C \subset P^w$ be any projective algebraic curve, with a choice of nonsingular model $C^{ns} \subset P^{w'}$ and birational morphism $\Phi : C^{ns} \rightarrow C$. Then;*

Desingularised Version of Lemma 5.3:

Let $U = \Phi^{-1}(NonSing(C)) \subset C^{ns}$ and let;

$V \subset U \times P^w$ be $\{(x,y) : x \in U \wedge y \in l_{\Phi(x)}\}$

Then V defines an irreducible algebraic variety and if $\overline{V} \subset C^{ns} \times P^w$ defines its Zariski closure, then, for $p_j \in C^{ns}$, corresponding to a branch γ_p^j of C, the fibre $\overline{V}(p_j)$ consists exactly of the tangent line $l_{\gamma_p^j}$.

Proof. The proof is merely a question of changing the parameter space from C to C^{ns} and adapting the argument of the previous lemma. The details are left to the reader.

□

In a similar vein, we also have;

Lemma 5.5. *Intuitive Construction of Tangent Lines*

Let $C \subset P^w$ be a projective algebraic curve and let $O \in C$ be a given fixed point, possibly singular. Let;

$V \subset (NonSing(C) \setminus \{O\}) \times P^w$ be $\{(x,y) : x \in NonSing(C) \wedge y \in l_{Ox}\}$

Then V defines an irreducible algebraic variety and, if $\overline{V} \subset C \times P^w$ defines its Zariski closure, then, if O is the origin of branches $\{\gamma_p^1, \ldots, \gamma_p^m\}$, the fibre $\overline{V}(O)$ consists exactly of the finite union $\bigcup_{1 \leq j \leq m} l_{\gamma_p^j}$ of the tangent lines to the m branches at O.

Proof. The proof is the same as Lemma 5.3, except that we use the definition of V^* given in Lemma 6.14 of [9] and the description of the fibre $\bar{V}^*(O)$ given in Lemma 6.14 of [9].

□

We also require the following result;

Lemma 5.6. *Convergence of Intersections of Tangents*

Let $C \subset P^2$ be a plane projective algebraic curve, not equal to a line, let O be a nonsingular point, which is the origin of a branch γ_O of character $(1, k-1)$, $(k \geq 2)$, and suppose that $char(L)$ is zero or coprime to k. Let;

$$V \subset (NSing(C) \setminus \{O\}) \times P^2 = \{(x,y) : x \in NSing(C) \wedge y \in l_O \cap l_x\}$$

Then V defines a generically finite cover of $NonSing(C)$. Moreover, if $\bar{V} \subset C \times P^2$ defines its Zariski closure, then the fibre $\bar{V}(O)$ consists exactly of the point O.

Proof. The fact that V defines a generically finite cover of $Nonsing(C)$ follows easily from the assumption that C is not a line. The infinite fibres of the cover V correspond to the, at most, finitely many points $\{p_1, \ldots, p_n\} \subset (NonSing(C) \setminus \{O\})$ such that l_O defines their tangent line. Let $U = (NonSing(C) \setminus \{O, p_1, \ldots, p_n\})$ and let V^{res} be the restriction of the cover V to U. We clearly have that $\bar{V}(O) = \bar{V}^{res}(O)$ and that the fibres of V^{res} consist of a unique intersection. In particular, the Zariski closure $\bar{V}^{res}$ is irreducible. Using the fact that O is nonsingular and an important property of Zariski structures, given in Theorem 3.3 of [16], it follows that the fibre $\bar{V}^{res}(O)$, hence the fibre $\bar{V}(O)$ also consists of a unique point O'. In order to show that $O = O'$, we employ an argument using infinitesimals;

We may assume that O lies at the origin $(0,0)$ of the affine coordinate system (x,y) and that the tangent line l_O corresponds to the line $y = 0$ in this coordinate system. Using Lemma 3.7, we can find a parametrisation of the branch γ_O, in the sense of Theorem 6.1 of [9], of the form $(t, y(t))$, where $y(t)$ is an algebraic power series. By the assumption on the tangent line l_O, the definition of a parametrisation in Theorem 6.1 of [9] and the assumption on the character of the branch, we must have that $ord_t(y(t)) = k$, so we can assume that $y(t) = t^k u(t)$, for the given $k \geq 2$, where $u(t) \in L[[t]]$ is a unit. Let $F(x,y)$ define C in the coordinate system (x,y), then we have that;

$$F(t, y(t)) = 0 \text{ and } F_x|_{(t,y(t))}.1 + F_y|_{(t,y(t))}.y'(t) = 0, \; (*)$$

as a formal identity in the power series ring $L[[t]]$, see [9] for similar calculations. We now work in the nonstandard model $K = L[[\epsilon]]^{alg}$. It follows easily from the paper [6] that we can interpret ϵ as an infinitesimal in $\mathcal{V}_0$. By construction, we clearly have that the identity $(*)$ holds, replacing t by ϵ, $(**)$. By definition of the specialisation map, given in [6], and $(**)$, we then have that $(\epsilon, y(\epsilon)) \in NonSing(C) \cap \mathcal{V}_{(0,0)}$. Using $(**)$ again, we also have that the equation of the tangent line $l_{(\epsilon,y(\epsilon))}$ in the nonstandard model K is given by;

$$(y - y(\epsilon)) = y'(\epsilon)(x - \epsilon) \ (***)$$

We now compute the intersection of $l_{(\epsilon,y(\epsilon))}$ with $y = 0$. By $(***)$, we obtain that;

$$l_{(0,0)} \cap l_{(\epsilon,y(\epsilon))} = (x_\epsilon, 0), \text{ where } x_\epsilon = \frac{(\epsilon y'(\epsilon) - y(\epsilon))}{y'(\epsilon)}$$

Using the assumption on $char(L)$, it is easy to calculate that $ord_t(ty'(t) - y(t)) \geq k$ and $ord_t(y'(t)) = k - 1$. It follows that $ord_t(x_t) \geq 1$, considered as an element of $L[[t]]$, hence, by the definition of the specialisation map in [6], we must have that $x_\epsilon \in \mathcal{V}_0$ and $(x_\epsilon, 0) \in \mathcal{V}_{(0,0)} = \mathcal{V}_O$. By construction, we have that $\bar{V}^{res}((\epsilon, y(\epsilon)), (x_\epsilon, 0))$, hence, by specialisation, we have that $\bar{V}^{res}(O, O)$. It follows that the fibre $\bar{V}^{res}(O)$ consists exactly of the point O as required. □

We now make the following definition;

Definition 5.7. *Let $C \subset P^2$ be a plane projective curve. We define a nonsingular point p, which is the origin of a branch γ_p having character $(1, r)$, for $r \geq 2$, to be a flex. We define p to be an ordinary flex, if $r = 2$.*

We now consider the duality construction applied to plane projective curves. We claim the following;

Lemma 5.8. *Let $C \subset P^2$ be a plane projective curve, defined in the coordinate system $\{X, Y, Z\}$ by the irreducible polynomial $F(X, Y, Z)$. Then, if C is not a line, the differential;*

$$dF : NonSing(C) \to P^{2*} : x \mapsto l_x = \left(\frac{\partial F}{\partial X}(x) : \frac{\partial F}{\partial Y}(x) : \frac{\partial F}{\partial Z}(x)\right)$$

defines a morphism, with the property that $C^* = \overline{Im(dF)} \subset P^{2*}$ is also a plane projective curve. If $char(L) \neq 2$, then the following conditions are equivalent;

(i). C has finitely many flexes.

(ii). $dF : C \leftrightsquigarrow C^*$ defines a birational morphism and $C = C^{**}$.

In particular, if $char(L) = 0$, then (ii) always holds, hence any plane projective curve over L, with $char(L) = 0$, can only have finitely many flexes, and if C has infinitely many flexes, the duality morphism dF is inseperable.

Proof. The fact that dF defines a morphism on $NonSing(C)$ follows easily from the fact that the partial derivatives $\{\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\}$ cannot all vanish at a nonsingular point of C. By basic results in algebraic geometry, the image $Im(dF) \subset P^{2*}$ is constructible and irreducible. As C is not a line, it must have infinite distinct tangent lines, hence, $C^* = \overline{Im(dF)}$ defines a plane projective algebraic curve. We first show;

$(i) \Rightarrow (ii)$

As C has finitely many flexes, there exists an open subset $U \subset NonSing(C)$ with the property that, for $x \in U$, the corresponding branch γ_x has character $(1, 1)$. As C^* is a projective algebraic curve, we can find a further open set $V \subset U$ such that $dF(V) \subset NonSing(C^*)$. If $G(U, V, W)$ is a defining equation for C^*, then, using the first part of this Lemma and Lemma 5.1, in order to identify P^{2**} with P^2, we obtain a morphism;

$dG : NonSing(C^*) \rightarrow P^2$

We now claim that;

$(dG \circ dF) = Id_V : V \rightarrow P^2$ $(*)$

This implies immediately that $C^{**} = C$ and $dF : C \leftrightsquigarrow C^*$ is a birational map, with birational inverse $dG : C^* \leftrightsquigarrow C$. We now show $(*)$. Suppose that $x_0 \in V$, with corresponding $y_0 = dF(x_0) \in NonSing(C^*)$. As the branch γ_{x_0} has character $(1, 1)$, and $char(L) \neq 2$, we have that the result of Lemma 5.6 holds for x_0, (replacing O in the Lemma). That is, if $V_1 \subset C \times P^2$ is the closed subvariety given in Lemma 5.6, then the fibre $V_1(x_0) = x_0$ and, if $x \in C$, with $x \neq x_0$, then $V_1(x) = l_{x_0} \cap l_x$. As $y_0 \in NonSing(C^*)$, we can apply Lemma 5.5, and obtain a closed subvariety $V_2 \subset C^* \times P^{2*}$ such that the fibre

$V_2(y_0) = l_{y_0}$ and, if $y \in C^*$, with $y \neq y_0$, then $V_2(y) = l_{y_0 y}$. Shrinking V if necessary, we can assume that if $x \in V$ with $x \neq x_0$, then $dF(x) \neq y_0$ and $l_x \cap l_{x_0}$ is a point, (†). Now, define the closed relation $S \subset V \times P^2 \times P^{2*}$ by;

$$S(x, x', y) \equiv V_1(x, x') \wedge V_2(dF(x), y)$$

By construction of $\{V_1, V_2\}$, we have that if $x \in V$ with $x \neq x_0$, then the fibre $S(x) = \{(x', y) : x' \in l_{x_0} \cap l_x, y \in l_{y_0 dG(x)}\}$. Using the assumption (†), the fibre $S(x)$ consists of a point and a line. Moreover, the point of intersection $l_{x_0} \cap l_x \in P^2$ is in dual correspondence with the line $l_{y_0 dF(x)} \subset P^{2*}$, $(**)$. Using Lemma 5.1, this follows from determining the intersection of the pencil of lines parametrised by $l_{y_0 dF(x)} = l_{dF(x_0) dF(x)}$. By construction of dF, this is exactly the intersection of tangent lines $l_{x_0} \cap l_x$.

We now use $(**)$ and a simple limiting argument, to show that this duality must also hold for the fibre $S(x_0)$. Let $R_2 \subset P^2 \times P^{2*}$ be the incidence relation, given by $R_2(u, v)$ iff $v \in H_u$, where H_u is the linear form with coefficients given by u. By $(**)$, we have that;

$$S(x) \subset R_2 \ (x \in V, x \neq x_0) \ (***)$$

Using elementary model theoretic arguments, $(***)$ is a closed condition on V. Hence, we must have that $S(x_0) \subset R_2$ as well, $(****)$.

Again, by construction of $\{V_1, V_2\}$, we have that the fibre $S(x_0) = \{(x_0, y) : y \in l_{y_0}\}$. By $(****)$, this shows that x_0 must be in dual correspondence with l_{y_0}. We, therefore, must have that $dG(y_0) = x_0$, hence $(*)$ is shown.

$(ii) \Rightarrow (i)$.

There are two approaches to this problem. We will show later in the section that if (ii) holds and $[(dF)^{-1}]^*$ is the induced bijection on branches guaranteed by Lemma 5.7 of [9], then, for a given branch γ of C with character (α, β), the corresponding branch γ^* of C^* has character (β, α). If C had infinitely many flexes, this would imply that C^* had infinitely many singular points, which is impossible. Hence, (i) must hold. For now, we will give a more direct algebraic proof.

Suppose that (ii) holds and C has infinitely many flexes. By Remarks 6.6 of [9] and using a similar argument to the above, we can assume that there exists an open $U \subset NonSing(C)$, with the property that $dF(U) \subset NonSing(C^*)$ and every $p \in U$ is a flex. For ease of notation, we abbreviate the dual morphism dF by F^{dual}. We will show directly that if p is in U, then;

$dF^{dual}_{(a,b)} : T_{p,C} \rightarrow T_{F^{dual}(p),C^*}$ is identically zero, $(*)$.

where we have used the differential and tangent space notation, given on p170 of [4]. By standard algebraic considerations, see Section 1 of [9], this implies that the morphism F^{dual} is ramified in the sense of algebraic geometry at every point of U. Using results of [10], see particularly Theorem 2.8, and a standard results about Zariski covers, that there can only exist finitely many ramification points, we conclude that the morphism F_{dual} is inseperable. In particular, it cannot be birational as required.

In order to show $(*)$, let $\{X, Y, Z\}$ be a choice of coordinates for $P^2(L)$ and, without loss of generality, assume that $U \subset (Z \neq 0)$. Working in the coordinate system $\{x = \frac{X}{Z}, y = \frac{Y}{Z}\}$,without loss of generality, we can assume that a given flex O of U is located at the origin $(0, 0)$ of the coordinate system (x, y) and its tangent line corresponds to $y = 0$. Arguing as in Lemma 5.5, we can find a parametrisation of the branch γ_O of the form $(t, y(t))$, where $ord_t y(t) \geq 3$. As in Lemma 5.6, we can consider $x_\epsilon = [\epsilon : y(\epsilon) : 1]$ as defining a point in $C \cap \mathcal{V}_O$, for an appropriate choice of non-standard model K. The coordinates of $F^{dual}(x_\epsilon)$ in the dual space P^{2*} are then given by taking the cross product $\phi(\epsilon) \times \phi'(\epsilon)$, where $\phi(\epsilon) = (\epsilon, y(\epsilon), 1)$. We have;

$$\phi(\epsilon) \times \phi'(\epsilon) = (\epsilon, y(\epsilon), 1) \times (1, y'(\epsilon), 0) = (-y'(\epsilon), 1, \epsilon y'(\epsilon) - y(\epsilon))$$

so $F^{dual}(x_\epsilon) = [-y'(\epsilon) : 1 : \epsilon y'(\epsilon) - y(\epsilon)] \in C^* \cap \mathcal{V}_{F^{dual}(O)}$, see also the corresponding calculation in Lemma 5.6. Now let $\{U, V, W\}$ be projective coordinates for $P^{2*}(L)$ and let $\{u = \frac{U}{V}, w = \frac{W}{V}\}$. Let F^{dual}_u and F^{dual}_w be the components of F^{dual} with respect to the affine coordinate system (u, w). Then;

$$F^{dual}_u(\epsilon, y(\epsilon)) = -y'(\epsilon) \qquad F^{dual}_v(\epsilon, y(\epsilon)) = \epsilon y'(\epsilon) - y(\epsilon)$$

Differentiating these expressions with respect to ϵ, see [9] for the justification of such calculations in non-zero characteristic, we obtain;

$\frac{\partial F_u^{dual}}{\partial x}|_{(\epsilon,y(\epsilon))}.1 + \frac{\partial F_u^{dual}}{\partial y}|_{(\epsilon,y(\epsilon))}.y'(\epsilon) = -y''(\epsilon)$

$\frac{\partial F_v^{dual}}{\partial x}|_{(\epsilon,y(\epsilon))}.1 + \frac{\partial F_v^{dual}}{\partial y}|_{(\epsilon,y(\epsilon))}.y'(\epsilon) = \epsilon y''(\epsilon)\ (**)$

Now setting $\epsilon = 0$ in $(**)$ and using the fact that $ord_t y(t) \geq 3$, we obtain immediately the result $(*)$, hence this direction of the lemma is shown.

In order to finish the result, observe that if $char(L) = 0$, one can use Lemma 5.6 directly and the argument of the first part of this Lemma $((i) \Rightarrow (ii))$ to show directly that the dual morphism $dF : C \leftrightsquigarrow C^*$ is birational. If C has infinitely many flexes, then one can use the argument of the second part of the Lemma $((ii) \Rightarrow (i))$ to show that the dual morphism $dF : C \rightarrow C^*$ is inseperable.

□

Remarks 5.9. *For the remainder of this section, we will always assume that* $char(L) \neq 2$ *a given projective algebraic curve* C *has* finitely *many flexes. As the duality morphism is then birational, this will allow us to use the theory of branches that we have have developed in previous papers. The exceptional case that* C *has infinitely many flexes was studied extensively in the paper* [5]. *We give a brief summary of the main results;*

(Corollary 2.2) Let C *be a non-singular projective algebraic curve of degree* n*, with infinitely many flexes. Then, if* $char(L) = p \neq 2$*, we have that* $p|n-1$.

(Proposition 3.7) Let C *be a non-singular projective algebraic curve of degree* $p+1$*, with infinitely many flexes, and* $char(L) = p \neq 2$*, then* C *is projectively equivalent to the plane curve with equation;*

$XY^p + YZ^p + ZX^p = 0$

In particular, by direct calculation, the duality morphism dF *is purely inseperable, hence, biunivocal, and* $C = C^{**}$.

(Corollary 4.3, Lemma 4.4 and Proposition 4.5)

Let C be a generic non-singular projective algebraic curve of degree $dp+1$, with $(d>1)$, then the duality morphism dF is biunivocal, but $C\neq C^{**}$.

The case of singular projective algebraic curves with infinitely many flexes is more difficult. For example, the graph of Frobenius, given by;

$YZ^{p-1}=X^{p}$

has degree p, infinitely many flexes and is singular. The dual curve C^{*} is given by $X=0$, in particular C^{**} is a point, hence $C\neq C^{**}$, and the duality morphism dF is purely inseperable, therefore, biunivocal.

We will return to these examples and the case when $char(L)=2$ in the final section of this paper.

We now make the following definition;

Definition 5.10. *Let $C\subset P^{2}$ be a plane projective curve. We define a multiple tangent line l of C to be a line which is tangent to* at least *2 branches of C.*

We now prove the following;

Lemma 5.11. *Let C be a projective algebraic curve, not equal to a line, with finitely many flexes, then every multiple tangent line of C corresponds to a singularity of C^{*}, in particular, C has finitely many multiple tangent lines.*

Proof. By the hypotheses and Lemma 5.7, the duality morphism dF is birational. By Lemma 5.7 of [9], the duality morphism induces a bijection $[(dF)^{-1}]^{*}$ between the branches of C and of C^{*}. We first claim the following;

Let γ be a branch of C with tangent line l_{γ}, and let O_{γ} be the point defined by l_{γ} in P^{2*}, then the corresponding branch $[(dF)^{-1}]^{*}(\gamma)$ of C^{*} passes through O_{γ}. $(*)$

The claim $(*)$ is trivially true, by definition of the duality morphism, if γ is centred at a non-singular point of C, $(**)$. Otherwise, we obtain the result by a straightforward limiting argument;

We use the notation and variety $\bar{V}$ of Lemma 5.4. Then define $S \subset C^{ns} \times P^2 \times P^{2*}$ by;

$$S(x,y,z) \equiv \overline{V}(x,y) \wedge \Gamma_{[\Phi \circ dF]}(x,z)$$

where $\Phi \circ dF : C^{ns} \to C^*$ is a birational morphism. Let $R_1 \subset P^2 \times P^{2*}$ be the incidence relation defined by $R_1(u,v)$ iff $u \in H_v$, where H_v is the linear form with coefficients given by v. If $x \in \Phi^{-1}(NonSing(C))$, then, by $(*)$ and Lemma 5.4, we have that the fibre $S(x)$ consists of the tangent line $l_{\Phi(x)} \subset P^2$ and its corresponding point of the dual space $dF \circ \Phi(x) \in P^{2*}$. In particular $S(x) \subset R_1$, $(***)$. As S and R_1 are closed varieties, the relation $(***)$ holds for all $x \in C^{ns}$. Now fix any branch γ of C with corresponding point $Q_\gamma \in C^{ns}$, then, by $(***)$, the fibre $S(Q_\gamma)$ consists of the tangent line l_γ and its corresponding point $O_\gamma \in P^{2*}$. From the definition of S we have that $\Phi \circ dF(Q_\gamma) = O_\gamma$. Hence, the corresponding branch $[(dF)^{-1}]^*(\gamma)$ of C^* must be centred at O_γ, see Lemma 5.7 of [9]. Therefore, the result $(*)$ is shown.

Now, if l is a multiple tangent line to C, by the previous definition, there exist at least 2 distinct branches $\{\gamma_1, \gamma_2\}$ of C such that $l = l_{\gamma_1} = l_{\gamma_2}$. By the claim $(*)$, if O_l is the corresponding point of P^{2*}, then the corresponding branches $\{[(dF)^{-1}]^*(\gamma_1), [(dF)^{-1}]^*(\gamma_2)\}$ of C^* both pass through O_l. By Lemma 5.4 of [9], O_l must be a singular point of C^*. Hence, the Lemma follows immediately from the fact that a plane projective curve can only have finitely many singular points.

□

Lemma 5.12. *Let C be a projective algebraic curve, with finitely many flexes, then $Cl(C)$, as defined in Definition 4.1, is the same as $deg(C^*)$ and $deg(C)$ is the same as $Cl(C^*)$. In particular, if C has at most nodes as singularities, then;*

$$deg(C^*) = n(n-1) - 2d$$

Proof. By Definition 1.12 of [9], $deg(C^*)$ is given by the number of distinct intersections $\{p_1, \ldots, p_n\}$ of C^* with a generic line $l \subset P^{2*}$. Let O_l be the corresponding generic point of P^2 and let $\{L_{p_1}, \ldots, L_{p_n}\}$ be the corresponding lines of P^2. We clearly have that each line L_{p_j} passes through O_l, for $1 \leq j \leq n$. If $\{U_{[dF]}, V_{[dF]}\}$ are the canonical open subsets of $\{C, C^*\}$, with respect to the birational morphism dF, see Section 1 of [9], we can assume that all the intersections $\{p_1, \ldots, p_n\}$ lie inside $V_{[dF]}$. Let $\{q_1, \ldots, q_n\}$ be the corresponding non-singular points of C. By the definition of the duality map dF, the tangent line l_{q_j} is exactly L_{p_j}, for $1 \leq j \leq n$. Hence, by Definition 4.1, we must have that $Cl(C) \geq n$. If $Cl(C) \geq n+1$, we could, without loss of generality, find a further q_{n+1}, distinct from $\{q_1, \ldots, q_n\}$, lying inside $U_{[dF]}$, witnessing the class of C, see Definition 4.1. In this case, the tangent line $l_{q_{n+1}}$ would pass through O, hence, again just using the definition of the duality map dF, the corresponding point p_{n+1} of P^{2*} would lie on $C^* \cap l \cap V_{[dF]}$. This would give at least $n+1$ intersections of C and l, which is a contradiction. Hence, $Cl(C) = deg(C^*) = n$ as required. The claim that $deg(C) = Cl(C^*)$ follows from the same argument, replacing C by C^* and C^* by C^{**}, and using the fact that $C = C^{**}$. Finally, the relation $deg(C^*) = n(n-1) - 2d$ follows immediately from the Plucker formula, proved in Theorem 4.3, and the relation $Cl(C) = deg(C^*)$, which we have just shown.

□

We now show the following important result;

Theorem 5.1. *Transformation of Branches by Duality*

Let C be a plane projective algebraic curve, with finitely may flexes, then, if γ is a branch of C with character (α, β), such that $\{\alpha, \alpha+\beta\}$ are coprime to $char(L) = p$, the corresponding branch of C^* has character (β, α).

Remarks 5.13. *One can find algebraic "proofs" of this result in the literature. These proofs use a local parametrisation of the branch and an analysis of the resulting parametrisation after applying the duality morphism. Unfortunately, such proofs fail to give the correct answer in the case when C has infinitely many flexes. The explanation of this discrepancy is that in such cases, the resulting "parametrisations" fail to give parametrisations in the sense of Theorem 6.1 of* [9]. *One requires the fact that the duality morphism is birational in order to show this stronger claim. This suggests that a geometric proof of this result is required. The one that we give is based on Severi's methods.*

Proof. Let γ^0 be a given branch of C, with character (α, β), as in the statement of the theorem, centred at O, and let l_{γ^0} be its tangent line. By $(*)$ of Lemma 5.11, the corresponding branch γ^{0*} of C^* is centred at the corresponding point $O' = O_{l^0_\gamma}$ of the dual space P^{2*}. Let l be a generic line through O', and let A_l be the corresponding point of the dual space P^2. As $O_{l^0_\gamma} \in l$, we have, by duality, that $A_l \in l_{\gamma^0}$, and, by genericity, that $A_l \neq O$. The line l then parametrises the pencil of lines passing through A_l. Now pick a further generic point B belonging to l and let l_B be the corresponding line in the dual space P^2. Again, as $B \in l$, we obtain, by duality that $A_l \in l_B$, and, by genericity, that $l_B \neq l_{\gamma^0}$. Now choose a parametrisation $\Phi : P^1 \rightarrow l_B$ such that $\Phi(0) = A_l$. We will denote the corresponding pencil of lines in P^{2*} by $\{l_t\}_{t\in P^1}$. By construction, we have that $l_0 = l$ and that the pencil $\{l_t\}_{t\in P^1}$ is centred at B. Now, considering P^{2*} as parametrising the set of lines in P^2, it defines a g_n^2 on C, where $n = deg(C)$. Each line $l_t \subset P^{2*}$, then defines a $g_n^1 \subset g_n^2$ on C. We will denote this g_n^1 by $g_n^{1,t}$.

By a suitable choice of coordinates, we may, without loss of generality, assume that the line l_B corresponds to the line l_∞ in the affine coordinate system $\{x = \frac{X}{Z}, y = \frac{Y}{Z}\}$, the point $\Phi(0) = A_l$ corresponds to $[0:1:0]$, the point O corresponds to $[0:0:1]$ and the point $\Phi(\infty)$ corresponds to $[1:0:0]$. By choice of l_B, we can assume that the line l_∞ intersects C transversely in ordinary simple points $\{p_1, \ldots, p_n\}$. Let $t_j = \Phi^{-1}(p_j)$ and let $U = P^1 \setminus \{t_1, \ldots, t_n, \infty\}$. By the choice of A_l, we can assume that $0 \in U$. Now, for $t \in U$, the corresponding $g_n^{1,t}$, defined above, has no fixed branch contribution. We then have that $g_n^{1,t} = (h_t)$, where h_t is the non-constant rational function on C defined by $h_t(x, y) = x - ty$, see Lemma 2.4. In particular, we have that $g_n^{1,0}$ is defined by $(x = 0)$. Now let $U_{[dF]}$ be the canonical set associated to the birational morphism $dF : C \leftrightsquigarrow C^*$, and let $V \subset U_{[dF]}$ be obtained by removing;

(i). All the finitely many flexes from $U_{[dF]}$.

(ii). All the finitely many points on $C \cap l_\infty$ from $U_{[dF]}$.

(iii). All the finitely many non-singular points of $U_{[dF]}$, whose tangent line is parallel to the y-axis.

Now, using the method before Lemma 3.9, we can, for $t \neq \infty$, define the rational function $\frac{dh_t}{dx}$. By the calculation there, if C is defined by $f(x, y) = 0$ in the coordinate system (x, y), we have that;

$$\frac{dh_t}{dx} = \frac{h_x f_y - h_y f_x}{f_y} = 1 + t\frac{f_x}{f_y}$$

Then, using the notation of Lemma 2.4, for $t \in U \setminus \{0\}$, $(\frac{dh_t}{dx} = 0)$ is the same as $(-\frac{f_y}{f_x} = t)$, considered as weighted sets on C. As $-\frac{f_y}{f_x}$ is a non-constant rational function on C, then, by Lemma 2.4, we can associate a $g_1^m = (-\frac{f_y}{f_x})$ to it.

Let $J_t = Jac(g_n^{1,t})$, see Definition 3.2 and Definition 3.22. Then J_t consists of a weighted set of branches $\{\alpha_1\gamma_1^t, \ldots, \alpha_r\gamma_r^t\}$. We claim the following;

$(J_t \cap V) = (g_1^m \cap V)$ for $(t \in U \setminus \{0\})$ $(\dagger)$

In order to show $(\dagger)$, let γ be a branch of V. By the definition of V, γ is centred at the finite position (a, b), has character $(1, 1)$ and its tangent line l_γ is not parallel to the y-axis. Let $(x, y(x))$ be a parametrisation of γ, in the form given by Lemma 3.7. If γ belongs to J_t, then $ord_\gamma(h_t) = 2$, $val_\gamma(h_t) < \infty$ and h_t determines an algebraic power series at the branch γ;

$h_t = \lambda + (x - a)^2\psi(x - a)$ with $\psi(0) \neq 0, \lambda < \infty$

We then have that;

$\frac{dh_t}{dx} = (x - a)[2\psi(x - a)] + [(x - a)\psi'(x - a)]$ $(char(L) \neq 2)$

At $x = a$, the expression in brackets reduces to $2\psi(0) \neq 0$, hence $ord_\gamma(\frac{dh_t}{dx}) = 1$ and $val_\gamma(\frac{dh_t}{dx}) = 0$. This shows that γ is counted once in the weighted set $(\frac{dh_t}{dx} = 0)$, and, therefore, once in the weighted set W_t of the corresponding g_m^1 defined above. If γ belongs to the g_m^1, then, $val_\gamma(\frac{dh_t}{dx}) = 0$, for some $t \in U \setminus \{0\}$. Reversing the above argument, using the fact that $val_\gamma(h_t) < \infty$, we obtain that $ord_\gamma(\frac{dh_t}{dx}) = 2$, hence γ belongs to J_t as required. Therefore, $(\dagger)$ is shown. Note that an almost identical argument to the above was carried out in Lemma 3.9.

We now consider the behaviour of the g_m^1 at the branch γ^0. We claim that γ^0 is counted β times in the weighted set W_0 of this g_m^1, $(\dagger\dagger)$. Let $(x(s), y(s))$ be a parametrisation of the branch at $(0, 0)$. Using the definition of a parametrisation in Theorem 6.1 of [9] and the fact that the tangent line l_{γ_0} is given by $x = 0$, we obtain immediately that $ord_s x(s) = \alpha + \beta$ and $ord_s y(s) = \alpha$. Using the fact that $f(x(s), y(s)) = 0$, we obtain that;

$f_x|_{(x(s),y(s))}x'(s) + f_y|_{(x(s),y(s))}y'(s) = 0$

If either f_x or f_y is identically zero on C, then the image of the map dF is contained in a line, which implies, by Lemma 5.8, that C has infinitely many flexes. Hence, we must have that $x'(s) \equiv 0$ iff $y'(s) \equiv 0$. If both $x'(s) \equiv 0$ and $y'(s) \equiv 0$, we can find algebraic power series $\{x_1(s), y_1(s)\}$ such that $x(s) = x_1(s^p)$ and $y(s) = y_1(s^p)$. This would contradict the construction of a parametrisation, as defined in Theorem 6.1 of [9], see also the method used in [10], Remarks 2.3. Hence, we have $x'(s) \neq 0$, and $y'(s) \neq 0$, and;

$-\frac{f_y}{f_x}|_{(x(s),y(s))} = \frac{x'(s)}{y'(s)}$

By the assumption on $\{\alpha, \beta\}$ in the statement of the theorem, we obtain that $ord_s(-\frac{f_y}{f_x}|_{(x(s),y(s))}) = (\alpha + \beta - 1) - (\alpha - 1) = \beta$. By definition of the g^1_m, this implies (††).

Now consider the $g^1_{m'}$ on C^*, given by the pencil $\{l_t\}_{t \in P^1}$. By Theorem 1.14, and the fact that C and C^* are birational, this transfers to a $g^1_{m'}$ on C. By choice of B, the $g^1_{m'}$ on C and C^* have no base branches. We claim that $g^1_m = g^1_{m'}$, (†††). Let $\{V, V'\}$ be canonical sets associated to the birational map $dF : C \leftrightsquigarrow C^*$, where V was defined above. Then, using Lemma 2.17 of [9], there exists an open subset $U' \subset U \subset P^1$ such that, for $t \in U'$, the corresponding weighted sets $\{W_t, W'_t\}$ of the $\{g^1_m, g^1_{m'}\}$ on C, consist of branches(points) which are simple for $\{W_t, W'_t\}$ respectively, based inside V or V'. In this case, a point $p \in W'_t$ corresponds to a transverse intersection between l_t and C^*. The line l_p in the dual space C is then a tangent line to a non-singular point p' of C, having character $(1, 1)$. We, therefore, have that $I_{p'}(C, l_p) = 2$, hence p' is counted once for $Jac(g^{1,t}_n)$. Combining this with the result (†), we obtain that $W_t = W'_t$, hence (†††) is shown.

We now finish the proof of the theorem. By Theorem 1.12, the fact that the g^1_m on C has no fixed branches, and the result (††), for generic $t \in \mathcal{V}_0$, the weighted set $W_t \cap C \cap \gamma^0$ of the g^1_m consists of β distinct branches(points), centred in $U' \cap \gamma^0$. By (†††), these correspond to β transverse intersections $C^* \cap l_t \cap \gamma^{0*}$. It follows immediately that $I_{\gamma^{0*}}(C^*, l_0) = \beta$, hence, as $l_0 = l$ was chosen to be generic through O', the order of the branch is β. We now follow through the same argument, replacing C by C^* and C^* by $C^{**} = C$. It follows immediately that the class of the branch γ^{0*} must be the order of the branch γ^0. Hence, the character of the branch γ^{0*} is (β, α) as required.

□

Remarks 5.14. *The assumption that* $\{\alpha, \alpha+\beta\}$ *are co-prime to* $char(L) = p$ *is necessary. Consider the projective algebraic curve* C *defined by;*

$$y = x^p + x^{p+2}$$

In projective coordinates, this is defined by $YZ^{p+1} = X^pZ^2 + X^{p+2}$ *and the duality morphism* dF *is given by;*

$$dF : [X : Y : Z] \mapsto [2X^{p+1} : -Z^{p+1} : 2X^pZ - YZ^p]$$

$$(x, y) \mapsto (-2x^{p+1}, y - 2x)$$

The duality morphism is clearly seperable, hence C *must have finitely many flexes.* C *is non-singular in the affine coordinate system* (x, y), *and the character of the unique branch* γ_0 *of* C *at the origin* $(0, 0)$ *is* $(1, p-1)$. *The proof of the theorem shows that the order of the corresponding branch* γ_0^* *of* C^* *is* $(p + 1) \neq (p - 1)$, *for* $char(L) \neq 2$.

One can easily construct further examples using appropriate rational parametrisations.

6. A Generalised Plucker Formula

The purpose of this section is to give a geometric proof of a generalisation of the Plucker formula of Section 4. We also discuss the question of representation of plane algebraic curves in greater detail. We first require the following definition;

Definition 6.1. *We will define a plane projective algebraic curve* C *to be normal if it is has finitely many flexes and all its branches* γ *have character* (α, β), *with* $\{\alpha, \alpha + \beta\}$ *coprime to* $char(L) = p$.

We first show the following;

Lemma 6.2. *Let* C *be a normal plane projective curve with* $class(C) = m$, *as defined in Definition 4.1,* $genus(C) = \rho$, *as defined in Definition 3.33 and* $order(C) = n$. *Then;*

$$\rho = \tfrac{1}{2}[m + \textstyle\sum_\gamma(\alpha(\gamma) - 1)] - (n - 1)$$

where the sum is taken over the finitely many singular branches and, for such a branch γ, $\alpha(\gamma)$ *gives the order of the branch. In particular,* *if C is a normal projective curve, having at most nodes as singularities, we obtain the formula shown earlier;*

$$\rho = \frac{m}{2} - (n-1)$$

Proof. By Remarks 4.2 and the fact that C has finitely many flexes, we can suppose that, for generic P, the m tangent lines of C passing through P are based at ordinary simple points. In particularly, P does not lie on any of the finitely many tangent lines belonging to the singular branches of C. We consider the g_n^1 defined by the pencil of lines passing through P. We have that $Jac(g_n^1)$ consists exactly of the m ordinary branches witnessing the class of C and the finitely many singular branches γ, each counted $\alpha(\gamma) - 1$ times. In particular;

$$order(Jac(g_n^1)) = m + \sum_{\gamma}(\alpha(\gamma) - 1)$$

Now we obtain the first part of the lemma from the fact that $order(g_n^1) = n$ and Definition 3.33 of the genus of C. If C has at most nodes as singularities, then it has *no* singular branches. Therefore, $(\alpha(\gamma) - 1) = 0$ for any branch γ of C. The second part of the lemma then follows from the previous formula.

□

Using duality, we have;

Lemma 6.3. *Let C be a normal plane projective algebraic curve, not equal to a line, with the invariants $\{m, n, \rho\}$ as defined above. Then;*

$$\rho = \frac{1}{2}[n + \sum_{\gamma}(\beta(\gamma) - 1)] - (m-1)$$

where the sum is taken over the finitely many flexes, and, for such a branch γ, $\beta(\gamma)$ gives the class of the branch.

Proof. As C is normal and not equal to a line, we may apply Lemma 5.8, Lemma 5.12 and Theorem 5.1. In particular, we have that C^* has finitely many flexes, and, as in the previous lemma, for generic P, the tangent lines of C^*, passing through P, are based at ordinary simple points. We consider the $g_{n'}^1$ on C^* defined by the pencil of lines passing through P. We have that $n' = order(C^*) = class(C) = m$ and $class(C^*) = order(C) = n$ by Lemma 5.12. We also have that $Jac(g_m^1)$ consists exactly of the n ordinary branches witnessing the class of C^* and the finitely many singular branches, each counted $(\beta(\gamma)-1)$ times, by Theorem 5.1. Hence;

$$order(Jac(g^1_m)) = n + \sum_\gamma (\beta(\gamma) - 1)$$

where the sum is taken over the finitely many flexes of C, using the fact that C and C^* are birational, given in Lemma 5.8. We also have that $\rho = genus(C) = genus(C^*)$ by Lemma 5.8 and Theorem 3.35. Hence, we obtain the first part of the theorem from the fact that $order(g^1_m) = m$ and Definition 3.33 of the genus of C^*.

□

Theorem 6.4. *Generalised Plucker Formula*

Let C be a normal plane projective algebraic curve, not equal to a line, with $\{m, n\}$ defined as in the previous lemmas. Then;

$$3m - 3n = \sum_\gamma (\beta(\gamma) - 1) - \sum_\gamma (\alpha(\gamma) - 1)$$

where the sums are taken over the finitely many flexes and finitely many singular branches of C respectively. In particular, if C is a plane projective curve, having at most nodes as singularities and no flexes, then it is either a line or a smooth conic.

Proof. The first part of the theorem follows immediately by combining the first formulas given in Lemma 6.2 and Lemma 6.3. If C is a smooth plane projective curve, having at most nodes as singularities and no flexes, then it must be normal as all its branches γ have character $(1, 1)$ $(char(L) \neq 2)$. If C is not a line, then we can apply the first part of the theorem, to obtain that $m = n$. Using the Plucker formula given in Theorem 4.3, we have that $m = n^2 - n - 2d$, where d is the number of nodes of C. Hence, $n^2 - n - 2d = n$, which gives that $d = \frac{n(n-2)}{2}$. If $n \geq 3$, this contradicts Theorem 3.37. Hence, $n = 2$ and $d = 0$. The second part of the theorem then follows.

□

We now apply the above formulas to the study of normal plane projective curves, having at most nodes as singularities.

Theorem 6.5. *Let C be a normal plane projective curve, not equal to a line, having at most nodes as singularities, with the convention on summation of branches given above and $\{m, n, \rho, d\}$ as defined in the previous lemmas. Then, we obtain the class formula;*

$3m - 3n = \sum_\gamma (\beta(\gamma) - 1)$ (1)

and the genus formula;

$6\rho + 3n - 6 = \sum_\gamma (\beta(\gamma) - 1)$ (2)

and the node formula;

$3n(n-2) - 6d = \sum_\gamma (\beta(\gamma) - 1)$ (3)

In particular, if C has at most ordinary flexes, and i is the number of these flexes, we obtain the class formula, referred to as Plucker III' in [14]*;*

$3m = 3n + i$ (4)

and the genus formula;

$6\rho = i - 3n + 6$ (5)

and the node formula, referred to as Plucker III in [14]*;*

$6d = 3n(n-2) - i$ (6)

Proof. The class formula (1) follows from the Generalised Plucker formula and the fact that $(\alpha(\gamma) - 1) = 0$ for any branch γ of C, as C has at most nodes as singularities. The genus formula (2) follows from (1) and the formula $\rho = \frac{m}{2} - (n-1)$, given in Lemma 6.2. The node formula (3) follows from (1) and the Plucker formula $m = n(n-1) - 2d$, given in Theorem 4.3. The formulas (4), (5), (6) all follow immediately from the corresponding formulas (1), (2), (3) and the fact that;

$i = \sum_\gamma (\beta(\gamma) - 1)$

as an ordinary flex has character $(1, 2)$, hence, for the corresponding branch γ, $(\beta(\gamma) - 1) = 1$.

□

Remarks 6.6. *For a nonsingular projective algebraic curve C, one can show that $C \cong W$, where W is a sphere with g attached handles. For an arbitrary projective algebraic curve C', one can define the genus g as* *the number of attached handles for $C' \leftrightsquigarrow C$, where C is nonsingular. This is how the genus g is defined in, for example,* [3].

Lemma 6.7. *For a nonsingular plane algebraic curve C, Severi's Definition 3.33 of genus g_1 coincides with the definition g_2 in Remark 6.6.*

Proof. By Theorem 3.35 and Lemma 6.7, both definitions are birationally invariant, so we can reduce to the case of plane curves C with at most nodes as singularities.

By the degree-genus formula (check for nodal case), (see [3]), we have that;

$g_2 = \frac{(n-1)(n-2)}{2} - d$, where d is the number of nodes. (i)

By the genus and node formula, (Theorem 6.5 (5),(6)), we obtain;

$6g_1 + 3n - 6 = 3n(n-2) - 6d \ (ii)$

Substituting $d = \frac{3n(n-2)-6g_1-3n+6}{6}$, from (ii) into (i), we obtain $g_1 = g_2$.

□

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Flat 1, 98 Prestbury Road, Cheltenham, GL52 2DJ
E-mail address: depiro100@gmail.com